ASYMPTOTIC STABILITY OF MARKOV OPERATORS CORRESPONDING TO THE DYNAMICAL SYSTEMS WITH MULTIPLICATIVE PERTURBATIONS

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Abstract. We consider discrete time dynamical systems with multiplicative perturbations. We give a sufficient condition for the asymptotic stability of Markov operators on measures generated by dynamical systems with multiplicative perturbations.

Introduction. In this paper we consider a stochastically perturbed discrete time dynamical system of the form $x_{n+1} = S(x_n)\xi_n$, $n = 0, 1, 2, \ldots$, where $S$ is a given Borel measurable transformation, and $\xi_n$ are random variables. The trajectories of our system are sequences of random variables $x_n$ with values in $\mathbb{R}^d$. Systems of this type has been examined recently by K. Horbacz ([1], [2]). She considered the case when $\xi_n$ are continuously distributed with a common density $g$. In this case $x_n$ are also continuously distributed. K. Horbacz gave a sufficient condition for the convergence of the densities of $x_n$ to a unique stationary density.

We study the same problem without assumption that the common distribution of $\xi_n$ is continuous. In our case $x_n$ are in general random vectors without density. Our aim is to found sufficient conditions for the weak convergence of the distributions of $x_n$ to a stationary measure. The Proof of the main result is based on a theorem of A. Lasota and J.A. Yorke [5] concerning Markov operator on measures.
Our paper is divided into two sections. Section 1 contains some notations and definitions. The main result is formulated in Section 2.

1. Formulation of the problem. Consider a stochastically perturbed discrete time dynamical system of the form

\[ x_{n+1} = S(x_n)\xi_n \quad \text{for } n = 0, 1, 2, \ldots \]  

where \( S \) is a Borel measurable transformation of \( \mathbb{R}^d \) into itself, and \( \xi_n \) are independent random variables with values in \( \mathbb{R}_+ \).

We assume the following conditions:

(i) The random variables \( \xi_0, \xi_1, \ldots \) are independent and have the same nontrivial distributions \( G \), i.e., \( G \) is not concentrated on a single point.

(ii) \( S \) is a function which satisfies the Lipschitz condition:

\[ |S(x) - S(z)| \leq L|x - z| \quad \text{for } x, z \in \mathbb{R}^d \]

where the symbol \( | \cdot | \) denotes a norm in \( \mathbb{R}^d \).

(iii) There is \( \alpha_0 \in (0, 1) \) such that

\[ L^{\alpha_0} \int_{\mathbb{R}_+} y^{\alpha_0} G(dy) < 1. \]

(iv) The vector \( x_0, \) and variables \( \xi_i \) are independent for \( i = 0, 1, 2, \ldots \).

According to (1.0) the random vector \( x_n \) is function of \( x_0 \) and \( \xi_0, \xi_1, \ldots, \xi_{n-1} \). From this and from condition (iv) it follows that \( x_n \) and \( \xi_n \) are independent. Using this fact we will derive a recurrence formula for the measures

\[ \mu_n(A) = \text{Prob} \ (x_n \in A), \quad A \in B(\mathbb{R}^d). \]

Let consider now a bounded Borel measurable function \( h : \mathbb{R}^d \rightarrow \mathbb{R} \). The expectation \( E(z_{n+1}) \) of the random vector \( z_{n+1} = h(x_{n+1}) \), (where \( n \geq 0 \)) is given by

\[ E(z_{n+1}) = E(h(x_{n+1})) = \int_{\mathbb{R}^d} h(x)\mu_{n+1}(dx). \]

Since \( z_{n+1} = h(S(x_n)\xi_n) \) this implies

\[ E(z_{n+1}) = E(h(S(x_n)\xi_n)) = \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}_+} h(S(x)y)G(dy) \right] \mu_n(dx). \]
Comparing (1.2) and (1.3) and setting \( h = 1_A \) we obtain:

\[
\mu_{n+1}(A) = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}_+} 1_A(S(x)y)G(dy) \right) \mu_n(dx) \quad \text{or} \quad \mu_{n+1}(A) = P \mu_n(A),
\]

where

\[
(1.4) \quad P \mu(A) = \int_{\mathbb{R}_+ \mathbb{R}^d} 1_A(S(x)y)\mu(dx)G(dy).
\]

The operator \( P \) given by formula (1.4) maps the space \( M_1 \), of all probabilistic measures on \( \mathbb{R}^d \) into itself and is called the \textit{Markov operator corresponding to the dynamical system} (1.0).

