LERAY–SCHAUDER DEGREE METHOD
IN ONE–PARAMETER FUNCTIONAL
BOUNDARY VALUE PROBLEMS

SVATOSLAV STÁNĚK

Abstract. Sufficient conditions for the existence of solutions of one–parameter functional boundary value problems of the type

$$x'' = f(t, x, x_t, x'_t, x'_t, \lambda),$$

$$(x_0, x'_0) \in \{(\varphi, \chi + c); c \in \mathbb{R}\}, \alpha(x|_J) = A, \beta(x(T) - x|_J) = B$$

are given. Here $f : J \times \mathbb{R} \times C_r \times \mathbb{R} \times C_r \times \mathbb{R} \to \mathbb{R}$ is continuous, $\varphi, \chi \in C_r$, $\alpha, \beta$ are continuous increasing functionals, $A, B \in \mathbb{R}$ and $x|_J$ is the restriction of $x$ to $J = [0, T]$. Results are proved by the Leray–Schauder degree method.

1. Introduction

Let $C_r$ ($r > 0$) be the Banach space of $C^0$–functions on $[-r, 0]$ with the norm $\|x\|_{[-r,0]} = \max\{|x(t)|; -r \leq t \leq 0\}$. Let $T$ be a positive constant. For every continuous function $x : [-r, T] \to \mathbb{R}$ and each $t \in [0, T] =: J$ denote by $x_t$ the element of $C_r$ defined by

$$x_t(s) = x(t + s), \quad s \in [-r, 0].$$

Let $X$ be the Banach space of $C^0$–functions on $J$ endowed with the norm $\|x\|_J = \max\{|x(t)|; t \in J\}$. Denote by $\mathcal{D}$ the set of all functionals $\gamma : X \to \mathbb{R}$ which are

a) continuous, $\gamma(0) = 0$,
b) increasing, i.e. \( x, y \in X, \quad x(t) < y(t) \) for \( t \in (0, T) \) \( \Rightarrow \gamma(x) < \gamma(y) \), and

c) \( \lim_{n \to \infty} \gamma(\varepsilon x_n) = \varepsilon \infty \) for each \( \varepsilon \in \{-1, 1\} \) and any sequence \( \{x_n\} \subset X \),
\( \lim_{n \to \infty} x_n(t) = \infty \) locally uniformly on \( (0, T) \).

This paper is concerned with the functional boundary value problem (BVP for short)

\[
\begin{align*}
x'' &= f(t, x, x_t, x', x'_t, \lambda), \\
(x_0, x'_0) &\in \{(\varphi, \chi + c); c \in \mathbb{R}\}, \quad \alpha(x|_J) = A, \quad \beta(x(T) - x|_J) = B
\end{align*}
\]

depending on the parameter \( \lambda \). Here \( f : J \times \mathbb{R} \times C_r \times \mathbb{R} \times C_r \times \mathbb{R} \to \mathbb{R} \) is a continuous operator, \( \varphi, \chi \in C_r \), \( \alpha, \beta \in D \), \( A, B \in \mathbb{R} \) and \( x|_J \) is the restriction of \( x \) to \( J \).

By a solution of BVP (1), (2) we mean a pair \( (x, \lambda_0) \), where \( \lambda_0 \in \mathbb{R} \) and \( x \in C^0([-r, T]) \cap C^2(J) \) is a solution of (1) for \( \lambda = \lambda_0 \) satisfying the last two conditions in (2) and
\[
x_t(s) = \varphi(t + s), \quad x'_t(s) = \chi(t + s) - \chi(0) + x'(0)
\]
for \( 0 > t + s (\geq -r) \) and
\[
x_t(s) = x(t + s), \quad x'_t(s) = x'(t + s)
\]
for \( 0 < t + s (\leq T) \).

This definition of BVP (1), (2) is motivated by the Haščák definitions for multipoint boundary value problems for linear differential equations with delays ([5]–[7]).

Our objective is to look for sufficient conditions imposed upon the nonlinearity \( f \) in order to obtain solutions of BVP (1), (2). The proofs are based on the Leray–Schauder degree theory (see e.g. [2]).

We observe that sufficient conditions for the existence (and uniqueness) of solutions of BVP

\[
y'' - q(t)y = g(t, y_t, \lambda),
y_0 = \varphi, \quad y(t_1) = y(T) = 0 \quad (0 < t_1 < T)
\]

were obtained in [8] with \( \varphi \in C_r \), \( \varphi(0) = 0 \). The proof of the existence theorem is based on a combination of the Schauder linearization technique and the Schauder fixed point theorem. In [10] was studied BVP

\[
x'' = F(t, x, x_t, x', x'_t, \lambda),
x_0 = \varphi, \quad x'(0) = x'(T) = 0
\]

with \( \varphi \in C^1([-r, 0]) \), \( \varphi(0) = 0 = \varphi'(0) \). The existence of solutions was proved by a combination of the Schauder quasilinearization technique and the Schauder fixed point theorem.
BVPs for second order differential and functional differential equations depending on the parameter were considered as a rule under linear boundary conditions using the shooting method ([1, 3]), by the Schauder linearization method and the Schauder fixed point theorem ([9], [11]), by a surjectivity result in $\mathbb{R}^n$ ([13]), by a combination of the Schauder quasilinearization technique and the Schauder fixed point theorem ([14]) and by the Leray–Schauder degree theory ([12]).

2. Lemmas

REMARK 1. By c) in the definition of $D$, $\text{Im} \gamma = \mathbb{R}$ for all $\gamma \in D$, where $\text{Im} \gamma$ denotes the range of $\gamma$.

REMARK 2. The following example shows that assumptions a) and b) in the definition of $D$ don't imply its assumption c).

