SELECTIONS OF BIADDITIVE SET-VALUED FUNCTIONS

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Abstract. In this paper we prove that there exists a biadditive selection \( f \) of a biadditive set-valued function \( F \) and a continuous selection when \( F \) is lower semicontinuous.

We begin with some notations and definitions. Let \( n(Y) \) denote the set of all nonempty subsets of a nonempty set \( Y \). If \( Y \) is a normed space then \( cc(Y) \) denotes the set of all compact and convex elements of \( n(Y) \).

Definition 1. Let \( X, Y, Z \) be real vector spaces. We say that a set-valued function \( F : X \to n(Z) \) (abbreviated to "s.v. function") in the sequel is additive iff

\[
F(x + y) = F(x) + F(y) \quad \text{for} \quad x, y \in X.
\]

A s.v. function \( F : X \times Y \to n(Z) \) is called biadditive iff \( F \) is additive with respect to each variable.

Definition 2. The point \( x_0 \) of a subset \( C \) of real vector space \( X \) is called an algebraic interior point of \( C \) (we write \( x_0 \in \text{core}C \)) iff for each \( x \in X \) there is a real positive \( \varepsilon \) such that

\[
\alpha x + (1 - \alpha)x_0 \in C \quad \text{for} \quad |\alpha| \leq \varepsilon.
\]

Definition 3. We say that a point \( x_0 \in C \), \( C \subseteq X \) is an extreme point of \( C \) iff there are no two different points \( x, y \in C \) and no number \( t \in (0, 1) \) such that

\[
x_0 = tx + (1 - t)y.
\]

The set of all extreme points of \( C \) is denoted by \( \text{Ext}C \).

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DEFINITION 4. A set $C \subseteq X$ is said to be a **convex cone** iff $C + C \subseteq C$ and $tC \subseteq C$ for all $t \in (0, \infty)$.

K. Nikodem in the paper [4] proved the following theorem.

**Theorem.** Let $X, Y$ be real vector spaces and $C$ be a convex cone in $X$. Assume that $F : C \to n(Y)$ is an additive s.v. function, $x_0 \in \text{core}C$ and $p \in \text{Ext}F(x_0)$. Then there exists exactly one additive selection $f : C \to Y$ of $F$ such that $f(x_0) = p$. In addition,

$$f(x) \in \text{Ext}F(x) \text{ for } x \in C.$$  

The following lemma (Nikodem [4]) will be useful for us.

**Lemma.** Let $B$ and $C'$ be subsets of a real vector space. If $p \in \text{Ext}(B+C')$, then there exists exactly one point $b \in B$ and exactly one point $c \in C'$ such that $b + c = p$. Moreover, $b \in \text{Ext}B$ and $c \in \text{Ext}C$, i.e. $\text{Ext}(B+C') \subseteq \text{Ext}B + \text{Ext}C$.

Now, we shall formulate a theorem, analogue to Nikodem's Theorem.

**Theorem 1.** Let $X, Y, Z$ be real vector spaces, $C, D$ be convex cones in $X, Y$, respectively, and $F : C \times D \to n(Z)$ be a biadditive s.v. function. Moreover, let $x_0 \in \text{core}C, y_0 \in \text{core}D$ and $p \in \text{Ext}F(x_0, y_0)$. Then there exists exactly one biadditive selection $f : C \times D \to Z$ of $F$ such that $f(x_0, y_0) = p$.

**Proof.** Let $U := C \cap (x_0 - C)$. If $u \in U$ then $x_0 - u \in U$. Fix any element $a \in U$. Since $p \in \text{Ext}F(x_0, y_0) = \text{Ext} \{F(a, y_0) + F(x_0 - a, y_0)\}$, there exist, by Nikodem's lemma, a unique point $p_a \in \text{Ext}F(a, y_0)$ and a unique point $p_{x_0-a} \in \text{Ext}F(x_0 - a, y_0)$ such that

$$p = p_a + p_{x_0-a}. \tag{1.1}$$

For the additive s.v. function $F(a, \cdot) : D \to n(Z)$, $y_0 \in \text{core}D$ and the point $p_a \in \text{Ext}F(a, y_0)$, the assumptions of Nikodem's Theorem hold. So there exists exactly one additive selection $f_a : D \to Z$ of $F(a, \cdot)$ such that

$$f_a(y_0) = p_a.$$

It holds for any $a \in U$. Now, let us define a function $g_0 : U \times D \to Z$ as follows:

$$g_0(a, y) := f_a(y) \text{ for } (a, y) \in U \times D.$$
It is easy to check that \( g_0 \) is properly defined and
\[
g_0(a, y) = f_a(y) \in F(a, y) \quad \text{for } (a, y) \in U \times D.
\]

Moreover,
\[
g_0(a, x + y) = f_a(x) + f_a(y) = g_0(a, x) + g_0(a, y) \quad \text{for } a \in U, \ x, y \in D.
\]