The equation (1.4) can be rewritten in the form

\[
(1.5) \quad P \mu(A) = \int \mu_1 A \mu(dx)
\]

where \( U : C_0(\mathbb{R}^d) \to C(\mathbb{R}^d) \) is the operator adjoint to the Markov operator \( P \). By \( C_0(\mathbb{R}^d) \) is denoted the space of all real valued continuous functions with compact support, and by \( C(\mathbb{R}^d) \) the space of all continuous functions.

The operator \( U \) satisfies the following equation:

\[
(1.6) \quad U f(x) = \int_{\mathbb{R}_+} f(S(x)y)G(dy).
\]

Let us define a sequence of functions \( T^n(x, y_1, \ldots, y_n) \) by setting:

\[
T(x, y) = S(x)y, \quad T^n(x, y_1, \ldots, y_n) = T(T^{n-1}(x, y_1, \ldots, y_{n-1}), y_n).
\]

Using this notation we obtain

\[
(1.7) \quad U^n f(x) = \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} f(T^n(x, y_1, \ldots, y_n))G(dy_1) \cdots G(dy_n)
\]

and

\[
(1.8) \quad P^n \mu(A) = \int_{\mathbb{R}^d} U^n 1_A \mu(dx).
\]
We introduce the class $ \Phi $ of functions $ \phi : \mathbb{R}_+ \to \mathbb{R}_+ $ satisfying the following conditions:

1° $ \phi $ is continuous and $ \phi(0) = 0; $  
2° $ \phi $ is nondecreasing and concave, i.e.

$$
\phi \left( \frac{t_1 + t_2}{2} \right) \geq \frac{1}{2} (\phi(t_1) + \phi(t_2)) \quad \text{for} \quad t_1, t_2 \in \mathbb{R}_+;
$$

3° $ \phi(t) > 0 $ for $ t > 0 $ and $ \phi(t) \to +\infty $ when $ t \to +\infty. $

We define the metric $ \varrho_{\phi} $ in $ \mathbb{R}^d $ by the formula:

$$
\varrho_{\phi}(x, y) = \phi(\varrho(x, y)) \quad \text{for} \quad x, y \in \mathbb{R}^d,
$$

where $ \varrho $ is Euclidean metric and in the space $ M_1 $ we define the distance between measures by:

$$
(1.9) \quad ||\mu_1 - \mu_2||_{\phi} = \sup_{F_{\phi}} \left| \int f(x)\mu_1(dx) - \int f(x)\mu_2(dx) \right|,
$$

where $ F_{\phi} $ is the set of functions such that $ |f| \leq 1 $ and $ |f(x) - f(y)| \leq \varrho_{\phi}(x, y) = \phi(\varrho(x, y)). $  

The space $ M_1 $ with the distance $ ||\mu_1 - \mu_2||_{\phi} $ is a complete metric space and

$$
(1.10) \quad \lim_{n \to +\infty} ||\mu_n - \mu||_{\phi} = 0 \quad \text{for} \quad \mu_n, \mu \in M_1
$$

holds if and only if the sequence $ \{\mu_n\} $ is weakly convergent to $ \mu. $ The sequence of measures $ \{\mu_n\} $ is convergent to $ \mu $ in $ ||\cdot||_{\phi} $ if and only if $ \{\mu_n\} $ is convergent to $ \mu $ in $ ||\cdot||_{id}, $ where id $ (x) = x. $ Indeed, the identity function id belongs to the set $ \Phi $ and the metrics $ \varrho_{id} $ and $ \varrho_{\phi} $ define the same topology. From now, $ ||\cdot|| = ||\cdot||_{id}. $