EXAMPLE 1. Consider the functional $\gamma : X \to \mathbb{R}$ defined by

$$\gamma(x) = x(0) + x(T) + \arctan x(T/2).$$

Obviously, $\gamma(0) = 0$, $\text{Im} \gamma = \mathbb{R}$, $\gamma$ is continuous increasing. Set $x_n(t) = n \sin(t \pi / T)$ for $t \in J$ and $n \in \mathbb{N}$. Then $\lim_{n \to \infty} x_n(t) = \infty$ locally uniformly on $(0, T)$ and

$$\lim_{n \to \infty} \gamma(\varepsilon x_n) = \lim_{n \to \infty} (\varepsilon x_n(0) + \varepsilon x_n(T) + \arctan(\varepsilon x_n(\pi/2)))$$

$$= \lim_{n \to \infty} \arctan(\varepsilon n \sin(\pi/2))$$

$$= \lim_{n \to \infty} \arctan(\varepsilon n) = \varepsilon \pi / 2$$

for $\varepsilon \in \{-1, 1\}$.

EXAMPLE 2. Special cases of boundary conditions (2) are conditions

(3) $x_0 = \varphi$, $x(\xi) = A$, $x(T) = B_1$ \quad ($A, B_1 \in \mathbb{R}$, $\xi \in (0, T))$,

(4) $x_0 = \varphi$, $\int_0^\tau x^{2n+1}(s)ds = A$, $x(T) = B + x(\xi)$ \quad ($A, B \in \mathbb{R}$, $n \in \mathbb{N}$, $\tau \in (0, T)$, $\xi \in (0, T))$. 

8 - Annales...
\[ x_0 = \varphi, \quad x^3(\xi_1) + x(\xi_2) = A, \quad x(T) = B_1 + (1/\tau) \int_0^T x(s)ds \]

\[(A, B_1 \in \mathbb{R}, \ 0 \leq \xi_1 < \xi_2 \leq T, \ \xi_2 - \xi_1 < T, \ \tau \in (0, T)),\]

\[ x_0 = \varphi, \quad \max\{x(t); \ t \in [a_1, a_2]\} = A, \quad \max\{x(T) - x(t); \ t \in [a_3, a_4]\} = B \]

\[(A, B \in \mathbb{R}, \ 0 < a_1 < a_2 < T, \ 0 < a_3 < a_4 < T).\]

Boundary conditions (3) (resp. (4); (5); (6)) we obtain setting (in (2))

\[
\begin{align*}
\alpha(x) &= x(\xi), \quad \beta(x) = x(\xi), \quad B = B_1 - A \\
(\text{resp.} \quad \alpha(x) &= \int_0^T x^{2n+1}(s)ds, \quad \beta(x) = x(\xi); \\
\alpha(x) &= x^3(\xi_1) + x(\xi_2), \quad \beta(x) = \int_0^T x(s)ds, \quad B = \tau B_1; \\
\alpha(x) &= \max\{x(t); \ t \in [a_1, a_2]\}, \quad \beta(x) = \max\{x(t); \ t \in [a_3, a_4]\}.
\end{align*}
\]

**Lemma 1.** Let \( u, v \in X, \ \alpha, \beta \in \mathcal{D}, \ c \in [0, 1]. \) Let

\[
\begin{align*}
\alpha(x + u) + (c - 1)\alpha(-x + u) &= c\alpha(u), \\
\beta(y(T) - y + v) + (c - 1)\beta(-y(T) + y + v) &= c\beta(v)
\end{align*}
\]

be satisfied for some \( x, y \in X. \) Then there exist \( \xi, \varrho \in (0, T) \) such that

\[ x(\xi) = 0, \quad y(\varrho) = y(T). \]

**Proof.** Define \( \alpha_1, \beta_1 \in \mathcal{D} \) by

\[
\begin{align*}
\alpha_1(z) &= \alpha(z + u) + (c - 1)\alpha(-z + u) - c\alpha(u), \\
\beta_1(z) &= \beta(z + v) + (c - 1)\beta(-z + v) - c\beta(v).
\end{align*}
\]

Assume \( x(t) \neq 0, y(T) - y(t) \neq 0 \) for \( t \in (0, T). \) Then \( \alpha_1(x) \neq 0, \beta_1(y(T) - y(t)) \neq 0 \) which contradicts the assumptions \( \alpha_1(x) = \alpha(x + u) + (c - 1)\alpha(-x + u) - c\alpha(u) = 0, \beta_1(y(T) - y) = \beta(y(T) - y + v) + (c - 1)\beta(-y(T) + y + v) - c\beta(v) = 0. \)

**Lemma 2.** Let \( \alpha, \beta \in \mathcal{D}, \ u_i, v_i \in X \ (i = 1, 2), \ A, B \in \mathbb{R} \) and \( v \in [0, \infty). \) Then there exist unique \( a, \mu \in \mathbb{R} \) such that the equalities

\[
\begin{align*}
\alpha \left( a \sin(\pi t/T) + \mu (\cos(\pi t/T) - 1) + u_1 \right) \\
- \nu \alpha \left( -a \sin(\pi t/T) - \mu (\cos(\pi t/T) - 1) + u_2 \right) &= A,
\end{align*}
\]

\[
\begin{align*}
\beta \left( a \sin(\pi t/T) + \mu (\cos(\pi t/T) - 1) + u_1 \right) \\
- \nu \beta \left( -a \sin(\pi t/T) - \mu (\cos(\pi t/T) - 1) + u_2 \right) &= 0.
\end{align*}
\]
\[
\beta (-a \sin(\pi t/T) - \mu (\cos(\pi t/T) + 1) + v_1) \\
-\nu \beta (a \sin(\pi t/T) + \mu (\cos(\pi t/T) + 1) + v_2) = B
\]

hold.

