ANDRZEJ KASPERSKI

APPROXIMATION OF ELEMENTS OF THE SPACES $X^t$ AND $X^p$ BY NONLINEAR, SINGULAR KERNELES

Abstract. Let $l^p$ be a Musielak-Orlicz sequence space. Let $X^t$ and $X^p$ be the modular spaces of multifunctions generated by $l^p$. Let $K_{w,j} : R \to R$ for $j = 0, 1, 2, \ldots, w \in W$, where $W$ is an abstract set of indices. Assuming certain singularity assumption on the nonlinear kernel $K_{w,j}$ and setting $T_w(F) = (T_w(F)(i))_{i=0}^\infty$ with $(T_w(F))(i) = \{ \sum_{j=0}^{i} K_{w,j}(f(j)) : f(j) \in F(j) \}$, convergence theorems $T_w(F) \xrightarrow{X^t} F$ in $X^t$ and $T_w(F) \xrightarrow{X^p} F$ in $X^p$ are obtained.

1. Introduction. In [3] a general approximation theorem in modular space was obtained and applied to translation operators and linear integral operators in Musielak-Orlicz space $L^p$ of periodic functions as well as in Musielak-Orlicz space $L^p$ of sequences. The application in $L^p$ was extended in [4] to some nonlinear integral operators and in [1] and [2] to some operators in the space $X^t$ of multifunctions generated by $l^p$. In [6] an extension of the results of [3] to the case of approximation by some nonlinear operators in the Musielak-Orlicz space $L^p$ of sequences was obtained. The aim of this note is to obtain an extension of the result of [6] to the case of approximation by some nonlinear operators in the spaces $X^t$ and $X^p$ generated by $l^p$.

Let $N$ be the set of all nonnegative integers. Let $l^p$ be the Musielak-Orlicz sequence space generated by a modular $\rho(x) = \sum_{i=0}^{\infty} \varphi_i(|x(i)|), x = (x(i))$, with $\varphi$-functions $\varphi_i$, i.e. $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ and it is a nondecreasing continuous function such that $\varphi_i(u) = 0$ iff $u = 0$ and $\varphi_i(u) \to \infty$ as $u \to \infty$ for every $i \in N$. Let

$$X = \{ F : N \to 2^\mathbb{N} : F(i) \text{ is compact nonempty set for all } i \in N \}.$$ 

Let $\ell^p(F)(i) = \min_{x \in F(i)} x$, $\ell^p(F)(i) = \max_{x \in F(i)} x$ for all $F \in X$ and all $i \in N$. Let

$$X^t = \{ F \in X : F(i) = [a(i), b(i)] \text{ for all } i \in N \text{ and } a, b \in l^p \}.$$
Let \( W \) be an abstract nonempty set of indices and let \( \mathcal{W} \) be a filter of subsets of \( W \).

**DEFINITION 1.** A function \( g: \mathcal{W} \to \mathbb{R} \) tends to zero with respect to \( \mathcal{W} \), written \( g(w) \not\to 0 \), if for every \( \varepsilon > 0 \) there is a set \( W \subset \mathcal{W} \) such that \( |g(w)| < \varepsilon \) for all \( w \in W \).

### 2. General Lemma.

**DEFINITION 2.** A family \( T = (T_w)_{w \in \mathcal{W}} \) of operators \( T_w: X^1_\phi \to X^1_\phi \) will be called \( \mathcal{W} \)-bounded if there exist positive constants \( k_1, \ldots, k_8 \) and a function \( g: \mathcal{W} \to \mathbb{R}_+ \) such that \( g(w) \not\to 0 \) and for all \( F, G \in X^1_\phi \) there is a set \( W_{F,G} \subset \mathcal{W} \) for which

\[
\rho(a (f(T_w(F)) - f(T_w(G)))) \\
\leq k_1 \rho(ak_2 (f(F) - f(G))) + k_3 \rho(ak_4 (f(F) - f(G))) + g(w),
\]

and

\[
\rho(a (f(T_w(F)) - f(T_w(G)))) \\
\leq k_5 \rho(ak_6 (f(F) - f(G))) + k_7 \rho(ak_8 (f(F) - f(G))) + g(w),
\]

for all \( w \in W_{F,G} \) and every \( a > 0 \).