Now, we shall show that \( g_0(a + b, x) = g_0(a, x) + g_0(b, x) \) for all \( x \in D \), \( a, b \in U \) such that \( a + b \in U \). Since \( p \in \text{Ext}\{F(a, y_0) + F(x_0 - a, y_0)\} \}, there exist exactly one \( a_1 \in F(a, y_0) \) and exactly one \( b_1 \in F(x_0 - a, y_0) \) such that
\[
p = a_1 + b_1.
\]
Similarly \( p \in \text{Ext}\{F(b, y_0) + F(x_0 - b, y_0)\} \}, whence \( p = a_2 + b_2 \), where \( a_2 \in F(b, y_0), b_2 \in F(x_0 - b, y_0) \) and \( p \in \text{Ext}\{F(a, y_0) + F(b, y_0) + F(x_0 - a - b, y_0)\} \} so \( p = a_3 + b_3 + c_3 \), where \( a_3 \in F(a, y_0), b_3 \in F(b, y_0) \) and \( c_3 \in F(x_0 - a - b, y_0) \). We get
\[
p = a_3 + (b_3 + c_3) = a_1 + b_1, \ a_1, a_3 \in F(a, y_0), \ b_1, b_3 + c_3 \in F(x_0 - a, y_0),
\]
whence, by the uniqueness of the representation (1.1), we infer that \( a_3 = a_1 = p_a \). In the same way we get that \( b_3 = a_2 = p_b \) and \( p_{a+b} = a_3 + b_3 \). That is \( p_a + p_b = p_{a+b} \). This means that
\[
f_a(y_0) + f_b(y_0) = f_{a+b}(y_0).
\]
Since the fact that \( f_a \) is a selection of \( F(a, \cdot) \) and \( f_b \) is a selection of \( F(b, \cdot) \) implies that \( f_a + f_b \) is a selection of \( F(a+b, \cdot) \), and by the uniqueness of selection passing through the point \( y_0 \), we deduce that
\[
f_{a+b}(y) = f_a(y) + f_b(y) \quad \text{for } y \in D
\]
and
\[
g_0(a + b, y) = f_{a+b}(y) = f_a(y) + f_b(y) = g_0(a, y) + g_0(b, y)
\]
for \( y \in D, \ a, b \in U \) such that \( a + b \in U \). So, we have proved that \( g_0 \) is a biadditive selection of \( F \) on the set \( U \times D \).

Now, we shall extend \( g_0 \) to a biadditive function defined on \( C \times D \). Fix any point \( x \in C \). Since \( x_0 \in \text{core} C \), there exists an \( \varepsilon > 0 \) such that
\[
x_0 + tx \in C \quad \text{for } |t| < \varepsilon.
\]
Let us take a number \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \varepsilon \). Then
\[
\frac{1}{n} x + x_0 \in C.
\]
Consequently
\[ \frac{x}{n} \in x_0 - C \quad \text{and} \quad \frac{x}{n} \in C. \]

It implies that \( \frac{x}{n} \in U. \) Put \( g(x, y) := ng_0 \left( \frac{x}{n}, y \right). \) This definition is correct. Indeed, if \( m \in \mathbb{N} \) is such a number that \( \frac{x}{m} \in U, \) then \( \frac{x}{nm} = (1 - \frac{1}{m}) \cdot \frac{x}{n} + \frac{1}{m} \cdot \frac{x}{n} \in x_0 - C \) as well as \( \frac{x}{mn} \in C \) thus \( \frac{x}{mn} \in U \) and
\[
m g_0 \left( \frac{x}{m}, y \right) = mng_0 \left( \frac{x}{nm}, y \right) = ng_0 \left( \frac{x}{n}, y \right).
\]

Moreover, the function \( g : C \times D \to Z \) defined above is biadditive. Indeed, let \( x \in C, \ y \in C, \ n \in \mathbb{N} \) be a number so large that \( \frac{x}{n}, \frac{y}{n}, \frac{x+y}{n} \in U. \) Then
\[
g(x + y, z) = ng_0 \left( \frac{x+y}{n}, z \right) = ng_0 \left( \frac{x}{n}, z \right) + ng_0 \left( \frac{y}{n}, z \right) = g(x, z) + g(y, z).
\]

Lastly, the function \( g \) is a selection of \( F. \) If \( x \in C, \ y \in D, \ n \in \mathbb{N} \) and \( \frac{x}{n} \in U, \) then
\[
g(x, y) = ng_0 \left( \frac{x}{n}, y \right) \in nF \left( \frac{x}{n}, y \right) \subseteq F \left( \frac{x}{n}, y \right) + \ldots + F \left( \frac{x}{n}, y \right) = F(x, y).
\]

To end the proof we have to show that \( g \) is a unique selection of \( F \) passing through the point \( (x_0, y_0), \) \( p. \) So, assume that there exists \( g_1 : C \times D \to Z \) biadditive selection of \( F \) such that \( g_1(x_0, y_0) = p. \) Fix any \( a \in U. \) Then
\[
p = g_1(x_0, y_0) = g_1(a, y_0) + g_1(x_0 - a, y_0).
\]

Since \( g_1(a, y_0) \in F(a, y_0) \) and \( g_1(x_0 - a, y_0) \in F(x_0 - a, y_0), \) by the uniqueness of representation (1.1), we have that
\[
g_1(a, y_0) = p_a = f_a(y_0) = g(a, y_0).
\]

Thus \( g_1(a, y_0) = g(a, y_0) \) for \( a \in U. \) Since \( g_1(a, \cdot), f_a \) are additive selections of \( F(a, \cdot) \) and \( g_1(a, y_0) = p_a = f_a(y_0), \) we deduce that
\[
g_1(a, y) = f_a(y) = g(a, y) \quad \text{for} \quad y \in D, \ a \in U
\]
(because the selection is unique). If \( a \in C, \ n \in \mathbb{N} \) and \( \frac{a}{n} \in U \) then
\[
g_1(a, y) = ng_1 \left( \frac{a}{n}, y \right) = ng \left( \frac{a}{n}, y \right) = g(a, y) \quad \text{for} \quad a \in C, \ y \in D.
\]