**PROOF.** Define the continuous functions \( p, q : \mathbb{R}^2 \rightarrow \mathbb{R} \) by

\[
p(x, y) = \alpha (x \sin(\pi t/T) + y(\cos(\pi t/T) - 1) + u_1) \\
-\nu \alpha (-x \sin(\pi t/T) - y(\cos(\pi t/T) - 1) + u_2),
\]

\[
q(x, y) = \beta (-x \sin(\pi t/T) - y(\cos(\pi t/T) + 1) + v_1) \\
-\nu \beta (x \sin(\pi t/T) + y(\cos(\pi t/T) + 1) + v_2).
\]

Since \( \alpha, \beta \in \mathbb{D}, 0 < \sin(\pi t/T) \leq 1, -2 < \cos(\pi t/T) - 1 < 0 \) and \( 0 < \cos(\pi t/T) + 1 < 2 \) for \( t \in (0, T) \), we see that (cf. the definition of \( \mathbb{D} \)) \( p(\cdot, y) \) is increasing on \( \mathbb{R} \) and \( p(x, \cdot), q(\cdot, y), q(x, \cdot) \) are decreasing on \( \mathbb{R} \) (for fixed \( x, y \in \mathbb{R} \)). Moreover,

\[
\lim_{x \to +\infty} p(x, y) = +\infty, \quad \lim_{y \to +\infty} p(x, y) = -\infty,
\]

\[
\lim_{x \to -\infty} q(x, y) = +\infty, \quad \lim_{y \to -\infty} q(x, y) = -\infty
\]

for \( \varepsilon \in \{-1, 1\} \) (and fixed \( x, y \in \mathbb{R} \)). Consequently, to each \( x \in \mathbb{R} \) there exists a unique \( y = r(x) \in \mathbb{R} \) such that \( p(x, r(x)) = A \). Evidently, \( r : \mathbb{R} \rightarrow \mathbb{R} \) is continuous increasing, \( \lim_{x \to +\infty} r(x) = +\infty \) for \( \varepsilon \in \{-1, 1\} \) and setting \( s(x) = q(x, r(x)) \) for \( x \in \mathbb{R} \), \( s \) is continuous decreasing, \( \lim_{x \to +\infty} s(x) = -\infty \) for \( \varepsilon \in \{-1, 1\} \). Hence \( s(a) = B \) for a unique \( a \in \mathbb{R} \) and if we set \( x = a, \mu = r(a) \), our lemma is proved. \( \square \)

**Lemma 3.** Let \( \alpha, \beta \in \mathbb{D}, a,A,B \in \mathbb{R} \). Then the system of nonlinear equations

\[
(7) \quad \alpha (a + x \sin(\pi t/T) + ty) = A, \quad \beta (-x \sin(\pi t/T) + (T - t)y) = B
\]

has a unique solution \((x, y) \in \mathbb{R}^2\).

**PROOF.** We shall consider the continuous functions \( p, q \in \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by

\[
p(x, y) = \alpha (a + x \sin(\pi t/T) + ty), \quad q(x, y) = \beta (-x \sin(\pi t/T) + (T - t)y).
\]

Since \( 0 < \sin(\pi t/T) \leq 1, 0 < t < T, 0 < T - t < T \) for \( t \in (0, T) \), \( p(\cdot, y), p(x, \cdot), q(x, \cdot), q(\cdot, y) \) are increasing on \( \mathbb{R} \) and \( q(\cdot, y) \) is decreasing on \( \mathbb{R} \) (for each
fixed $x, y \in \mathbb{R}$). Moreover, $\lim_{x \to \infty} p(x, y) = \varepsilon \infty$, $\lim_{y \to \infty} p(x, y) = \varepsilon \infty$, $\lim_{y \to \infty} q(x, y) = \varepsilon \infty$ and $\lim_{x \to \infty} q(x, y) = -\varepsilon \infty$ for $\varepsilon \in \{-1, 1\}$. In the same manner as in the proof of Lemma 2 we can verify that system (7) has a unique solution. □

3. Existence theorems

Let $u, v \in X$ and $\chi \in C_r$. Consider BVP

(8) $x'' = h(t, x, x_t, x_{tt}, x_t', \lambda),$

(9) $(x_0, x_0') 
\in \{(0, \chi + c); c \in \mathbb{R}\}, \quad \alpha(u + x_j) = \alpha(u), \quad \beta(x(T) - x_j + v) = \beta(v)$
depending on the parameter $\lambda$. Here $h : J \times \mathbb{R} \times C_r \times \mathbb{R} \times C_r \times \mathbb{R} \to \mathbb{R}$ is a continuous operator and $\alpha, \beta \in D$.

Set $S_K = \{x : x \in C_r, \|x\|_{[-r,0]} \leq K\}$ for each positive constant $K$ and $\|x\|_I = \max\{|x(t)|; t \in I\}$ for each compact $I \subset \mathbb{R}$ and $x \in C^0(I)$.

THEOREM 1. Let $\chi \in C_r$, $m = \|\chi\|$. Assume there exist constants $K > 0$, $\Lambda > 0$, $M > 0$ and a function $w_1 : [0, \infty) \times [0, \infty) \to (0, \infty)$ nondecreasing in both its arguments such that

(10) $h(t, x, \psi, 0, \varepsilon, \Lambda) \geq 0$ for $(t, x, \psi, \varepsilon) \in J \times [0, K] \times S_K \times S_{M+2m},$

(10') $h(t, x, \psi, 0, \varepsilon, -\Lambda) \leq 0$

for $(t, x, \psi, \varepsilon) \in J \times [-K, 0] \times S_K \times S_{M+2m},$

(10'') $h(t, -K, \psi, 0, \varepsilon, \Lambda) \leq h(t, K, \psi, 0, \varepsilon, \Lambda)$

for $(t, \psi, \varepsilon, \Lambda) \in J \times S_K \times S_{M+2m} \times [-\Lambda, \Lambda],$

(11) $|h(t, x, \psi, y, \varepsilon, \Lambda)| \leq w_1(|y|, \|\varepsilon\|_{[-r,0]})$

for $(t, x, \psi, \Lambda) \in J \times [-K, K] \times S_K \times [-\Lambda, \Lambda], (y, \varepsilon) \in \mathbb{R} \times C_r$

and

(12) $\int_0^M \frac{s ds}{w_1(s, M + 2m) + (3K/2)(\pi/T)^2} > 2K.$
Then BVP (8), (9) has at least one solution \((x, \lambda_0)\) satisfying
\[
\|x\|_J \leq K, \quad \|x\|_J \leq M, \quad |\lambda_0| \leq \Lambda.
\]