2. Asymptotic stability. Let $ P $ be a Markov operator; a measure $ \mu \in M_1 $ is called stationary or invariant if $ P\mu = \mu. $ A Markov operator is called asymptotically stable if there exists a stationary distribution $ \mu_* $ such that

$$
(2.1) \quad \lim_{n \to +\infty} ||P^n\mu - \mu_*|| = 0 \quad \text{for} \quad \mu \in M_1.
$$

From now we consider $ \mathbb{R}^d $ with metric $ \varrho_{\phi}. $
We introduce the following definitions:
A Markov operator $P$ is called nonexpansive if
$$\|P\mu_1 - P\mu_2\|_\phi \leq \|\mu_1 - \mu_2\|_\phi \quad \text{for } \mu_1, \mu_2 \in M_1.$$ 

A Markov operator $P : M_1 \to M_1$, satisfies the Prochorov condition if
there exists a compact set $Y \subset \mathbb{R}^d$ and a number $\beta > 0$ such that

$$(2.3) \quad \lim_{n \to +\infty} \inf P^n\mu(Y) \geq \beta \quad \text{for } \mu \in M_1.$$ 

From [5] it follows that, if $P$ satisfies the Prochorov condition and $P$ is
nonexpansive then the Markov operator $P$ has an invariant measure $\mu_*$. 

We can use the following theorem [5]:

**Theorem.** Let $P$ be a nonexpansive Markov operator Assume that for
every $\varepsilon > 0$ there is a number $\lambda > 0$ having the following property: for every
$\mu_1, \mu_2 \in M_1$ there exists a Borel set $A$ with $\text{diam} A < \varepsilon$ and an integer $n_0$
such that

$$(2.4) \quad P^{n_0}\mu_i(A) \geq \lambda \quad \text{for } i = 1, 2.$$ 

Then $P$ satisfies the following condition

$$(2.5) \quad \lim_{n \to +\infty} \|P^n(\mu_1 - \mu_2)\| = 0 \quad \text{for } \mu_1, \mu_2 \in M_1.$$ 

Now we proof the following auxiliary lemma:

**Lemma 1.** Assume that conditions (i), (ii), (iv) hold for equation (1.0). 
Suppose that the Markov operator $P$ corresponding to the dynamical system
(1.0) satisfies Prochorov condition and the following inequality holds:

$$(2.6) \quad L \left( \int F^\alpha G(dy) \right) \leq 1$$

for some $\alpha \in (0, 1)$. Then the Markov operator defined by equation (1.4) is
asymptotically stable.

**Proof.** First, we prove that the operator $P$ is nonexpansive i.e.

$$\sup_{f \in F^+} \left| \int_{\mathbb{R}^d} Uf(x)\mu_1 - \int_{\mathbb{R}^d} Uf(x)\mu_2 \right| \leq \sup_{f \in F^+} \int_{\mathbb{R}^d} |f(x)\mu_1 - f(x)\mu_2|$$

for some $\alpha \in (0, 1)$. Then the Markov operator defined by equation (1.4) is
asymptotically stable.
for $\phi(t) = |t|^\alpha$. In order to check it we show that if $f \in F_\phi$ than $Uf \in F_\phi$. Indeed

$$
|Uf(x) - Uf(z)| \leq \int_{\mathbb{R}^+} (f(yS(x)) - f(yS(z)))G(dy)
$$

$$
\leq \int_{\mathbb{R}^+} \phi(y|S(x) - S(z)|)G(dy)
$$

$$
\leq \int_{\mathbb{R}^+} y^\alpha |S(x) - S(z)|^\alpha G(dy)
$$

$$
\leq |S(x) - S(z)|^\alpha \int_{\mathbb{R}^+} y^\alpha G(dy)
$$

$$
\leq L^\alpha |x - z|^\alpha \int_{\mathbb{R}^+} y^\alpha G(dy)
$$

$$
\leq |x - z|^\alpha = \phi(|x - z|).
$$

Since $P$ is nonexpansive and $P$ satisfies Prochorov condition, the operator $P$ has an invariant measure $\mu_*$.