Hence \( g = g_1 \) on the set \( C \times D. \) This completes the proof. \( \square \)
Remark 1. The last proof implies that 
\[ f(x, y) \in \text{Ext} \, F(x, y) \quad \text{for} \quad (x, y) \in C \times D, \]
whenever \( F : C \times D \to \text{conv}(Z) \), where \( \text{conv}(Z) \) denotes the set of nonempty convex subsets of \( Z \). Indeed, if \( x \in U \) and \( y \in D \), then \( g_0(x, y) \in \text{Ext} \, F(x, y) \).

Fix \( x \in C, \ y \in D, \ n \in \mathbb{N} \) such that \( \frac{x}{n} \in U \). Then 
\[ g(x, y) = n \, g_0 \left( \frac{x}{n}, y \right) \in n \text{Ext} \, F \left( \frac{x}{n}, y \right) \subseteq \text{Ext} \left( n \, F \left( \frac{x}{n}, y \right) \right) \subseteq \text{Ext} \, F(x, y). \]

Theorem 2. Let \( X, Y, Z \) be real vector spaces, and \( C, D \) convex cones in \( X, Y \), respectively. Assume that \( F : C \times D \to \text{conv}(Z) \) is a biadditive s.v. function and \( x_0 \in \text{core} \, C, \ y_0 \in \text{core} \, D \) and \( p \in \text{conv}[\text{Ext} \, F(x_0, y_0)] \). Then there exists a biadditive function \( f : C \times D \to Z \) such that \( f(x_0, y_0) = p \) and 
\[ f(x, y) \in \text{conv}[\text{Ext} \, F(x, y)] \quad \text{for} \quad (x, y) \in C \times D. \]

Proof. The point \( p \) belongs to \( \text{conv}[\text{Ext} \, F(x_0, y_0)] \), so there exist a number \( n \in \mathbb{N} \), points \( p_1, \ldots, p_n \in \text{Ext} \, F(x_0, y_0) \) and nonnegative numbers \( \lambda_1, \ldots, \lambda_n \) such that \( \sum_{i=1}^n \lambda_i = 1 \) and \( p = \sum_{i=1}^n \lambda_i p_i \). By Theorem 1, there exist biadditive functions \( f_i : C \times D \to Z \) for which \( f_i(x_0, y_0) = p_i \) and 
\[ f_i(x, y) \in \text{Ext} \, F(x, y) \quad \text{for} \quad (x, y) \in C \times D, \quad i = 1, \ldots, n. \]

It is easy to check that the function \( f : C \times D \to Z \) given by formula 
\[ f(x, y) := \sum_{i=1}^n \lambda_i f_i(x, y) \quad \text{for} \quad (x, y) \in C \times D \]
is biadditive, \( f(x_0, y_0) = \sum_{i=1}^n \lambda_i p_i = p \) and \( f(x, y) \in \text{conv}[\text{Ext} \, F(x, y)] \) for all \( (x, y) \in C \times D \).

Definition 5. Assume that \( X, Y \) are topological vector spaces and \( C \) is an open subset of \( X \). We say that a s.v. function \( F : C \to n(Y) \) is lower semicontinuous (l.s.c.) at a point \( x_0 \in C \) iff for any neighbourhood \( V \) of zero in \( Y \), there exists a neighbourhood \( U \) of zero in \( X \) such that 
\[ F(x_0) \subseteq F(x) + V \quad \text{for} \quad x \in x_0 + U. \]
We say that \( F \) is upper semicontinuous (u.s.c.) at \( x_0 \in C \) iff for every neighbourhood \( V \) of zero in \( Y \) there exists a neighbourhood \( U \) of zero in \( X \) such that 
\[ F(x) \subseteq F(x_0) + V \quad \text{for} \quad x \in x_0 + U. \]

\( F \) is called continuous at \( x_0 \in C \) iff it is both l.s.c. and u.s.c. at \( x_0 \).
THEOREM 3. Let $X, Y, Z$ be topological vector spaces and $Z$ be locally convex, $C, D$ open convex cones in $X, Y$, respectively. A s.v. function $A : C \times D \to \text{cc}(Z)$ is biadditive if and only if there exist a biadditive continuous s.v. function $L : C \times D \to \text{cc}(Z)$ and a biadditive function $a : C \times D \to Z$ such that

$$A(x, y) = a(x, y) + L(x, y) \quad \text{for} \quad (x, y) \in C \times D.$$ 

PROOF. By Theorem 1, there exists a biadditive selection $a : C \times D \to Z$ of $A$. Let us define an s.v. function $L : C \times D \to \text{cc}(Z)$ as follows:

$$L(x, y) := A(x, y) - a(x, y) \quad \text{for} \quad (x, y) \in C \times D.$$ 

Obviously $0 \in L(x, y)$ for all $(x, y) \in C \times D$. Fix any $(x_0, y_0) \in C \times D$. Let $W$ be a neighbourhood of zero in $Z$. $L(x_0, y_0)$ is bounded, so there is a positive integer $n \geq 3$ such that