**DEFINITION 3.** Let \( F_w \in X^1_\phi \) for every \( w \in W \) and let \( F \in X^1_\phi \). We write \( F_w \xrightarrow{\mathcal{W}} F \) if for every \( \varepsilon > 0 \) and every \( a > 0 \) there is a \( W \subset \mathcal{W} \) such that

\[
\rho(a (f(F_w) - f(F))) < \varepsilon \quad \text{and} \quad \rho(a (\hat{f}(F_w) - \hat{f}(F))) < \varepsilon \quad \text{for every} \ w \in W.
\]

**DEFINITION 4.** Let \( S \subset X^1_\phi \). We denote

\[
S_{\mathcal{W}} = \{F \in X^1_\phi: F_w \xrightarrow{\mathcal{W}} F \text{ for some } F_w \in S, w \in W\}.
\]

**LEMMA 1.** Let \( S \subset X^1_\phi \) and let \( T = (T_w)_{w \in \mathcal{W}} \) be \( \mathcal{W} \)-bounded. If \( T_w(F) \xrightarrow{\mathcal{W}} F \) for every \( F \in S \), then \( T_w(F) \xrightarrow{\mathcal{W}} F \) for every \( F \in S_{\mathcal{W}} \).

**Proof.** Let \( a, \varepsilon > 0 \) be arbitrary and let \( F \in S_{\mathcal{W}} \) be given. Then there exist \( G \in S \) and \( W_1 \subset \mathcal{W} \) such that:

\[
\rho(3ak_2 (f(F) - f(G))) < \varepsilon/6k_1, \quad \rho(3ak_4 (\hat{f}(F) - \hat{f}(G))) < \varepsilon/6k_2, \quad \rho(3a (f(T_w(G)) - f(G))) < \varepsilon/6, \quad \rho(3a (\hat{f}(T_w(G)) - \hat{f}(G))) < \varepsilon/6, \quad g(w) < \varepsilon/6 \quad \text{for every} \ w \in W_1.
\]

where we may assume \( k_1, k_3 \geq 1 \). Let \( W_{F,G} \) be chosen for \( (T_w)_{w \in \mathcal{W}} \) and \( F, G \) according to the definition of \( \mathcal{W} \)-boundedness. Then we have

\[
\rho(a (f(T_w(F)) - f(F))) \leq \rho(3a (f(T_w(F)) - f(T_w(G)))) \\
+ \rho(3a (f(T_w(G)) - f(G))) + \rho(3a (f(F) - f(G))) \\
\leq k_1 \rho(3ak_2 (f(F) - f(G))) + k_3 \rho(3ak_4 (\hat{f}(F) - \hat{f}(G))) \\
+ \rho(3a (f(T_w(G)) - f(G))) + \rho(3a (f(F) - f(G))) + g(w).
\]

Taking \( W = W_1 \cap W_{F,G} \) we obtain \( \rho(a (f(T_w(F)) - f(F))) < \varepsilon \) for all \( w \in W \). We prove analogously that there exists a \( W \subset \mathcal{W} \) such that \( \rho(a (\hat{f}(T_w(F)) - \hat{f}(F))) < \varepsilon \) for every \( w \in W \). Hence \( T_w(F) \xrightarrow{\mathcal{W}} F \) because \( W_0 = W \cap W \subset \mathcal{W} \).
3. The application. Let now $W = \mathbb{N}$ and let the filter $\mathcal{W}$ consist of all sets $W \subset W$ which are complements of finite sets.

Let now $\varphi_i$ be convex for $i \in W$. Let for every $w \in W K_{w,j} : \mathbb{R} \to \mathbb{R}$ for $j \in W$, and let $K_{w,j}(0) = 0$ for all $w, j \in W$. We define for all $F \in X_\varphi^1$ and $i \in W$

\[(T_w(F))(i) = \{ \sum_{j=0}^{i} K_{w,i-j}(f(j)) : f(j) \in F(j), \ j = 0, 1, ..., i \}, \]

(1) \[T_w(F) = ((T_w(F))(i))_{i=0}^\infty.\]

We shall call $K$ a semisingular kernel, if the following conditions are satisfied, where

\[L_{w,i} = \sup_{u \neq v} \frac{|K_{w,i}(u) - K_{w,i}(v)|}{|u - v|},\]

(i) \[L(w) = (\sum_{i=0}^{\infty} L_{w,i}) \leq \sigma < \infty,\]

(ii) \[L_{w,j}/L(w) \not\to 0 \text{ for } j = 1, 2, ...\]

If moreover $(1/c)K_{w,0}(c) \not\to 1$ for every $c \neq 0$, then $K$ will be called a singular kernel.