Now we show that condition (2.4) holds. Fix an $\varepsilon > 0$. Then there exists an integer $m$ such that

$$
\phi(r^m \text{ diam}_\phi Y) \leq \varepsilon
$$

where $0 < r < 1$, $Y$ - compact set satisfying Prochorov condition. Notice that

$$
\text{Prob } (\xi_n < (\int_{\mathbb{R}^+} y^\alpha G(dy))^{\frac{1}{\alpha}}) > 0.
$$

Thus there exists

$$
c < (\int_{\mathbb{R}^+} y^\alpha G(dy))^{\frac{1}{\alpha}}
$$

such that $\text{Prob } (\xi_n \leq c) > 0$.

Fix $\tilde{y} \in [O, c]$. According to (ii) we have

$$
|T(x, \tilde{y}) - T(z, \tilde{y})| = |S(x)\tilde{y} - S(z)\tilde{y}|
$$

$$
= |\tilde{y}|S(x) - S(z)| \leq cL|x - z|.
$$
Conditions (2.8) and (2.6) imply that \( cL < 1 \). Thus, we can set in (2.7) \( r = cL \), \((0 < r < 1)\). Observe that

\[
|T^m(x, y_1, \ldots, y_m) - T^m(z, y_1, \ldots, y_m)|
= |T(T^{-1}(x, y_1, \ldots, y_{m-1}), y_m) - T(T^{-1}(z, y_1, \ldots, y_{m-1}), y_m)|
\leq r|T^{-1}(x, y_1, \ldots, y_{m-1}) - T^{-1}(z, y_1, \ldots, y_{m-1})| \leq r^m|x - z|.
\]

where \((\tilde{y}_1, \ldots, \tilde{y}_m) \in [0, c]^m\) is fixed. Condition (2.9) implies that

\[
diam_\varepsilon(T^m(Y, \tilde{y}_1, \ldots, \tilde{y}_m)) \leq r^m \text{ diam}_\varepsilon Y.
\]

Define

\[
A = T^m(Y, \tilde{y}_1, \ldots, \tilde{y}_m).
\]

Then

\[
diam_\varepsilon(A) \leq \phi(\text{diam}_\varepsilon A) \leq \phi(r^m \text{ diam}_\varepsilon Y) < \varepsilon.
\]

According to Prochorov condition there exists \( \bar{n} = \bar{n}(\mu_i) \) such that

\[
P^n\mu_i(Y) \geq \beta \quad \text{for} \quad n \geq \bar{n}, \quad i = 1, 2.
\]

Set \( n_0 = \bar{n} + m \), then

\[
P^{n_0}\mu_i(A) = \int \int \cdots \int A(T^{n_0}(x, y_1, \ldots, y_{n_0})) \mu_i(dx) G(dy_1) \ldots G(dy_{n_0})
\geq \int \int \cdots \int A(T^m(T^\bar{m}(x, y_{k_1}, \ldots, y_{k_{\bar{n}}}), y_{k_{\bar{n}+1}}, \ldots, y_{k_{n_0}})) \mu_i(dx)
\times G(dy_{k_1}) \ldots G(dy_{k_{n_0}}).
\]