$$\frac{2}{n} L(x_0, y_0) \subseteq W.$$ 

There exist a balanced neighbourhood $U$ of 0 in $X$ such that $\frac{1}{n} x_0 + u \in C$, $x_0 + u \in C$ for all $u \in U$ and a neighbourhood $V$ of 0 in $Y$ such that $\frac{1}{n} y_0 + v \in D$, $y_0 + v \in D$ for $v \in V$. Then

$$L(x_0, y_0) = L\left(\frac{n-2}{n} x_0, y_0\right) + \frac{2}{n} L(x_0, y_0)$$

$$\subseteq L\left(\frac{n-2}{n} x_0, y_0\right) + L\left(\frac{1}{n} x_0 + \frac{n-1}{n} u, y_0\right) + W$$

$$= L\left(\frac{n-1}{n} x_0 + \frac{n-1}{n} u, y_0\right) + W = L(x_0 + u, \frac{n-1}{n} y_0) + W$$

$$\subseteq L(x_0 + u, \frac{n-1}{n} y_0) + L(x_0 + u, \frac{1}{n} y_0 + v) + W$$

$$= L(x_0 + u, y_0 + v) + W,$$

where $(u, v) \in U \times V$. So, $L(x_0, y_0) \subseteq L(x, y) + W$ for $(x, y) \in (x_0, y_0) + U \times V$. Hence the function $L$ is lower semicontinuous at $(x_0, y_0)$ and $L$ is l.s.c. in $C \times D$.

Since $(\frac{1}{n} x_0, \frac{1}{n} y_0) \in C \times D$ and $C \times D$ is open, there exist a balanced neighbourhood $U$ of 0 in $X$ and a balanced neighbourhood $V$ of 0 in $Y$ such that $\frac{1}{n} x_0 - u \in C, x_0 + u \in C$ for $u \in U$, $\frac{1}{n} y_0 - \frac{n+1}{n} v \in D, y_0 + v \in D$ for
Let \((u, v) \in U \times V. Then
\[
\begin{align*}
L(x_0 + u, y_0 + v) & \subseteq L(x_0 + u, y_0 + v) + L\left(\frac{1}{n}x_0 - u, y_0 + v\right) \\
& = L\left(\frac{n+1}{n}x_0, y_0 + v\right) = L\left(x_0, \frac{n+1}{n}y_0 + \frac{n+1}{n}v\right) \\
& \subseteq L\left(x_0, \frac{n+1}{n}y_0 + \frac{n+1}{n}v\right) + L\left(x_0, \frac{1}{n}y_0 - \frac{n+1}{n}v\right) \\
& = L\left(x_0, \frac{n+2}{n}y_0\right) = L(x_0, y_0) + \frac{2}{n}L(x_0, y_0) \\
& \subseteq L(x_0, y_0) + W.
\end{align*}
\]
So, \(L(x_0 + u, y_0 + v) \subseteq L(x_0, y_0) + W\) for \((u, v) \in U \times V. Hence L is upper semicontinuous at \((x_0, y_0)\). By the first part of the proof \(L\) is continuous in \(C \times D\).

For the next theorem we need some Banach-Steinhaus-type theorems for a bilinear function, which are probably known, however we will give them here for convenience of readers.

**Definition 6.** Let \(X, Y, Z\) be real normed spaces. A bilinear map \(T : X \times Y \to Z\) is called **bounded** iff there exists a real number \(M > 0\) such that
\[
\|T(x, y)\| \leq M \|x\| \cdot \|y\| \quad \text{for } (x, y) \in X \times Y.
\]

The norm of a bilinear bounded map \(T\) is defined by the formula
\[
\|T\| = \sup_{\|x\| \leq 1, \|y\| \leq 1} \|T(x, y)\|.
\]
A bilinear map is bounded if and only if it is continuous.

**Theorem 4.** Let \(X, Y\) be Banach spaces and \(Z\) be a normed space. Assume that bilinear maps \(T_n : X \times Y \to Z\) are continuous, \(n \in \mathbb{N}\). If the sequence \(\{T_n(x, y)\}_{n \in \mathbb{N}}\) is bounded for all \((x, y) \in X \times Y\), then the sequence \(\{\|T_n\|\}_{n \in \mathbb{N}}\) is bounded.

**Proof.** Let \(A_k := \{(x, y) \in X \times Y : \|T_n(x, y)\| \leq k, n \in \mathbb{N}\}, \ k \in \mathbb{N}\. It is easy to verify that
\[
X \times Y = \bigcup_{k \in \mathbb{N}} A_k.
\]
The continuity of the maps \(T_n\) and the norm implies that sets \(A_k\) are closed, \(k \in \mathbb{N}\). Since \(X, Y\) are Banach spaces, we deduce by Baire's theorem that
\(X \times Y\) is the second category set; this means that there exists a number \(k_0 \in \mathbb{N}\) such that \(A_{k_0}\) is not a nowhere dense set; in other words \(\text{Int} A_{k_0} \neq \emptyset\). so there exist real numbers \(r_1 > 0, r_2 > 0\) such that

\[
\text{cl} K_1(x_0, r_1) \times \text{cl} K_2(y_0, r_2) \subseteq A_{k_0}
\]