**Proof.** Define the continuous operator \(h^* : J \times \mathbb{R} \times C_r \times \mathbb{R} \times C_r \times \mathbb{R} \to \mathbb{R}\) by
\[
h^*(t, x, \psi, y, \varphi, \lambda) = h(t, x, \psi, y, \hat{\varphi}, \lambda)
\]
where \((s \in [-r, 0])\)
\[
\hat{\varphi}(s) = \begin{cases} 
M + 2m & \text{for } \varphi(s) > M + 2m \\
\varphi(s) & \text{for } |\varphi(s)| \leq M + 2m \\
-(M + 2m) & \text{for } \varphi(s) < -(M + 2m).
\end{cases}
\]
Consider the equation
\[
x'' = c_h^*(t, x, x_t, x', x_t', \lambda) + (1 - c)(\varepsilon^2 x + k\lambda), \quad \in [0, 1],
\]
where
\[
\varepsilon = \frac{\pi}{T}, \quad k = \frac{\pi^2 K}{2T^2 \Lambda}.
\]
Let \((x_c, \lambda_c)\) be a solution of BVP \((16_c), (16'_c)\) with \(c \in [0, 1)\) such that
\[
\|x_c\|_J \leq K, \quad |\lambda_c| \leq \Lambda, \text{ where}
\]
\[
(x_{c_0}, x'_{c_0}) \in \{(0, \chi + d); \ d \in \mathbb{R}\},
\]
\[
(16'_c) \quad \alpha(u + x_c|_J) + (c - 1)\alpha(u - x_c|_J) = c\alpha(u),
\]
\[
\beta(x_c(T) - x_c|_J + v) + (c - 1)\beta(-x_c(T) + x_c|_J + v) = c\beta(v).
\]
We shall prove
\[
\|x_c\|_J < K, \quad \|x'_c\|_J < M,
\]
\[
\|x''_c\|_J \leq w_1(M, M + 2m) + (3K/2)(\pi/T)^2, \quad |\lambda_c| \leq \Lambda.
\]
Assume \(\lambda_c = \Lambda\). By Lemma 1 (with \(c = 1\)) \(x_c(v) = 0, \ x_c(T) = x_c(\xi)\) for some \(v, \xi \in (0, T)\) and therefore \(0 \leq \max\{x_c(t); \ t \in J\} = x_c(\tau)\) for \(\tau \in (0, T)\). Then \(x''_c(\tau) = 0, \ x''_c(\tau) \leq 0 \) which contradicts (cf. \((10')\) and \((15)\)) \(x''_c(\tau) = c_h^*(\tau, x_c(\tau), x_{c\tau}, 0, x_{c\tau}', \Lambda) + (1 - c)(\varepsilon^2 x_c(\tau) + k\Lambda) > 0).\)
Let \(\lambda_c = -\Lambda\). Then \(0 \geq \min\{x_c(t); \ t \in J\} = x_c(\mu)\) for a \(\mu \in (0, T)\) and \(x'_c(\mu) = 0, \ x''_c(\mu) \geq 0 \) which contradicts (cf. \((10'')\) and \((15)\)) \(x''_c(\mu) =\)
\[ c.h^*(\mu, x_c(\mu), x_{cm}, 0, x'_{cm}, -\Lambda) + (1 - c)(\varepsilon^2 x_c(\mu) - k\Lambda) < 0. \text{ Hence } |\lambda_c| < \Lambda. \]

Let \( |x_c|^J = K \), for example let \( x_c(\kappa) = K \) with a \( \kappa \in (0, T) \) (see Lemma 1 with \( c = 1 \)). Then \( x'_c(\kappa) = 0, x''_c(\kappa) \leq 0 \) which contradicts (cf. (11) and (15)) \( x''_c(\kappa) = c.h^*(\kappa, K, x_{cm}, 0, x_c, \lambda_c) + (1 - c)(\varepsilon^2 K + k\lambda_c) \geq (1 - c)(\varepsilon^2 K - k\Lambda) = (1 - c)(\pi^2 K/2T^2) > 0. \) Hence \( |x_c|^J < K. \) Since \( x_c(v) = 0 \) and \( x_c(0) = 0, x'_c(\eta) = 0 \) for an \( \eta \in (0, v) \) and, moreover,

\begin{equation}
|x''_c(t)| \leq c|h^*(t, x_c(t), x_{ct}, x'_c(t), x''_c, \lambda_c)| + (1 - c)(\varepsilon^2 K + k\Lambda) < w_1(|x'_c(t)|, M + 2m) + (3K/2)(\pi/T)^2
\end{equation}

for \( t \in J \) by (12) and (15). So, using (13), (18) and a standard procedure (see e.g. [4]) we can prove \( |x'_{ct}|^J < M. \) Finally, \( |x''_c|^J < w_1(|x'_c|^J, M + 2m) + (3K/2)(\pi/T)^2 \leq w_1(M, M + 2m) + (3K/2)(\pi/T)^2 \) and (17) is proved.

