

## DIFFERENTIAL INCLUSIONS – THE THEORY INITIATED BY CRACOW MATHEMATICAL SCHOOL

LECH GÓRNIOWICZ

*Annual Lecture dedicated to the memory of Professor Andrzej Lasota*

### 1. Historical remarks

The theory of differential inclusions was initiated in 1934–1936 with 4 papers. Two of them by the French mathematician A. Marchaud:

1. *Sur les champs de demi-droites et les équations différentielles du premier ordre*, Bull. Soc. Math. France **62** (1934), 1–38,
2. *Sur les champs continus de demi-cônes convexes et leurs intégrales*, Compositio Math. **3** (1936), 89–127,

and the remaining two by S. K. Zaremba a mathematician from Cracow:

3. *O równaniach paratyngensowych* [On paratingent equations], Dodatek do rocznika PTM, **9** (1935), rozprawa doktorska (PhD thesis),
4. *Sur les équations au paratingent*, Bull. Sci. Math. **60** (2) (1936), 139–160.

It is noteworthy that A. Marchaud called the equations in question contingent equations. The rapid development of the theory took place at the beginning of the sixties of the previous century, when Cracow Mathematical School headed by Tadeusz Ważewski started working on these issues. The following mathematicians belonging to the group are worth mentioning:

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It is worth adding that Ważewski himself used another term, that is orientor equation [“równanie orientorowe”]. The fundamental role for the theory was played by Ważewski’s work titled:

1. *On an optimal control problem*, Prague, 1964, 692–704.

In that paper, Ważewski demonstrated that each problem of controlling ordinary differential equations of the first order can be articulated with orientor equation terms. That observation served as an essential stimulus to study orientor differential equations and consequently, it contributed to the introduction of the new term, still valid, and that is “differential inclusions”. We shall return to the subject of the relations to the control theory in the forthcoming passages of the present lecture.

The theory of differential inclusions is located within the mainstream of non-linear analysis – or to put it more precisely – multi-valued analysis. This theory is intensively developed especially in the countries such as France, Germany, Russia, Italy, Canada and USA. In Poland, there is a large group of mathematicians working on these issues. That group is mainly located in the following centres: Gdańsk, Toruń, Warszawa and Zielona Góra.

Moreover, the references listed below – because of the nature of the lecture – are limited to the works by Professor Andrzej Lasota exclusively relating directly or indirectly to the theory of differential inclusions. Rich literature on the subject can be found in the references in the particular monographs.

## 2. Multivalued mappings

In this section, we shall survey the most important properties of multi-valued mappings which we use in the sequel. There are several monographs devoted to multivalued mappings; see e.g. [1]–[3], [7], [8].

In what follows, we assume that all topological spaces are the Tikhonov  $T_{3\frac{1}{2}}$ -spaces.

Let  $X$  and  $Y$  be two spaces and assume that, for every point  $x \in X$ , a nonempty closed (sometimes we will assume only that  $\varphi(x) \neq \emptyset$ ) subset  $\varphi(x)$  of  $Y$  is given; in this case, we say that  $\varphi$  is a *multivalued mapping* from  $X$  to  $Y$  and we write  $\varphi: X \multimap Y$ . In what follows, the symbol  $\varphi: X \rightarrow Y$  is reserved for single-valued mappings, i.e.,  $\varphi(x)$  is a point of  $Y$ .

Let  $\varphi: X \multimap Y$  be a multivalued map. We associate with  $\varphi$  the *graph*  $\Gamma_\varphi$  of  $\varphi$  by putting:

$$\Gamma_\varphi = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$$

and two natural projections  $p_\varphi: \Gamma_\varphi \rightarrow X$ ,  $q_\varphi: \Gamma_\varphi \rightarrow Y$  defined as follows:  $p_\varphi(x, y) = x$  and  $q_\varphi(x, y) = y$ , for every  $(x, y) \in \Gamma_\varphi$ .

Let us also present some more general examples stimulating our consideration of multivalued maps.

**EXAMPLE 2.1** (Inverse functions). Let  $f: X \rightarrow Y$  be a (single-valued) continuous map from  $X$  onto  $Y$ . Then its inverse can be considered as a multivalued map  $\varphi_f: Y \multimap X$  defined by:

$$\varphi_f(y) = f^{-1}(y), \quad \text{for } y \in Y.$$

**EXAMPLE 2.2** (Implicit functions). Let  $f: X \times Y \rightarrow Z$  and  $g: X \rightarrow Z$  be two continuous maps such that, for every  $x \in X$ , there exists  $y \in Y$  such that  $f(x, y) = g(x)$ .

The implicit function (defined by  $f$  and  $g$ ) is a multivalued map  $\varphi: X \multimap Y$  defined as follows:

$$\varphi(x) = \{y \in Y \mid f(x, y) = g(x)\}.$$

**EXAMPLE 2.3.** Let  $f: X \times Y \rightarrow \mathbb{R}$  be a continuous map. Assume that there is  $r > 0$  such that for every  $x \in X$  there exists  $y \in Y$  such that  $f(x, y) \leq r$ . Then we let  $\varphi_r: X \multimap Y$ ,  $\varphi_r(x) = \{y \in Y \mid f(x, y) \leq r\}$ .