(where \(K_1\) is a ball in \(X\), \(K_2\) is a ball in \(Y\)). If \(\|x - x_0\| \leq r_1\) and \(\|y - y_0\| \leq r_2\), then \(\|T_n(x, y)\| \leq k_0\) for all \(n \in \mathbb{N}\). Fix \((x, y) \in X \times Y\) such that \(x \neq 0\) and \(y \neq 0\). Since \(\left\| \left( \frac{x}{\|x\|} r_1 + x_0 \right) - x_0 \right\| = r_1\) and \(\left\| \left( \frac{y}{\|y\|} r_2 + y_0 \right) - y_0 \right\| = r_2\) one has

\[
\left\| T_n \left( \frac{x}{\|x\|} r_1 + x_0, \frac{y}{\|y\|} r_2 + y_0 \right) \right\| \leq k_0
\]

and

\[
\left\| T_n(x, y) \right\| = \left\| T_n \left( \frac{x}{\|x\|} r_1, y \right) \right\| \cdot \frac{\|x\|}{r_1}
\]

\[
= \frac{\|x\|}{r_1} \left\| T_n \left( \frac{x}{\|x\|} r_1 + x_0, y \right) - T_n(x_0, y) \right\|
\]

\[
\leq \frac{\|x\|}{r_1} \left\{ \left\| T_n \left( \frac{x}{\|x\|} r_1 + x_0, y \right) \right\| + \left\| T_n(x_0, y) \right\| \right\}
\]

\[
= \frac{\|x\|}{r_1} \left\{ \frac{\|y\|}{r_2} \left\| T_n \left( \frac{x}{\|x\|} r_1 + x_0, \frac{y}{\|y\|} r_2 + y_0 \right) - T_n(x_0, y_0) \right\| \right\}
\]

\[
\leq \frac{4k_0}{r_1 \cdot r_2} \|x\| \cdot \|y\|
\]

for \((x, y) \in X \times Y\) such that \(x \neq 0, y \neq 0\). Hence

\[
\left\| T_n \right\| = \sup_{\|x\| = \|y\| = 1} \left\| T_n(x, y) \right\| \leq \frac{4k_0}{r_1 r_2} \text{ for } n \in \mathbb{N}.
\]

\(\square\)

**Definition 7.** A subset \(A\) of a normed space \(X\) is called **linearly dense** in \(X\) iff the set

\[
\left\{ \sum_{i=1}^{n} \lambda_i a_i; \quad a_i \in A, \quad \lambda_i \in \mathbb{R}, \quad i = 1, \ldots, n; \quad n \in \mathbb{N} \right\}
\]

is dense in \(X\).
THEOREM 5. Let $X,Y,Z$ be Banach spaces and $A_1,A_2$ be linearly dense sets in $X,Y$, respectively. Assume that $T_n : X \times Y \to Z, n \in \mathbb{N}$ is a sequence of bilinear and continuous maps. The sequence $\{T_n(x,y)\}_{n \in \mathbb{N}}$ is convergent for all $(x,y) \in X \times Y$ iff $\{T_n(x,y)\}_{n \in \mathbb{N}}$ is convergent for all $(x,y) \in A_1 \times A_2$ and the sequence $\{|| T_n |||\}_{n \in \mathbb{N}}$ is bounded.

PROOF. If the sequence $\{T_n(x,y)\}_{n \in \mathbb{N}}$ is convergent in $X \times Y$ then it is in $A_1 \times A_2$. Since $\{T_n(x,y)\}_{n \in \mathbb{N}}$ is convergent, the sequence $\{|| T_n(x,y) |||\}_{n \in \mathbb{N}}$ is bounded for any $(x,y) \in X \times Y$. Hence, by Theorem 4, the sequence $\{|| T_n |||\}_{n \in \mathbb{N}}$ is bounded.

Now we assume that $\{T_n(x,y)\}_{n \in \mathbb{N}}$ is convergent in $A_1 \times A_2$ and $\{|| T_n |||\}_{n \in \mathbb{N}}$ is bounded by $M$. Fix any pair $(x_0,y_0) \in X \times Y$ and let $a \in A_1$ be an element of the set $A_1$. Then the map $F_n : Y \to Z$, given by the formula $F_n(y) := T_n(a,y)$ for $y \in Y$, is linear and continuous in $Y$. Moreover, the sequence $\{F_n(y)\}_{n \in \mathbb{N}}$ is convergent for any $y \in A_2$ and $\{|| F_n |||\}_{n \in \mathbb{N}}$ is bounded. Indeed,

$$|| F_n || = \sup_{\| y \|=1} || F_n(y) || = \sup_{\| y \|=1} || T_n(a,y) || \leq \sup_{\| y \|=1} || T_n || || a || || y || = M \cdot || a ||, \quad n \in \mathbb{N}.$$ 

So, by Theorem 16.8 ([3] p.156), we get the convergence of the sequence $\{F_n(y)\}_{n \in \mathbb{N}}$ for all $y \in Y$. Hence, in particular, $\{F_n(y_0)\}_{n \in \mathbb{N}}$ is convergent. Since $a \in A_1$ is arbitrary, the sequence $\{T_n(a,y_0)\}_{n \in \mathbb{N}}$ is convergent for any $a \in A_1$.