Let \( Y_i \) (\( i = 1, 2 \)) be the Banach space of \( C^1 \)-functions on \( J \) with the norm

\[ \|x\|_i = \sum_{j=0}^i \|x^{(j)}\|^J, \quad Y_0 = \{x; x \in Y_i, x(0) = 0\}. \]

Define the operators

\[ U, H, V : Y_0 \times \mathbb{R} \rightarrow X \times \mathbb{R}^2 \]

by

\[ (U(x, \lambda))(t) = (x''(t) + \varepsilon^2 x(t) + k\lambda, \alpha(x + u) - \alpha(-x + u), \beta(x(T) - x + v) - \beta(-x(T) + x + v)), \]

\[ (H(x, \lambda))(t) = (h^*(t, x(t), x_{ct}, x'_c(t), \lambda), \alpha(u) - \alpha(-x + u), \beta(v) - \beta(-x(T) + x + v)), \]

\[ (V(x, \lambda))(t) = (\varepsilon^2 x(t) + k\lambda, 0, 0), \]

where

\[ x_t(s) = \begin{cases} 0 & \text{for } t + s < 0 \\ x(t + s) & \text{for } t + s \geq 0, \end{cases} \]

\[ x'_t(s) = \begin{cases} \chi(t + s) - \chi(0) + x'(0) & \text{for } t + s < 0 \\ x'(t + s) & \text{for } t + s \geq 0. \end{cases} \]

Consider the operator equation

\[ (19_c) \quad U(x, \lambda) = c(H(x, \lambda) + V(x, \lambda)) + 2(1 - c)V(x, \lambda), \quad c \in [0, 1]. \]

We see that BVP (8), (9) with \( h = h^* \) has a solution \((x, \lambda_0)\) if \((x|^J, \lambda_0)\) is a solution of (19_1) and conversely, if \((x, \lambda_0)\) is a solution of (19_1), then \((z, \lambda_0)\) is a solution of BVP (8), (9) with \( h = h^* \) where \((z_0, z'_0) = (0, \chi - \chi(0) + x'(0)), z|^J = x.\) So, to prove the existence of solutions of BVP (8), (9) with \( h = h^* \) it is sufficient to show that (19_1) has a solution.
We shall prove that $U : Y_{02} \times \mathbb{R} \rightarrow \mathbb{X} \times \mathbb{R}^2$ is one to one and onto. Let $(z, a, b) \in \mathbb{X} \times \mathbb{R}^2$ and consider the operator equation

\[ U(x, \lambda) = (z, a, b), \]

that is the equations

\[ (20') \quad x'' + \varepsilon^2 x + k\lambda = z(t), \]

\[ (20'') \quad \alpha(x + u) - \alpha(-x + u) = a, \quad \beta(x(T) - x + v) - \beta(-x(T) + x + v) = b, \]

where $x \in Y_{02}$, $\lambda \in \mathbb{R}$. The function $x(t) = c_1 \sin(\varepsilon t) + c_2 \cos(\varepsilon t) - (k\lambda/\varepsilon^2) + w(t)$ is the general solution of $(20')$ where $w(t) = (1/\varepsilon) \int_0^t z(s) \sin(\varepsilon(t-s)) \, ds$ and $c_1, c_2$ are integration constants. The function $x$ satisfies $(20'')$ and $x(0) = 0$ if and only if $c_2 = k\lambda/\varepsilon^2$ and $(c_1, \lambda)$ is a solution of the system

\[ \alpha (c_1 \sin(\varepsilon t) + (k\lambda/\varepsilon^2)(\cos(\varepsilon t) - 1) + w + u) - \alpha (-c_1 \sin(\varepsilon t) - (k\lambda/\varepsilon^2)(\cos(\varepsilon t) - 1) - w + u) = a, \]

\[ \beta (-c_1 \sin(\varepsilon t) - (k\lambda/\varepsilon^2)(1 + \cos(\varepsilon t)) + w(T) - w + v) - \beta (c_1 \sin(\varepsilon t) + (k\lambda/\varepsilon^2)(1 + \cos(\varepsilon t)) - w(T) + w + v) = b, \]

since $\varepsilon T = \pi$. By Lemma 2 (with $a = c_1$, $\mu = k\lambda/\varepsilon^2$, $u_1 = w + u$, $u_2 = -w + u$, $v_1 = w(T) - w + v$, $v_2 = -w(T) + w + v$, $A = a$, $B = b$), there exists a unique solution $(\tilde{c}, \tilde{\lambda})$ of the above system. Hence $U^{-1} : \mathbb{X} \times \mathbb{R}^2 \rightarrow Y_{02} \times \mathbb{R}$ exists. Let $(x, \lambda) \in Y_{02} \times \mathbb{R}$ and set $U(x, \lambda) = (z, a, b)$, $U(-x, -\lambda) = (z_1, a_1, b_1)$. Then

\[ x''(t) + \varepsilon^2 x(t) + k\lambda = z(t), \quad -x''(t) - \varepsilon^2 x(t) - k\lambda = z_1(t) \tag{for t \in J} \]

and

\[ \alpha(x + u) - \alpha(-x + u) = a, \quad \beta(x(T) - x + v) - \beta(-x(T) + x + v) = b, \]

\[ \alpha(-x + u) - \alpha(x + u) = a_1, \quad \beta(-x(T) + x + v) - \beta(x(T) - x + v) = b_1. \]

Therefore $z_1 = -z$, $a_1 = -a$, $b_1 = -b$ and consequently

\[ U(x, \lambda) = -U(-x, -\lambda) \]