Define

\[
T_{(y_1, \ldots, y_m)}^{-m}(\omega) = \{ \omega \in \mathbb{R}^d : T^m(\omega, y_1, \ldots, y_m) \in A \}
\]

and notice that condition

\[
T^m(T^\bar{m}(x, y_{k_1}, \ldots, y_{k_{\bar{n}}}), y_{k_{\bar{n}+1}}, \ldots, y_{k_{n_0}}) \in A
\]

gives

\[
T^\bar{m}(x, y_{k_1}, \ldots, y_{k_{\bar{n}}}) \in T^{-m}_{(y_{k_{\bar{n}+1}}, \ldots, y_{k_{n_0}})}(A).
\]
This implies:
\[ P^m \mu_i(A) \geq (G[0, c]^m)^m \int \int \limits_{A} 1_{T_{[0,c]}^- m}(A)(T^m(x, y_{k_1}, \ldots, y_{k_m}))\mu_i(dx) \times G(dy_{k_1}) \ldots G(dy_{k_m}), \]

where
\[
T_{[0,c]}^- m(A) = \{ \omega \in \mathbb{R}^d \text{ such that there is } (y_1, \ldots, y_m) \in [0, c]^m : T^m(\omega, y_1, \ldots, y_m) \in A \}. 
\]

From the definition of the set \( A \) it follows that:
\[
Y \subset T_{[0,c]}^- m(A). 
\]

Consequently
\[ P^m \mu_i(A) \geq (G[0, c]^m)^m \int \int \limits_{A} 1_Y(T^m(x, y_{k_1}, \ldots, y_{k_m}))\mu_i(dx)G(dy_{k_1}) \ldots G(dy_{k_m}) \]
\[
= (G[0, c]^m)^m P^m \mu_i(Y) \geq (G[0, c]^m)^m \beta > 0, \text{ where } m \text{ is fixed.} 
\]

If \( \lambda = (G[0, c]^m)^m \beta \), than \( \lambda \) satisfies conditions (2.4). Since \( P \) is nonexpansive and satisfies conditions (2.4), operator \( P \) is asymptotically stable which completes the proof.

A continuous \( V : \mathbb{R}^d \rightarrow [0, +\infty) \) is called a Liapunov function if
\[
\lim_{\varepsilon(x, x_0) \rightarrow +\infty} V(x) = +\infty
\]
for some \( x_0 \in \mathbb{R}^d \).

Now we present an auxiliary proposition concerning the Prochorov condition ([5]).

**Proposition 1.** Let \( P \) be a Markov operator and let \( U \) be a operator dual to \( P \). Assume that there is a Liapunov function \( V \) such that
\[ UV(x) \leq aV(x) + b \quad \text{for} \quad x \in \mathbb{R}^d 
\]
where $a, b$ are nonnegative constants and $a < 1$. Then $P$ satisfies the Prochorov condition.

From Lemma 1 and Proposition 1 we have the following.

**Theorem 1.** If conditions (i)-(iv) hold for equation (1.0), then the operator $P$ given by equation (1.4) is asymptotically stable.

**Proof.** Setting $V(x) = |x|^\alpha_0$ we have

$$ UV(x) = \int_{\mathbb{R}^n_+} |S(x)y|^\alpha_0 G(dy) = |S(x)|^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy) $$

$$ = |S(x) - S(x_0) + S(x_0)|^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy) $$

$$ \leq |S(x) - S(x_0)|^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy) + |S(x_0)|^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy). $$

Since $S$ satisfies Lipschitz condition (ii), it is easy to notice that following inequalities hold:

$$ UV(x) \leq L^\alpha_0 |x - x_0|^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy) + |S(x_0)|^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy) $$

$$ \leq L^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy) |x|^\alpha_0 + L^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy) |x_0|^\alpha_0 $$

$$ + |S(x_0)|^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy). $$

Thus condition (2.15) holds with

$$ a = L^\alpha_0 \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy) $$

and

$$ b = (L^\alpha_0 |x_0|^\alpha_0 + |S(x_0)|^\alpha_0) \int_{\mathbb{R}^n_+} y^\alpha_0 G(dy). $$

Consequently Markov operator $P$ corresponding to the dynamical system (1.0) satisfies the Prohorov condition (2.3). According to Lemma 1 the Markov operator $P$ is asymptotically stable. The proof is completed. $\square$
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