DEFINITION 5. The sequence $(\varphi_i)_{i=0}^\infty$ is called $\tau_+$-bounded if there exist constants $k_1, k_2 \geq 1$ and a double-sequence $(\epsilon_{i,j})$ such that $\varphi_i(k_2 u) + \epsilon_{i,j}$ for $u \geq 0$, $i, j \in W$, where $\epsilon_{i,j} \geq 0$, $\epsilon_{i,0} = 0$, $\epsilon_j = \sum_{i=0}^{\infty} \epsilon_{i,j} \to 0$ as $j \to \infty$, $\epsilon = \sup \epsilon_j < \infty$.

THEOREM 1. If $K$ is a semisingular kernel such that $K_{w,i}(s) \geq K_{w,i}(t)$ for all $i, w \in W$ and $s > t$, $\varphi = (\varphi_i)_{i=0}^\infty$ is $\tau_+$-bounded, then $T_w : X_\varphi^1 \to X_\varphi^1$ for every $w \in W$ and the family $T$ of operators defined by (1) is $\mathcal{W}$-bounded.

Proof. It is easy to see that

\[f(T_w(F))(i) = \sum_{j=0}^{i} K_{w,i-j}(f(F)(j)) \]

and

\[f(T_w(F))(i) = \sum_{j=0}^{i} K_{w,i-j}(f(F)(j)).\]

$T_w(F)(i)$ is convex because $K_{w,i}$ is continuous for all $w, i \in W$. Let $c > 0$ be arbitrary. Then for $F, G \in X_\varphi^1$ we have (see the proof of Theorem 1 in [6])

\[\rho(c(f(T_w(F)) - f(T_w(G)))) \leq \sum_{i=0}^{\infty} \varphi_i(c \sum_{j=0}^{i} L_{w,j}|f(F)(i-j) - f(G)(i-j)|) \]

\[\leq k_1 \rho(ck_2 \sigma(f(F) - f(G))) + g(w),\]
\[
\rho(c(f(T_w(F)) - f(T_w(G)))) \leq k_1 \rho(ck_2 \sigma(f(F) - f(G))) + g(w),
\]
where \( g(w) = \frac{1}{L(w)} \sum_{j=1}^{\infty} L_{w,j} c_j \neq 0 \). So \( T_w: X^1_\phi \to X^1_\phi \) and \( T \) is \( \mathcal{W} \)-bounded.

Now, given a kernel \( K \) and a number \( c \neq 0 \), let us denote
\[
x^j_w(c) = (0, 0, \ldots, 0, K_{w,1}(c), K_{w,2}(c), \ldots).
\]
Moreover, let us write
\[
e_k = (\delta_{i,k})^{\infty}_{i=0} \text{ with } \delta_{i,k} = 1 \text{ for } i = k, \quad \delta_{i,k} = 0 \text{ for } i \neq k,
\]
\[
E_k = (\Delta_{i,k})^{\infty}_{i=0} \text{ with } \Delta_{i,k} = [0, 1] \text{ for } i = k, \quad \Delta_{i,k} = 0 \text{ for } i \neq k.
\]