for all $(x, \lambda) \in Y_{02} \times \mathbb{R}$. So $U$ is an odd operator and then $U^{-1}$ is odd as well.
In order to prove that $U^{-1}$ is a continuous operator let $\{(z_n, a_n, b_n)\} \subset X \times \mathbb{R}^2$ be a convergent sequence, $(z_n, a_n, b_n) \to (z, a, b)$ as $n \to \infty$. Set $(x_n, \lambda_n) = U^{-1}(z_n, a_n, b_n)$, $(x, \lambda) = U^{-1}(z, a, b)$. Then
\begin{align*}
x_n''(t) + \varepsilon^2 x_n(t) + k\lambda_n = z_n(t), \quad x''(t) + \varepsilon^2 x(t) + k\lambda = z(t) \quad \text{for } t \in J, \ n \in \mathbb{N}
\end{align*}
and there exist sequences $\{c_n\}, \{d_n\} \subset \mathbb{R}$ and $c, d \in \mathbb{R}$ such that
\begin{align*}
(21') & \quad \alpha (c_n \sin(\varepsilon t) + d_n (\cos(\varepsilon t) - 1) + w_n + u) \\
& \quad - \alpha (-c_n \sin(\varepsilon t) - d_n (\cos(\varepsilon t) - 1) - w_n + u) = a_n,
(21'') & \quad \beta (-c_n \sin(\varepsilon t) - d_n (1 + \cos(\varepsilon t)) + w_n(T) - w + v) \\
& \quad - \beta (c_n \sin(\varepsilon t) + d_n (1 + \cos(\varepsilon t)) - w_n(T) + w + v) = b_n,
(22') & \quad \alpha (c \sin(\varepsilon t) + d (\cos(\varepsilon t) - 1) + w + u) \\
& \quad - \alpha (-c \sin(\varepsilon t) - d (\cos(\varepsilon t) - 1) - w + u) = a,
(22'') & \quad \beta (-c \sin(\varepsilon t) - d (1 + \cos(\varepsilon t)) + w(T) - w + v) \\
& \quad - \beta (c \sin(\varepsilon t) + d (1 + \cos(\varepsilon t)) - w(T) + w + v) = b,
\end{align*}
and
\begin{align*}
x_n(t) = c_n \sin(\varepsilon t) + d_n (\cos(\varepsilon t) - 1) + w_n(t), \\
x(t) = c \sin(\varepsilon t) + d (\cos(\varepsilon t) - 1) + w(t)
\end{align*}
for $t \in J$ and $n \in \mathbb{N}$ where
\begin{align*}
w_n(t) &= (1/\varepsilon) \int_0^t z_n(s) \sin(\varepsilon (t - s))ds, \\
w(t) &= (1/\varepsilon) \int_0^t z(s) \sin(\varepsilon (t - s))ds, \quad t \in J, \ n \in \mathbb{N}
\end{align*}
and
\begin{align*}
\lambda_n &= \varepsilon^2 d_n/k, \quad \lambda = \varepsilon^2 d/k, \quad n \in \mathbb{N}.
\end{align*}
Evidently, $\lim w_n = w$ in $Y_2$ and $\{c_n\}, \{d_n\}$ are bounded sequences since $\text{Im} \alpha = \mathbb{R} = \text{Im} \beta$ and $\{a_n\}, \{b_n\}$ and $\{w_n\}$ are bounded in $\mathbb{R}$ and $X$, respectively. Assume, on the contrary, that for example $\{c_n\}$ is not convergent
(the convergence of \( \{d_n\} \) can be proved similarly). Then there exist convergent subsequences \( \{c_{k_n}\} , \{c_{l_n}\} \), \( \lim_{n \to \infty} c_{k_n} = c^* , \lim_{n \to \infty} c_{l_n} = \tilde{c}, c^* \neq \tilde{c} \).

Without loss of generality we can assume that \( \{d_{k_n}\} , \{d_{l_n}\} \) are convergent, \( \lim_{n \to \infty} d_{k_n} = d^* , \lim_{n \to \infty} d_{l_n} = \tilde{d} \), where \( d^* \) equals \( d \) or not. Taking the limits in (21'), (21'') as \( k \to \infty \) and \( l \to \infty \) we obtain

\[
\begin{align*}
\alpha (c^* \sin(\epsilon t) + d^* (\cos(\epsilon t) - 1) + w + u) \\
- \alpha (-c^* \sin(\epsilon t) - d^* (\cos(\epsilon t) - 1) - w + u) &= a, \\
\beta (-c^* \sin(\epsilon t) - d^* (1 + \cos(\epsilon t)) + w(T) - w + v) \\
- \beta (c^* \sin(\epsilon t) + d^* (1 + \cos(\epsilon t)) - w(T) + w + v) &= b,
\end{align*}
\]

and

\[
\begin{align*}
\alpha (\tilde{c} \sin(\epsilon t) + \tilde{d} (\cos(\epsilon t) - 1) + w + u) \\
- \alpha (-\tilde{c} \sin(\epsilon t) - \tilde{d} (\cos(\epsilon t) - 1) - w + u) &= a, \\
\beta (-\tilde{c} \sin(\epsilon t) - \tilde{d} (1 + \cos(\epsilon t)) + w(T) - w + v) \\
- \beta (\tilde{c} \sin(\epsilon t) + \tilde{d} (1 + \cos(\epsilon t)) - w(T) + w + v) &= b,
\end{align*}
\]

respectively. Hence \( c^* = \tilde{c}, d^* = \tilde{d} \) by Lemma 2 (with \( u_1 = w + u, u_2 = -w + u, v_1 = w(T) - w + v, v_2 = -w(T) + w + v \), a contradiction. Let \( \lim_{n \to \infty} c_n = c_0, \lim_{n \to \infty} d_n = d_0 \). Taking the limits in (21'), (21'') as \( n \to \infty \) we see that (22'), (22'') hold with \( c = c_0, d = d_0 \) and consequently \( c = c_0, d = d_0 \) by Lemma 2. Then

\[
\lim_{n \to \infty} x_n^{(i)}(t) = \lim_{n \to \infty} (c_n \sin(\epsilon t) + d_n (\cos(\epsilon t) - 1) + w_n(t))^{(i)} \\
= (c \sin(\epsilon t) + d (\cos(\epsilon t) - 1) + w(t))^{(i)}
\]

uniformly on \( J (i = 0, 1, 2) \) and \( \lim_{n \to \infty} \lambda_n = \lambda \); hence \( \lim_{n \to \infty} U^{-1}(z_n, a_n, b_n) = U^{-1}(z, a, b) \) and consequently \( U^{-1} \) is a continuous operator.