Let us define maps $G_n : X \to Z$ as follows:

$$G_n(x) := T_n(x,y_0) \quad \text{for} \quad x \in X, \; n \in \mathbb{N}.$$ 

$G_n$ are linear and continuous maps and the sequence $\{G_n(x)\}_{n \in \mathbb{N}}$ is convergent for any $x \in A_1$. Moreover,

$$|| G_n || = \sup_{\| x \|=1} || G_n(x) || \leq M \cdot || y_0 ||, \quad n \in \mathbb{N}.$$ 

Hence, by the same theorem, the sequence $\{G_n(x)\}_{n \in \mathbb{N}}$ is convergent for any $x \in X_1$, in particular for $x = x_0$. Consequently $\{T_n(x_0,y_0)\}_{n \in \mathbb{N}}$ is convergent.

THEOREM 6. Let $X,Y,Z,A_1,A_2$ be just like in the last theorem. If a sequence $T_n : X \times Y \to Z$ of bilinear and continuous maps is convergent in $A_1 \times A_2$ and the sequence $\{|| T_n |||\}_{n \in \mathbb{N}}$ is bounded then the function $T : X \times Y \to Z$ given by

$$T(x,y) := \lim_{n \to \infty} T_n(x,y) \quad \text{for} \quad (x,y) \in X \times Y$$ 

is continuous.
is a bilinear as well as continuous map and

\[ \| T \| \leq \sup_{n \in \mathbb{N}} \| T_n \|. \]

**Proof.** Theorem 5 implies the convergence of the sequence \( \{ T_n(x, y) \}_{n \in \mathbb{N}} \) for all \((x, y) \in X \times Y\) and hence, the correctness of definition of the map \( T \). Its bilinearity and continuity follow from the Theorem 48.4 ([1] p.139).

Let \( x \in X, \ y \in Y \) and \( \| x \| \leq 1, \| y \| \leq 1 \). Then

\[ \| T(x, y) \| \leq \| T(x, y) - T_n(x, y) \| + \| T_n(x, y) \| \]
\[ \leq \| T(x, y) - T_n(x, y) \| + M \| x \| \| y \| \]
\[ \leq \| T(x, y) - T_n(x, y) \| + M \]

for \( n \in \mathbb{N} \), where \( M = \sup_{n \in \mathbb{N}} \| T_n \| \). By letting \( n \to \infty \), we obtain \( \| T(x, y) \| \leq M \) for \((x, y) \in X \times Y, \| x \| \leq 1, \| y \| \leq 1\). Thus

\[ \| T \| = \sup_{\| x \| \leq 1, \| y \| \leq 1} \| T(x, y) \| \leq M = \sup_{n \in \mathbb{N}} \| T_n \|. \]

\[ \square \]

**Lemma 1.** Let \( X, Y, Z \) be real vector spaces, \( C, D \) convex cones in \( X, Y \), respectively. Let \( f : C \times D \to Z \) be a biadditive function. Then there exists a biadditive function \( \bar{f} : X \times Y \to Z \) such that \( \bar{f}(x, y) = f(x, y) \) for \((x, y) \in C \times D\). If \( C, D \) are open then

\[ \bar{f}(x, y) := f(x_1, y_1) - f(x_2, y_1) - f(x_1, y_2) + f(x_2, y_2), \]

where \( x = x_1 - x_2, \ y = y_1 - y_2, \ x_1, x_2 \in C, y_1, y_2 \in D \).

**Proof.** If \( C, D \) are cones then \((C \times D) - (C \times D) = (C - C) \times (D - D)\) is a subspace of \( X \times Y \). Let us define a function \( f_0 \) on \((C - C) \times (D - D)\) as follows:

\[ f_0(x, y) := f(x_1, y_1) - f(x_2, y_1) - f(x_1, y_2) + f(x_2, y_2), \]

where \( x = x_1 - x_2, \ y = y_1 - y_2, \ x_1, x_2 \in C, y_1, y_2 \in D \).

At first we shall show that the definition of \( f_0 \) is correct. Assume that \( x = x_1 - x_2 = z_1 - z_2 \) and \( y = y_1 - y_2 \) where \( x_1, x_2, z_1, z_2 \in C \) and \( y_1, y_2 \in D \). Then \( x_1 + z_2 = z_1 + x_2 \) and

\[
[f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2)]
\]
\[- [f(z_1, y_1) - f(z_1, y_2) - f(z_2, y_1) + f(z_2, y_2)]
\]
\[= f(x_1 + z_2, y_1) + f(x_2 + z_1, y_2) - f(x_2 + z_1, y_1) - f(x_1 + z_2, y_2)
\]
\[=[f(x_1 + z_2, y_1) - f(x_1 + z_2, y_1)] + [f(x_2 + z_1, y_2) - f(x_1 + z_2, y_2)] = 0.
\]
The case when $x = x_1 - x_2$ and $y = y_1 - y_2 = u_1 - u_2$, $(x_1, x_2 \in C, y_1, y_2, u_1, u_2 \in D)$ is similar.