**Lemma 2.** If \( F = c_0 e_0 + c_0 E_0 + \ldots + c_n e_n + c_n E_n \), then for every \( b > 0 \)
\[
\rho \left( b 2^{-1}(n+1)^{-1} (f(T_w(F)) - f(F)) \right)
\leq \frac{3}{2} \sum_{j=0}^{n} \rho \left( bx^j_w(d_j) \right) + \frac{1}{2} \sum_{j=0}^{n} \varphi_j(b \mid K_{w,j}(d_j) - d_j) \]
and
\[
\rho \left( b 2^{-1}(n+1)^{-1} (f(T_w(F)) - f(F)) \right)
\leq \frac{3}{2} \sum_{j=0}^{n} \rho \left( b x^j_w(d_j) \right) + \frac{1}{2} \sum_{j=0}^{n} \varphi_j(b \mid K_{w,j}(d_j) - d_j),
\]
where
\[
d_j = \begin{cases} c_j + c_j & \text{for } c_j \leq 0, \\
c_j & \text{for } c_j > 0, \end{cases} \quad d_j = \begin{cases} c_j & \text{for } c_j \leq 0, \\
c_j + c_j & \text{for } c_j > 0. \end{cases}
\]

**Proof.** It is easily seen that
\[
f(T_w(F))(i) - f(F)(i) = \begin{cases} \sum_{j=0}^{i} K_{w,i-j}(d_j) - d_i & \text{for } i \leq n, \\
\sum_{j=0}^{n} K_{w,i-j}(d_j) & \text{for } i > n, \end{cases}
\]
and
\[
f(T_w(F))(i) - f(F)(i) = \begin{cases} \sum_{j=0}^{i} K_{w,i-j}(d_j) - d_i & \text{for } i \leq n, \\
\sum_{j=0}^{n} K_{w,i-j}(d_j) & \text{for } i > n. \end{cases}
\]
So the proof is quite analogous to that of Lemma in [6] and we omit it.
We easily obtain (see [5, 8.13 and 8.14]) the following

**Lemma 3.** Let \( \varphi = (\varphi_j)_{j=0}^\infty \) satisfy the condition \((\delta)\). Let \( F \in X^1_\varphi \) and \( F = (F(i))_{i=0}^\infty \). Let \( F_w \) be such that \( F_w(i) = F(i) \) for \( i = 0, 1, \ldots, w \), and \( F_w(i) = 0 \) for \( i > w \) for every \( w \in \mathbb{N} \), then \( F_w \overset{\varphi,w}{\longrightarrow} F \).

**Theorem 2.** Let \( \varphi = (\varphi_j)_{j=0}^\infty \) satisfy the condition \((\delta)\). Let \( K \) be a singular kernel such that \( \rho(bx_j) \overset{\mathbb{R}}{\longrightarrow} 0 \) for every \( j \in \mathbb{N} \) and all \( b > 0 \). Let the assumptions of Theorem 1 hold. Then \( T_w(F) \overset{\varphi,w}{\longrightarrow} F \) for every \( F \in X^1_\varphi \).

**Proof.** Let \( S = \{c_0 e_0 + c_1 e_1 + \cdots + c_n e_n + c_n E_n : n \in \mathbb{N} \} \). From the assumptions and from Lemma 2, we easily obtain that \( T_w(F) \overset{\varphi,w}{\longrightarrow} F \) for every \( F \in S \). From the assumptions and from Lemma 3, \( S \overset{\varphi,w}{\longrightarrow} X^1_\varphi \), so, from Lemma 1 and Theorem 1, \( T_w(F) \overset{\varphi,w}{\longrightarrow} F \) for every \( F \in X^1_\varphi \).


**Let**

\[ X_\varphi = \{F \in X : f(F), \overline{f(F)} \in l^\varphi \}. \]

**Remark 1.** If \( F, G \in X_\varphi \) and \( a \in \mathbb{R} \), then \( F + G \in X_\varphi \).

**Proof.** Let \( F, G \in X_\varphi \) and \( a \in \mathbb{R} \). If \( F(i) \) and \( G(i) \) are compact, then \( F(i) + G(i) \) and \( aF(i) \) are compact. \( f(F + G)(i) = f(F)(i) + f(G)(i) \) and \( \overline{f(F + G)}(i) = \overline{f(F)}(i) + \overline{f(G)}(i) \) for every \( i \in \mathbb{N} \), so \( F + G \in X_\varphi \). If \( a = 0 \), then \( aF \in X_\varphi \). If \( a > 0 \), then \( f(aF)(i) = af(F)(i) \), \( \overline{f(aF)}(i) = a\overline{f(F)}(i) \). If \( a < 0 \), then \( f(aF)(i) = a\overline{f(F)}(i) \), \( \overline{f(aF)}(i) = af(F)(i) \). So \( aF \in X_\varphi \).

