A NOTE ON THE FRÉCHET THEOREM

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Abstract. We give conditions under which every measurable function is the limit almost everywhere of a sequence of continuous functions.

The following classical result is well known (see e.g. [7, p.110]):

FRÉCHET THEOREM. Let $E$ be a Lebesgue subset of $\mathbb{R}^m$, and $f$ a Lebesgue measurable extended real-valued function on $E$. Then there exists a sequence of finite continuous functions which converges to $f$ almost everywhere in $E$.

Recently the problem of the approximation of measurable functions by continuous ones was studied in [8]. In this note we present some more general results.

Let $T$ be a topological space and let $\mu$ be a measure defined on a $\sigma$–algebra $\mathcal{T}$ of subsets of $T$ such that $\mathcal{T}$ contains $\mathcal{B}(T)$, the Borel sets of $T$. We say that $\mu$ is regular if for each $A \in \mathcal{T}$ and each $\epsilon > 0$ there exist closed $F$ and open $G$ such that $F \subset A \subset G$ and $\mu(G \setminus F) < \epsilon$ (cf. [2, p.7]).

We shall use a general version of the Lusin theorem given in [9].

LUSIN THEOREM. Let $(T, \mathcal{T}, \mu)$ be a space of a regular measure, and $X$ a topological space with a countable base. If $f : T \to X$ is measurable, then for each $\epsilon > 0$ there exists a closed set $T_\epsilon \subset T$ such that $\mu(T \setminus T_\epsilon) < \epsilon$ and $f | T_\epsilon$ is continuous.

The following theorem is the main result of the paper. Its proof is rather standard (cf. the proof of the Fréchet theorem in [7]).

THEOREM 1. Let $(T, \mathcal{T}, \mu)$ be a space of a regular measure with $T$ metrizable; let $X$ be a locally convex, separable and metrizable space. Then for

each measurable function \( f : T \rightarrow X \) there exists a sequence of continuous functions \( f_n : T \rightarrow X, n \in \mathbb{N} \), convergent to \( f \) \( \mu \)-a.e.

**Proof.** By the Lusin theorem, for each \( n \in \mathbb{N} \) there is closed \( F_n \subset T \) such that \( \mu(T \setminus F_n) < 1/n \) and \( f | F_n \) is continuous. Let \( A_n = F_1 \cup \ldots \cup F_n \). Then \( A_n \subset A_{n+1} \) and \( f | A_n \) is continuous. By the Dugundji extension theorem (see e.g. [4, p.92]), \( f | A_n \) has a continuous extension \( f_n : T \rightarrow X \). Let \( A = \bigcup \{ A_n : n \in \mathbb{N} \} \). It is immediate that \( \mu(T \setminus A) = 0 \), and \( f_n(t) \) converges to \( f(t) \) for each \( t \in A \). It completes the proof. \( \square \)

For real functions we can relax the assumption of the metrizability of \( T \).

**Theorem 2.** Let \((T, \mathcal{T}, \mu)\) be a space of a regular measure, where \( T \) is normal. Each measurable extended real-valued function \( f \) on \( T \) is the limit \( \mu \)-a.e. of a sequence of continuous functions \( f_n : T \rightarrow \mathbb{R}, n \in \mathbb{N} \).

**Proof.** Let \( h \) be a homeomorphism of the extended real line and the interval \([-1,1]\), and let \( g = h \circ f \). There exists a sequence of continuous functions \( g_n : T \rightarrow [-1,1], n \in \mathbb{N} \), which converges to \( g \) \( \mu \)-a.e. It can be constructed in the same way as in the previous proof, but with the use of the Tietze–Urysohn extension theorem. Now we define \( f_n(t) = \max\{-n, \min\{h^{-1}(g_n(t)), n\}\}, n \in \mathbb{N}, t \in T \). It is obvious that the functions \( f_n \) are finite, continuous and converge \( \mu \)-a.e. to \( f \). \( \square \)

**Remarks.**

1. If \( \mu \) is a regular measure then its completion is also regular. Hence, in both theorems it suffices to assume that \( f \) is measurable with respect to \( \mathcal{T}_\mu \), the completion of \( \mathcal{T} \). Note that if \( f \) is the \( \mu \)-a.e. limit of a sequence of continuous functions, then \( f \) is \( \mathcal{T}_\mu \)-measurable.

2. If \( T \) is a Polish space and \( \mathcal{T} = \mathcal{B}(T) \), then the assumption of the separability of \( X \) in Theorem 1 is superfluous. In fact, for every Borel function \( f \) from such \( T \) into a metric space the range \( f(T) \) is separable (see [3, p.164]).

3. There is a less known extension theorem which says that a continuous function from a closed subset of a normal space \( T \) into a separable Banach space can be extended to a continuous function on \( T \). It follows from the result of Hanner [6] and the Dugundji extension theorem. By an application of this result we obtain a variant of Theorem 1 with \( T \) normal and \( X \) separable Banach.

4. It is well known that a finite Borel measure on a metric space is regular (see e.g. [2, Th.1.1]). More general, if \( \mu \) is a measure on \( \mathcal{B}(T) \), where \( T \) is metrizable and can be represented as the union of countably many open sets of finite measure, then \( \mu \) is regular (see [5, p.61]). It implies that the \( m \)-th dimensional Lebesgue measure is regular (cf. also Remark 1).
We conclude this paper with a variant of Theorem 1 for Baire measures. The smallest $\sigma$-algebra in a topological space $T$ with respect to which all continuous real-valued functions are measurable is called the $\sigma$-algebra of Baire sets in $T$, and denoted by $B_o(T)$. Always $B_o(T) \subset B(T)$; if $T$ is metrizable then these two $\sigma$-algebras are equal.

**Theorem 3.** Let $T$ be a normal topological space, $\mu$ a finite measure on $B_o(T)$, and $X$ a separable Banach space. Each $B_o(T)$-measurable function $f: T \to X$ is the limit $\mu$-a.e. of a sequence of continuous functions.

**Proof.** A finite measure $\mu$ on $B_o(T)$ is regular in this sense, that for every Baire set $A \subset T$ and every $\epsilon > 0$ there exist closed $F$ and open $G$ such that $F, G \in B_o(T)$, $F \subset A \subset G$ and $\mu(G \setminus F) < \epsilon$ (see [1, Cor.7.2.1]). Thus we have a version of the Lusin theorem with $\mathcal{T} = B_o(T)$ and such $\mu$. Now we argue in the same way as in the proof of Theorem 1, using the extension theorem from Remark 3.

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**References**


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