We shall check that $f_0$ is a biadditive map on $(C - C) \times (D - D)$ to $Z$ and $f_0(x, y) = f(x, y)$ for $(x, y) \in C \times D$. Indeed, let $x, z \in C - C$ and $y \in D - D$. Then there exist $x_1, x_2, z_1, z_2 \in C$ and $y_1, y_2 \in D$ such that $x = x_1 - x_2$, $y = y_1 - y_2$, $z = z_1 - z_2$. By definition of $f_0$

\[
\begin{align*}
&f_0(x + z, y) = f_0((x_1 + z_1) - (x_2 + z_2), y_1 - y_2) \\
&= f((x_1 + z_1, y_1) - f(x_1 + z_1, y_2) \\
&- f(x_2 + z_2, y_1) + f(x_2 + z_2, y_2) \\
&= [f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2)] \\
&+ [f(z_1, y_1) - f(z_1, y_2) - f(z_2, y_1) + f(z_2, y_2)] \\
&= f_0(x, y) + f_0(z, y).
\end{align*}
\]

In the same way we can prove the additivity of $f_0$ with respect to the second variable. Finally, we shall check that $f_0$ is an extension of $f$. Let $(x, y) \in C \times D$. Then $(x, y) = (2x, 2y) - (x, y)$ and

\[
\begin{align*}
f_0(x, y) &= f((2x, 2y) - (x, y)) \\
&= f(2x, 2y) - f(x, 2y) - f(2x, y) + f(x, y) \\
&= f(x, 2y) - [f(2x, y) - f(x, y)] = f(x, 2y) - f(x, y) = f(x, y).
\end{align*}
\]

Let $X_1$ be a subspace of $X$, and $Y_1$ be a subspace of $Y$ such that $(C - C) \oplus X_1 = X$ and $(D - D) \oplus Y_1 = Y$. So, if $(x, y) \in X \times Y$ then

$(x, y) = (x_1 + x_2, y_1 + y_2)$, where $x_1 \in C - C, x_2 \in X_1, y_1 \in D - D, y_2 \in Y_2$.

Let us define a function $\tilde{f} : X \times Y \rightarrow Z$ as follows:

$\tilde{f}(x, y) = f_0(x_1, y_1)$.

It is easy to check that $\tilde{f}$ is properly defined biadditive extension of $f$. 

**Remark 2.** With respect to the above lemma we may assert in Theorem 3 that the biadditive function $a$ is given on $X \times Y$. Similarly in the next theorem.

Now, we shall prove the following theorem, analogue to Theorem 2 from [6].

**Theorem 7.** Let $X, Y$ be real separable Banach spaces, $C, D$ be open, convex cones in $X, Y$, respectively, and let $Z$ be a real Banach space. Assume that $F : C \times D \rightarrow \text{cc}(Z)$ is a biadditive s.v. function, $x_0 \in C$, $y_0 \in D$ and
\[ p \in F(x_0, y_0). \] Then there exists a biadditive selection \( f : C \times D \to Z \) of \( F \) such that \( f(x_0, y_0) = p \). Moreover, if \( F \) is lower semicontinuous, then \( f \) is continuous.

**Proof.** Since \( F \) is compact and convex valued in \( Z \), by the Krein-Milman Theorem ([5])

\[ p \in F(x_0, y_0) = \text{cl}[\text{convExt} F(x_0, y_0)]. \]

Then for each \( n \in \mathbb{N} \) there is an element \( p_n \in \text{convExt} F(x_0, y_0) \) such that

\[ \| p_n - p \| < \frac{1}{n}. \]

Theorem 2 guarantees the existence of biadditive functions \( f_n : C \times D \to Z \) such that

\[ f_n(x_0, y_0) = p_n \]

and

\[ f_n(x, y) \in \text{convExt} F(x, y) \subseteq F(x, y) \quad \text{for} \quad (x, y) \in C \times D. \]

The set \( C \times D \) is an open cone in \( X \times Y \) and the set \((C \times D) - (C \times D)\) is an open subspace of \( X \times Y \), whence

\[ (C \times D) - (C \times D) = \text{lin} (C \times D) = (C - C) \times (D - D) = X \times Y. \]

By Lemma 1

\[ \tilde{f}_n(x, y) := f_n(x_1, y_1) - f_n(x_2, y_1) - f_n(x_1, y_2) + f_n(x_2, y_2), \]

where \( x = x_1 - x_2, \ y = y_1 - y_2, \ x_1, x_2 \in C, \ y_1, y_2 \in D \), is a biadditive map from \( X \times Y \) to \( Z \) and \( \tilde{f}_n(x, y) = f_n(x, y) \) for \( (x, y) \in C \times D \).

Now, we assume that \( F \) is a lower semicontinuous s.v.function. For a fixed \( x \in C \) a function \( y \to F(x, y) \) is additive and \( \mathbb{Q}_+ \)-homogeneous on \( D \) (see Lemma 5.1 in [4]). There exists a constant \( M(x) > 0 \) such that

\[ \| F(x, y) \| \leq M(x) \| y \|, \]

where \( \| F(x, y) \| = \sup \{ \| u \| ; u \in F(x, y) \} \) for \( y \in D \) (see Theorem 4 in [7]). Then, for each \( x \in C \), the set

\[ F(x, \Sigma) = \bigcup_{y \in \Sigma} F(x, y), \]

where \( \Sigma = \{ y \in D; \| y \| \leq 1 \} \) is bounded. By Smajdor's theorem from [7] there exists a constant \( M \) such that

\[ \sup_{y \in \Sigma} \| F(x, y) \| \leq M \| x \| \quad \text{for} \quad x \in C. \]
Let us take a point \( y \in D \) and let \( \{r_n\}_{n \in \mathbb{N}} \) be a sequence of rational numbers such that \( \lim_{n \to \infty} r_n = \| y \| \) and \( \| y \| < r_n \) for \( n \in \mathbb{N} \). Since \( \frac{y}{r_n} \in \Sigma \), \( \| F(x, \frac{y}{r_n}) \| \leq M \| x \| \) for all \( n \in \mathbb{N}, x \in C \). Hence \( \| F(x, y) \| \leq M r_n \| x \| \).