Let

\[ d(A, B) = \max \{\max_{x \in A} \min_{y \in B} |x - y|, \max_{y \in B} \min_{x \in A} |x - y|\} \]

for all compact nonempty \( A, B \subset \mathbb{R} \).

For all \( F, G \in X_\varphi \) we define the function \( D(F, G) \) by the formula

\[ D(F, G)(i) = d(F(i), G(i)) \]

for every \( i \in \mathbb{N} \).

Now, we introduce the function \( O \) by the formula

\[ O(i) = 0 \]

for every \( i \in \mathbb{N} \).

**Remark 2.** If \( F, G \in X_\varphi \), then \( D(F, G) \in l^\varphi \).

**Proof.** Let \( F, G \in X_\varphi \). We have for every \( a > 0 \)

\[ \rho(aD(F, G)) \leq \rho(a(D(F, O) + D(G, O))) \leq \rho(2aD(F, O)) + \rho(2aD(G, O)) \]

\[ \leq \rho(4a\overline{f(F)}) + \rho(4a\overline{f(G)}) + \rho(4af(G)) + \rho(4af(G)). \]

So \( D(F, G) \in l^\varphi \).

**Definition 2.** A family \( T = (T_w)_{w \in W} \) of operators \( T_w : X_\varphi \rightarrow X_\varphi \) will be called \((d, W)\) - bounded if there exist positive constants \( k_1, k_2 \) and a function \( g : W \rightarrow \mathbb{R}_+ \) such that \( g(w) \neq 0 \) and for \( F, G \in X_\varphi \) there is a set \( W_{F,G} \in W \) for which
\[ \rho(aD(T_w(F), T_w(G))) \leq k_1 \rho(ak_2 D(F, G)) + g(w) \quad \text{for all } w \in W_{F, G}, \ a > 0. \]

**DEFINITION 3'.** Let \( F_w \in X_\varphi \) for every \( w \in W \) and let \( F \in X_\varphi \). We write \( F \xrightarrow{d, \varphi, W} F \) if for every \( \varepsilon > 0 \) and every \( a > 0 \) there is a \( W \in W' \) such that \( \rho(aD(F_w, F)) < \varepsilon \) for every \( w \in W \).

**DEFINITION 4'.** Let \( S \subset X_\varphi \). We denote

\[ S_{d, \varphi, W} = \{ F \in X_\varphi : F \xrightarrow{d, \varphi, W} F \text{ for some } F_w \in S, \ w \in W \}. \]

**LEMMA 1'.** Let \( S \subset X_\varphi \) and let \( T = (T_w)_{w \in W} \) be \((d, W')\)-bounded. If \( T_w(F) \xrightarrow{d, \varphi, W} F \) for every \( F \in S \), then \( T_w(F) \xrightarrow{d, \varphi, W} F \) for every \( F \in S_{d, \varphi, W} \).

**Proof.** Let \( a, \varepsilon > 0 \) be arbitrary and let \( F \in S_{d, \varphi, W} \) be given. Then there exist \( G \in S \) and \( W_1 \in W' \) such that \( \rho(3ak_2 D(T_w(F), F)) < \varepsilon / 6k_1 \), \( \rho(3aD(T_w(G), G)) < \varepsilon / 6 \), \( \rho(3aD(F, G)) < \varepsilon / 6 \), \( g(w) < \varepsilon / 6 \) for every \( w \in W_1 \), where we may assume \( k_1 \geq 1 \). Let \( W_{F, G} \) be chosen for \( T \) and \( F, G \) according to the definition of \((d, W')\)-boundedness. We have

\[ \rho(aD(T_w(F), F)) \leq \rho(3aD(T_w(F), T_w(G))) + \rho(3aD(T_w(G), G)) + \rho(3aD(F, G)) \]
\[ \leq k_1 \rho(3ak_2 D(F, G)) + g(w) + \rho(3aD(F, G)) + \rho(3aD(T_w(G), G)). \]

Taking \( W = W_1 \cap W_{F, G} \) we obtain \( \rho(aD(T_w(F), F)) < \varepsilon \) for all \( w \in W \).