Applying \( U^{-1} \) we can rewrite (19c) as

\[
(x, \lambda) = U^{-1} \left( c(Hj(x, \lambda) + Vj(x, \lambda)) + 2(1 - c)Vj(x, \lambda) \right), \\
c \in [0, 1],
\]

where \( j : Y_{01} \times \mathbb{R} \to Y_{02} \times \mathbb{R} \) is the natural embedding, which is completely continuous by the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem. Set

\[
\Omega = \{(x, \lambda) ; (x, \lambda) \in Y_{02} \times \mathbb{R}, \|x\|_J < K, \|x'\|_J < M, \\
\|x''\|_J < w_1(M, M + 2m) + (3M/2)(\pi/T)^2, |\lambda| < \Lambda \}.
\]
Then \( \Omega \) is a bounded open convex and symmetric with respect to \( 0 \in \Omega \) subset of \( Y_{02} \times \mathbb{R} \), \( U^{-1}(Hj + Vj) \) is a compact operator on \( \Omega \) and \( U^{-1}(2Vj) \) is a completely continuous operator on \( Y_{02} \times \mathbb{R} \). To prove that BVP (8), (9) with \( h = h^* \) has a solution \( (x, \lambda_0) \) satisfying (14) it is sufficient to show that \( U^{-1}(Hj + Vj) \) has a fixed point in \( \overline{\Omega} \), that is (23i) has a solution in \( \overline{\Omega} \). If \( U^{-1}(Hj + Vj) \) has a fixed point on \( \partial \Omega \), our theorem is proved. Assume \( (U^{-1}(Hj + Vj))(x, \lambda) \neq (x, \lambda) \) for all \( (x, \lambda) \in \partial \Omega \). Define \( W : [0,1] \times \overline{\Omega} \rightarrow Y_{02} \times \mathbb{R} \) by \( W(c,x,\lambda) = U^{-1}(c(Hj(x,\lambda) + Vj(x,\lambda)) + 2(1 - c)Vj(x,\lambda)) \). \( W \) is a compact operator and (cf. (17)) \( W(c,x,\lambda) \neq (x,\lambda) \) for \( (x,\lambda) \in \partial \Omega \) and \( c \in [0,1] \); hence (cf. e.g. [2]) \( D(I - U^{-1}(Hj + Vj),\Omega,0) = D(I - U^{-1}(2Vj),\Omega,0) \), where "D" denotes the Leray–Schauder degree. Since \( U^{-1} \) is odd and \( Vj \) is linear, \( U^{-1}(2Vj) \) is odd and consequently \( D(I - U^{-1}(2Vj),\Omega,0) \neq 0 \) by the Borsuk theorem (see e.g. [2, Theorem 8.3, p. 58]). Thus there exists a solution \( (x, \lambda_0) \in \overline{\Omega} \) of (23i) and since \( \|x'(t)\|_{[-r,0]} \leq \|x'(t) + \|\chi - \chi(0)(t)\|_{[-r,0]} \leq M + 2m \) for \( t \in J \) we see that

\[
h^*(t, x(t), x_t, x'(t), x'_t, \lambda_0) = h(t, x(t), x_t, x'(t), x'_t, \lambda_0)
\]
on \( J \). This completes the proof. \( \square \)

**Remark 3.** Let \( \varphi \in C_r \) and \( (x_0, y_0) \in \mathbb{R}^2 \) be the unique solution of system (7) with \( a = \varphi(0) \), \( A, B \in \mathbb{R} \) (see Lemma 3). Then the function

\[
* x(t) = \begin{cases} 
\varphi(t) & \text{for } t \in [-r,0], \\
\varphi(0) + x_0 \sin(\pi t/T) + y_0 t & \text{for } t \in (0,T)
\end{cases}
\]
satisfies boundary conditions \( x_0 = \varphi, \alpha(x|_J) = A, \beta(x(T) - x|_J) = B \).

**Theorem 2.** Assume that \( f \) satisfies the following assumptions:

\((H_1)\) (Sign conditions): For each constant \( E > 0 \) there exist constants \( K > 0 \) and \( \Lambda > 0 \) such that

\[
f(t, x - E, \psi, y, \varphi, \Lambda) \geq -E
\]

for \( (t, x, \psi, y, \varphi) \in J \times [0, K + 2E] \times S_{K+E} \times [-E, E] \times C_r ,
\]

\[
f(t, x + E, \psi, y, \varphi, -\Lambda) \leq E
\]

for \( (t, x, \psi, y, \varphi) \in J \times [-K - 2E, 0] \times S_{K+E} \times [-E, E] \times C_r ,
\]

\[
f(t, x, \psi, y, \varphi, \lambda) \geq -E
\]

for \( (t, x, \psi, y, \varphi, \lambda) \in J \times [K - E, K + E] \times S_{K+E} \times [-E, E] \times C_r \times [-\Lambda, \Lambda] ,
\]

\[
f(t, x, \psi, y, \varphi, \lambda) \leq E
\]

for \( (t, x, \psi, y, \varphi, \lambda) \in J \times [-K - E, -K + E] \times S_{K+E} \times [-E, E] \times C_r \times [-\Lambda, \Lambda] ;
\]
(H₂) (Bernstein–Nagumo growth condition): A nondecreasing function
\( w(\cdot, A) : [0, \infty) \rightarrow (0, \infty) \) exists to any bounded subset \( A \) of \( \mathbb{R} \times C_r \times \mathbb{R} \) such that

\[
\int_0^\infty \frac{sdS}{w(s, A)} = \infty
\]

and

\[
|f(t, x, \psi, y, \varphi, \lambda)| \leq w(|y|, A) \quad \text{for} \quad (t, x, \psi, \lambda) \in J \times A, \ (y, \varphi) \in \mathbb{R} \times C_r.
\]

Then BVP (1), (2) has at least one solution for each \( \varphi, \chi \in C_r \) and \( A, B \in \mathbb{R} \).