Passing to the limit with \( n \to \infty \), we get

\[
(7.1) \quad \| F(x, y) \| \leq M \| x \| \| y \| \quad \text{for} \quad (x, y) \in C \times D.
\]

Hence and by the relation \( f_n(x, y) \in F(x, y) \) we deduce that

\[
(7.2) \quad \| f_n(x, y) \| \leq M \| x \| \| y \| \quad \text{for} \quad (x, y) \in C \times D, n \in \mathbb{N}.
\]

For every \( x \in X \), the function \( \tilde{f}_n(x, \cdot) : Y \to Z \) is additive in \( Y \) and bounded in some neighbourhood of any point of \( D \), so by the Mehdi theorem (Theorem 4 in [2]) \( \tilde{f}_n(x, \cdot) \) is continuous. Similarly, we get continuity of \( \tilde{f}_n(\cdot, y) \) for any \( y \in Y \). Thus \( \tilde{f}_n \) is a bilinear and continuous map on \( X \times Y \).

Now, we shall show that the sequence \( \{\| \tilde{f}_n \|\}_{n \in \mathbb{N}} \) is bounded. Let us fix \( (x, y) \in X \times Y \) and \( x_1, x_2 \in C, y_1, y_2 \in D \) such that \( x = x_1 - x_2, y = y_1 - y_2 \). Then

\[
\| \tilde{f}_n(x, y) \| = \| f_n(x_1, y_1) - f_n(x_2, y_2) - f_n(x_2, y_1) + f_n(x_2, y_2) \|
\leq \| f_n(x_1, y_1) \| + \| f_n(x_2, y_2) \|
+ \| f_n(x_2, y_1) \| + \| f_n(x_2, y_2) \|,
\]

whence and by (7.2) we get

\[
\| \tilde{f}_n(x, y) \| \leq M (\| x_1 \| \| y_1 \| + \| x_1 \| \| y_2 \| + \| x_2 \| \| y_1 \| + \| x_2 \| \| y_2 \|).
\]

Thus, by Theorem 4 the sequence \( \{\| \tilde{f}_n \|\}_{n \in \mathbb{N}} \) is bounded.

Let sets \( A \) and \( B \) be dense and countable in \( C \) and \( D \), respectively. The set

\[
S := A \times B = \{(x_1, y_1), (x_2, y_2), \ldots\}
\]

is dense in \( C \times D \) and linearly dense in \( X \times Y \). We choose a subsequence \( \{\tilde{f}_{n_k}\}_{n \in \mathbb{N}} \) of the sequence \( \{\tilde{f}_n\}_{n \in \mathbb{N}} \) convergent to the point \((x_1, y_1)\). We are able to do it because \( \{\tilde{f}_n(x_1, y_1)\}_{n \in \mathbb{N}} \) is a sequence of elements of the compact set \( F(x_1, y_1) \). Next, we choose a subsequence of \( \{\tilde{f}_{n_k}\}_{n \in \mathbb{N}} \) convergent to \((x_2, y_2)\), etc. Using the diagonal method we get the subsequence \( \{\tilde{f}_{n_k}\}_{k \in \mathbb{N}} \) of \( \{\tilde{f}_n\}_{n \in \mathbb{N}} \) convergent on \( S \). The sequence \( \{\tilde{f}_{n_k}\}_{k \in \mathbb{N}} \) is convergent on the linearly dense in \( X \times Y \) set \( S \) and the sequence \( \{\| \tilde{f}_{n_k} \|\}_{k \in \mathbb{N}} \) is bounded, so by Theorem 5 it converges to some bilinear and continuous map \( \tilde{f} : X \times Y \to Z \). For any \((x, y) \in C \times D \) we have

\[
\tilde{f}(x, y) \in \text{cl}[\text{convExt} F(x, y)] = F(x, y).
\]
Therefore \( f := \tilde{f} |_{C \times D} \) is a selection of \( F \) on the cone \( C \times D \).

If \( F : C \times D \to \text{cc}(Z) \) is a biadditive s.v. function, then there exist a biadditive function \( a : X \times Y \to Z \) and a biadditive continuous s.v. function \( L : C \times D \to \text{cc}(Y) \) such that

\[
F(x, y) = a(x, y) + L(x, y) \quad \text{for } (x, y) \in C \times D
\]

(cf. Theorem 3 and Remark 2). By the first part of the proof there exists a bilinear and continuous function \( f : X \times Y \to Z \) such that \( f|_{C \times D} \) is a selection of \( L \) on the cone \( C \times D \) and

\[
f(x_0, y_0) = p - a(x_0, y_0).
\]

Then the function \( f_1 : X \times Y \to Z \) given by

\[
f_1(x, y) := a(x, y) + f(x, y) \quad \text{for } (x, y) \in X \times Y,
\]

restricted to \( C \times D \), is a biadditive selection of \( F \) satisfying the condition

\[
f_1(x_0, y_0) = a(x_0, y_0) + f(x_0, y_0) = p.
\]

This completes the proof. \( \square \)

**References**


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