5. **The application.** Let \( \varphi, W, W' \) be such that as in section 3. Let for every \( w \in W, \ K_{w,j} : R \to R \) for \( j \in N \) and let \( K_{w,j}(0) = 0 \) for all \( w, j \in W \). We define for all \( F \in X_\varphi \) and all compact nonempty \( A \subset R \) the operators \( K_{w,i} \), \( T_w \) by the formulas

\[ K_{w,i}(A) = \{ K_{w,i}(x) : x \in A \} \quad \text{for all } w, i \in W, \]
\[ (T_w(F))(i) = \sum_{j=0}^{i} K_{w,i-j}(F(j)), \quad T_w(F) = \left( (T_w(F))(i) \right)_{i=0}^{\infty} \]

for all \( i, w \in W \).

We shall call \( K \) d-semisingular kernel, if the following conditions are satisfied:

(i) \( L(w) = \left( \sum_{i=0}^{\infty} L_{w,i} \right) \leq \sigma < \infty \),

(ii) \( L_{w,j} / L(w) \not\leq 0 \) for \( j = 1, 2, \ldots \)

where

\[ L_{w,i} = \max_{A \neq B} \frac{d(K_{w,i}(A), K_{w,i}(B))}{d(A, B)} \]

for all \( w \in W \) and \( i \in N \), with compact nonempty \( A, B \subset R \).

If moreover \( d(K_{w,0}(A), A) \not\leq 0 \) for every compact nonempty \( A \subset R \), then \( K \) will be called the d-singular kernel.
THEOREM 1'. Let $K$ be a $d$-semisingular kernel. Let $\varphi = (\varphi_i)_{i=0}^\infty$ be $\tau_+^*$-bounded. If $K_{w,i}(s) \geq K_{w,i}(t)$ for all $w, i \in W$ and $s > t$, then $T_w: X_* \to X_*$ for every $w \in W$ and the family $T$ given by (3) is $(d, W)$-bounded.

Proof. It is easy to see that $T_w: l^* \to l^*$ for every $w \in W$ (see [6, Theorem 1]). $K_{w,i}$ is continuous for all $w, i \in W$ so we easily obtain that $T_w: X_* \to X_*$ for every $w \in W$. Now, we prove that $T$ is $(d, \mathcal{W})$-bounded. Let $a > 0$ be arbitrary. Then for $F, G \in X_*$ we have (see also [6, the proof of Theorem 1])

$$
\rho(aD(T_w(F), T_w(G))) = \sum_{i=0}^{\infty} \varphi_i(a d(\sum_{j=0}^{i} K_{w,i-j}(F(j)), \sum_{j=0}^{i} K_{w,i-j}(G(j))))
$$

$$
\leq \sum_{i=0}^{\infty} \varphi_i(a \sum_{j=0}^{i} d(K_{w,i-j}(F(j)), K_{w,i-j}(G(j))))
$$

$$
= \sum_{i=0}^{\infty} \varphi_i(a \sum_{j=0}^{i} d(K_{w,j}(F(i-j)), K_{w,j}(G(i-j))))
$$

$$
\leq \sum_{i=0}^{\infty} \varphi_i(a \sum_{j=0}^{i} L_{w,j} d(F(i-j), G(i-j)))
$$

$$
\leq \frac{1}{L(w)} \sum_{j=0}^{\infty} \sum_{i=0}^{j} L_{w,j} \varphi_i(a L(w) d(F(i-j), G(i-j)))
$$

$$
= \frac{1}{L(w)} \sum_{j=0}^{\infty} L_{w,j} \sum_{i=0}^{j} \varphi_i a L(w) d(F(i), G(i)))
$$

$$
\leq k_1 \rho(a k_2 \sigma D(F, G)) + g(w),
$$

where

$$
g(w) = \frac{1}{L(w)} \sum_{j=1}^{\infty} L_{w,j} e_j \not\equiv 0.
$$

Now, let us write $E_k = (A_{i,k})_{i=0}^\infty$ with $A_{i,k} = A_k$ for $i = k$, $k \in \mathbb{N}$, where $A_k \subset \mathbb{R}$ and $A_k$ is compact nonempty for every $k \in \mathbb{N}$ and $A_{i,k} = 0$ for $i \neq k$. Moreover, let us write for every compact nonempty $A \subset \mathbb{R}$