**Proof.** Let \( \varphi, \chi \in C_r \), \( A, B \in \mathbb{R} \) and \( p \in C^0([0, T]) \cap C^2(J) \) satisfy boundary conditions \( p_0 = \varphi, \ \alpha(p|J) = A, \ \beta(p(T) - p|J) = B \) (see Remark 3). Set \( E_1 = \max \{||p|-\tau, \ ||p'||, \ ||p''||\} \) and

\[
h(t, x, \psi, y, \varphi, \lambda) = f(t, x + p(t), \psi + p_t, y + p'(t), \varphi + \psi_t, \lambda) - p''(t)
\]

for \( (t, x, \psi, y, \varphi, \lambda) \in J \times \mathbb{R} \times C_r \times \mathbb{R} \times C_r \times \mathbb{R} \) where

\[
z_t(s) = \begin{cases} 
p'(0) & \text{for } t + s < 0 \\
p'(t + s) & \text{for } t + s \geq 0.
\end{cases}
\]

We see that \( (x + p, \lambda_0) \) is a solution of BVP (1), (2) if and only if \( (x, \lambda_0) \) is a solution of BVP (8), (9) with \( u = p|J \), and \( v = p(T) = p|J \). Thus to prove our theorem it is sufficient to show that BVP (8), (9) has a solution which occurs if \( h \) satisfies the assumptions of Theorem 1.

Let \( K > 0, \ \Lambda > 0 \) be constants corresponding to \( E = E_1 \) in assumption (H₁). Then

\[
h(t, x, \psi, 0, \varphi, \Lambda) = f(t, x + p(t), \psi + p_t, p'(t), \varphi + \psi_t, \Lambda) - p''(t)
\]

\[
\geq E_1 - p''(t) \geq 0
\]

for \( (t, x, \psi, \varphi) \in J \times [0, K] \times S_K \times C_r \),

\[
h(t, x, \psi, 0, \varphi, -\Lambda) = f(t, x + p(t), \psi + p_t, p'(t), \varphi + \psi_t, -\Lambda) - p''(t)
\]

\[
\leq - E_1 - p''(t) \leq 0
\]

for \( (t, x, \psi, \varphi) \in J \times [-K, 0] \times S_K \times C_r \), and

\[
h(t, K, \psi, 0, \varphi, \lambda) = f(t, K + p(t), \psi + p_t, p'(t), \varphi + \psi_t, \lambda) - p''(t) \geq E_1 - p''(t) \geq 0
\]
\[ h(t, -K, \psi, 0, \varphi, \lambda) = f(t, -K + p(t), \psi + p_t, p'(t), \varphi + z_t, \lambda) - p''(t) \leq -E_1 - p''(t) \leq 0 \]

for \((t, \psi, \varphi, \lambda) \in J \times S_K \times C_r \times [-\Lambda, \Lambda].\) 

Set \(A = [-K - E_1, K + E_1] \times S_K + E_1 \times [-\Lambda, \Lambda].\) By \((H_2),\) a nondecreasing function \(w(. , A) : [0, \infty) \rightarrow (0, \infty)\) exists such that (24) and (25) hold. Then

\[ |h(t, x, \psi, y, \varphi, \lambda)| = |f(t, x + p(t), \psi + p_t, y + p'(t), \varphi + z_t, \lambda) - p''(t)| \leq w(|y + p'(t)|, A) + E_1 \leq w(|y| + E_1, A) + E_1 \]

for \((t, x, \psi, y, \varphi, \lambda) \in J \times [-K, K] \times S_K \times C_r \times [-\Lambda, \Lambda] \) and \(y \in \mathbb{R}.)\) Since the function \(w_1(s) = w(s + E_1, A) + E_1\) is positive nondecreasing on \([0, \infty)\) and (cf. (24))

\[
\int_0^M \frac{s ds}{w_1(s) + (3K/2)(\pi/T)^2} = \int_0^M \frac{s ds}{w(s + E_1, A) + E_1 + (3K/2)(\pi/T)^2} > 2K
\]

for a positive constant \(M,\) the assumptions of Theorem 1 are satisfied. This completes the proof. \(\square\)

**Example 3.** Consider the functional differential equation

\[ x''(t) = a(t) + b(t)x^3(t) + c(t)x(t - r) + d(t)x'(t) + (1 + |\sin t|)\lambda \]

depending on the parameter \(\lambda\) together with boundary conditions (2). Here \(a, b, c, d \in C^0(J),\) \(b(t) > 0\) on \(J.\) Equation (25) is the special case of (1) with \(f(t, x, \psi, y, \varphi, \lambda) = a(t) + b(t)x^3 + c(t)\psi(-r) + d(t)y + (1 + |\sin t|)\lambda\) and satisfies the assumptions of Theorem 2. Indeed, let \(b = \min\{b(t); t \in J\}(> 0)\) and fix \(E > 0.\) Then

\[ K = \max \left\{ \frac{1}{3} + \left( \frac{1}{27} + \frac{S}{2} + \left( \frac{S^2}{4} + \frac{S}{27} \right)^{\frac{1}{2}} \right)^{\frac{3}{2}} + \left( \frac{1}{27} + \frac{S}{2} - \left( \frac{S^2}{4} + \frac{S}{27} \right)^{\frac{1}{2}} \right)^{\frac{3}{2}}, \right. \]

\[ \left. \frac{24C}{b} , 2E \right\} \]

and \(\Lambda = Q + KC\) are constants corresponding to \(E\) in \((H_1)\) where \(C = ||c||_J, S = (8/b) (3||a||_J + 3E(C + ||d||_J + 1) + 2E^3||b||_J), Q = ||a||_J + E(C + ||d||_J + 1) + E^3||b||_J\) and \(w(s, A) = HS + P\) satisfies assumption \((H_2)\) for suitable positive constants \(P = P(A), H = H(A).\) Hence, there exists at least one solution of BVP (25), (2) for each \(\varphi, \chi \in C_r\) and \(A, B \in \mathbb{R}.\)
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DEPARTMENT OF MATHEMATICAL ANALYSIS
FACULTY OF SCIENCE, PALACKÝ UNIVERSITY
TOMKOVA 40, 779 00 OLOMOUC
CZECH REPUBLIC