JENSEN CONVEX FUNCTIONS BOUNDED ABOVE ON NONZERO CHRISTENSEN MEASURABLE SETS

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Abstract. We prove that every Jensen convex function mapping a real linear Polish space into $\mathbb{R}$ bounded above on a nonzero Christensen measurable set is convex.

Functions satisfying

$$f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2}$$

for $x, y$ from the domain being a convex set are called Jensen convex and they play very important role in many branches of mathematics (more information on such functions can be find in [5]). A lot of authors were interested in finding conditions which implies the continuity of $f$ satisfying (1). Among others, W. Sierpiński, A. Ostrowski and M.R. Mehdi showed that every Jensen convex function which is Lebesgue measurable, or bounded above on a set of positive Lebesgue measure, or bounded above on a set of second category with the Baire property, has to be continuous (see [5, Theorems 9.3.1, 9.3.2, p.232 and Theorem 9.4.2, p.241]). P. Fischer and Z. Słodkowski generalized the result of Sierpiński; they proved that each Christensen measurable Jensen convex function mapping a real linear Polish space into $\mathbb{R}$ is continuous and convex (see [4, Theorem 2]). However the following problem seems to be open: does

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each Jensen convex function bounded above on a nonzero Christensen measurable set have to be continuous? This problem was formulated by K. Baron and R. Ger at the 21st International Symposium on Functional Equations (1983, Konolfingen, Switzerland) (see [6, 44, Problem (P239), pp. 285–286]).

We prove that each Jensen convex function \( f: X \to \mathbb{R} \) mapping a real linear Polish space \( X \) into \( \mathbb{R} \) bounded above on a nonzero Christensen measurable set is convex.

First, let us recall some basic definitions (cf. [2]–[4]) concerning Christensen measurability.

Let \( X \) be a real linear Polish space and let \( \mathcal{M} \) be the \( \sigma \)-algebra of all universally measurable subsets of \( X \); i.e. \( \mathcal{M} \) is the intersection of all completions of the Borel \( \sigma \)-algebra of \( X \) with respect to probability Borel measures. In the following by a measure we mean a countable additive Borel measure extended to \( \mathcal{M} \).

**Definition 1.** A set \( B \in \mathcal{M} \) is a Haar zero set iff there exists a probability measure \( u \) on \( X \) such that \( u(B + x) = 0 \) for each \( x \in X \). A set \( P \subset X \) is a Christensen zero set iff \( P \) is a subset of a Haar zero set. A set \( D \subset X \) is a Christensen measurable set iff there are \( B \in \mathcal{M} \) and a Christensen zero set \( P \) such that \( D = B \cup P \). Finally, a function \( f: X \to \mathbb{R} \) is said to be Christensen measurable iff \( f^{-1}(U) \) is a Christensen measurable set for each open set \( U \subset \mathbb{R} \).

**Lemma 1 ([1, Lemma 14]).** Let \( D \subset X \) be a nonzero Christensen measurable set and \( x \in X \setminus \{0\} \). Then there exist a Borel set \( D_x \subset D \) and \( y_x \in X \) such that the set \( k_x^{-1}(y_x + D_x) \subset \mathbb{R} \) has positive Lebesgue measure, where \( k_x: \mathbb{R} \to X \) is given by \( k_x(a) = ax \).

Now we prove the announced result.

**Theorem 1.** Assume \( f: X \to \mathbb{R} \) is Jensen convex. If

\[
\sup f(C) < \infty
\]

for a nonzero Christensen measurable \( C \subset X \), then \( f \) is convex.

**Proof.** Fix \( x \in X \setminus \{0\} \) and \( z \in X \), define \( \varphi: \mathbb{R} \to \mathbb{R} \) by

\[
\varphi(\alpha) = f(\alpha x + z) \quad \text{for } \alpha \in \mathbb{R}
\]

and note that it is Jensen convex. According to Lemma 1 there are a Borel set \( B \subset \mathbb{R} \) of positive Lebesgue measure and a \( y \in X \) such that

\[\alpha x - y \in C \quad \text{for } \alpha \in B.\]
Consequently, for $\alpha \in B$ we have
\[
\varphi \left( \frac{\alpha}{2} \right) = f \left( \frac{(\alpha x - y) + (y + 2z)}{2} \right) \\
\leq \frac{f(\alpha x - y) + f(y + 2z)}{2} \leq \sup f(C) + f(y + 2z)
\]

This shows that $\sup \varphi \left( \frac{1}{2} B \right) < \infty$ and, according to theorem of Ostrowski [5, Theorem 9.3.1, p.232], $\varphi$ is continuous. Hence, by [5, Theorem 5.3.5, p.133], $\varphi$ is convex and to finish the proof it is enough to apply the following simple remark:

If $X$ is a real linear space, then $f : X \to \mathbb{R}$ is convex if and only if for every $x \in X \setminus \{0\}$, $z \in X$ the function (3) is convex. \hfill \square

**Corollary 1.** Assume $X$ is a real linear Polish space and $f : X \to \mathbb{R}$ is additive. If (2) holds for a nonzero Christensen measurable set $C \subset X$, then $f$ is linear.

**References**


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