$$
x_w^+(A) = (0, 0, \ldots, 0, K_{w,1}(A), K_{w,2}(A), \ldots).
$$

LEMMA 2'. If $F = E_0 + E_1 + \ldots + E_n$, then for every $b > 0$ the inequality

$$
\rho(b^{-1} (n+1)^{-1} D(T_w(F), F))
$$

$$
\leq \frac{3}{2} \sum_{j=0}^{n} \rho(b D(x_w^+(A_{i,j}), O)) + \frac{1}{2} \sum_{j=0}^{n} \varphi_j(b d(K_{w,0}(A_{i,j}), A_{i,j}))
$$

holds.

Proof. For any $a > 0$ we have (see also [6, the proof of Lemma])

27
\[
\rho(aD(T_w(F), F))
\]
\[
= \sum_{j=0}^{n} \varphi_j(\{a d(\sum_{j=0}^{l} K_{w,l-j}(A_{j,j}), A_{j,j})\}) + \sum_{l=0}^{\infty} \varphi_l(\{a d(\sum_{j=0}^{n} K_{w,l-j}(A_{j,j}), 0)\})
\]
\[
\leq \sum_{j=1}^{n} \varphi_j(\{a (\sum_{j=0}^{l} d(K_{w,l-j}(A_{j,j}), 0) + d(K_{w,0}(A_{j,j}), A_{j,j}))\})
\]
\[
+ \varphi_0(\{a d(K_{w,0}(A_{0,0}), A_{0,0})\}) + \sum_{l=n+1}^{\infty} \varphi_l(\{a (\sum_{j=0}^{n} d(K_{w,l-j}(A_{j,j}), 0))\})
\]
\[
\leq \frac{3}{2} \sum_{j=0}^{n} \sum_{i=1}^{\infty} \varphi_{i+j}(2a(n+1) d(K_{w,i}(A_{j,j}), 0))
\]
\[
+ \frac{1}{2} \sum_{i=0}^{n} \varphi_i(2a(n+1) d(K_{w,0}(A_{j,j}), A_{j,j})).
\]

We obtain the assertion after writing \(b = 2a(n+1)\).

**Lemma 3'.** Let \(\varphi = (\varphi_j)_{j=0}^{\infty}\) satisfy the condition \((\delta_2)\). Let \(F \in X_\varphi\) and \(F = (F(i))_{i=0}^{\infty}\). Let \(F_w\) be such that \(F_w(i) = F(i)\) for \(i = 0, 1, \ldots, w\) and \(F_w(i) = 0\) for \(i > w\) for every \(w \in \mathbb{W}\), then \(F \xrightarrow{d_{\varphi,\varphi}} F\).

**Proof.** We have for every \(a > 0\)

\[
\rho(a D(F_w, F)) = \sum_{i=w+1}^{\infty} \varphi_i(a D(F(i), 0))
\]
\[
= \sum_{i=w+1}^{\infty} \varphi_i(a \max(|f(F)(i)|, |\overline{f}(F)(i)|)) \leq 0.
\]

**Theorem 2'.** Let the assumptions of Lemmas 2', 3' and Theorem 1' hold.
Let \(K\) be the \(d\)-singular kernel. If for every compact and nonempty \(A \subset \mathbb{R}\) \(\rho(b D(x'_\varphi(A), O)) \leq 0\) for all \(b > 0\) and every \(j \in \mathbb{N}\), then \(T_w(F) \xrightarrow{d_{\varphi,\varphi}} F\) for every \(F \in X_\varphi\).

**Proof.** Let \(S = \{E_0 + E_1 + \ldots + E_n: n \in \mathbb{N}\}\). From the assumptions \(T_w(F) \xrightarrow{d_{\varphi,\varphi}} F\) for every \(F \in S\) and \(S_{d_{\varphi,\varphi}} = X_\varphi\) so we obtain the assertion from Theorem 1'.

**Remark 3.** If \(T_w: X^1_\varphi \to X^1_\varphi\) and the assumptions of Theorem 2' hold, then \(T_w(F) \xrightarrow{d_{\varphi,\varphi}} F\) for every \(F \in X^1_\varphi\).

**References**