**EXAMPLE 2.4** (Multivalued dynamical systems). Dynamical systems determined by autonomous ordinary differential equations without the uniqueness property are multivalued maps.

**EXAMPLE 2.5** (Metric projection). Let  $A$  be a compact subset of a metric space  $(X, d)$ . Then, for every  $x \in X$ , there exists  $a \in A$  such that

$$d(a, x) = \text{dist}(x, A).$$

We define the metric projection  $P: X \multimap A$  by putting:

$$P(x) = \{a \in A \mid d(a, x) = \text{dist}(x, A)\}, \quad x \in X.$$

Note that the metric retraction is a special case of the metric projection.

Let  $K$  be a compact subset of the euclidean space  $\mathbb{R}^n$ . We shall say that  $K$  is a proximative retract if there exists an open subset  $U$  of  $\mathbb{R}^n$  such that  $K \subset U$  and a proximative retraction  $r: U \rightarrow K$  defined as follows:

$$\|r(x) - x\| = \text{dist}(x, K).$$

Note that any convex compact set  $K \subset \mathbb{R}^n$  or any compact  $C^2$ -manifold with or without boundary is a proximative retract.

We shall need also the notion of the Bouligand cone. Let  $C$  be a closed subset of  $\mathbb{R}^n$ . We say that the set:

$$T_C(x) = \left\{ v \in \mathbb{R}^n \mid \liminf_{h \rightarrow 0^+} \frac{\text{dist}(x + hv, C)}{h} = 0 \right\}$$

is the Bouligand contingent cone to  $C$  at  $x \in C$ .

Let  $\varphi: X \multimap Y$  be a multivalued map and  $f: X \rightarrow Y$  be a single-valued map. We say that  $f$  is a *selection* of  $\varphi$  (written  $f \subset \varphi$ ) if  $f(x) \in \varphi(x)$ , for every  $x \in X$ .

The problem of existence of good selections for multivalued mappings is very important in the fixed point theory.

The concept of upper semicontinuity is related to the notion of the small counter image of open sets. On the other hand, the concept of lower semicontinuity is related to the large counter image of open sets.

**DEFINITION 2.6.** A multivalued map  $\varphi: X \multimap Y$  is called *upper semicontinuous* (u.s.c.) map if for every open  $U \subset Y$  the set  $\varphi^{-1}(U)$  is open in  $X$ , where  $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \subset U\}$ .

In terms of closed sets, we have:

**PROPOSITION 2.7.** A multivalued map  $\varphi: X \multimap Y$  is u.s.c. iff for every closed set  $A \subset Y$  the set  $\varphi_+^{-1}(A)$  is a closed subset of  $X$ .

**PROPOSITION 2.8.** If  $\varphi: X \multimap Y$  is u.s.c., then the graph  $\Gamma_\varphi$  is a closed subset of  $X \times Y$ .

**PROPOSITION 2.9.** Let  $\varphi: X \multimap Y$  be a u.s.c. map with compact values and let  $A$  be a compact subset of  $X$ . Then  $\varphi(A)$  is compact.

**PROPOSITION 2.10.** If  $\varphi: X \multimap Y$  and  $\psi: Y \multimap Z$  are two u.s.c. mappings with compact values, then the composition  $\psi \circ \varphi: X \multimap Z$  of  $\varphi$  and  $\psi$  is a u.s.c. map with compact values.

Using the large counter image, instead of a small one, we get:

**DEFINITION 2.11.** Let  $\varphi: X \multimap Y$  be a multivalued map. If, for every open  $U \subset Y$ , the set  $\varphi_+^{-1}(U)$  is open in  $X$ , then  $\varphi$  is called a *lower semicontinuous* (l.s.c.) map, where  $\varphi_+^{-1}(U) = \{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ .

Note that, for  $\varphi = f: X \rightarrow Y$ , the notion of upper semicontinuity coincides with the lower semicontinuity which means nothing else than the continuity of  $f$ .

In what follows, we also say that a multivalued map  $\varphi: X \multimap Y$  is *continuous map* if it is both u.s.c. and l.s.c.

The most famous selection theorem is the following result proved by E. A. Michael.

**THEOREM 2.12** (E. A. Michael). *Let  $X$  be a paracompact space,  $E$  a Banach space and  $\varphi: X \multimap E$  a l.s.c. map with closed convex values. Then there exists  $f: X \rightarrow E$ , a continuous selection of  $\varphi$  ( $f \subset \varphi$ ).*

We would like to point out that in the following part, we will present the Kuratowski–Ryll–Nardzewski selection theorem frequently used in the theory of differential inclusions.

Apart from semicontinuous multivalued mappings, multivalued measurable mappings will be of the great importance in the sequel. Throughout this section, we assume that  $Y$  is a separable metric space, and  $(\Omega, \mathcal{U}, \mu)$  is a measurable space, i.e., a set  $\Omega$  equipped with  $\sigma$ -algebra  $\mathcal{U}$  of subsets and a countably additive measure  $\mu$  on  $\mathcal{U}$ . A typical example is when  $\Omega$  is a bounded domain in the Euclidean space  $\mathbb{R}^k$ , equipped with the Lebesgue measure.

**DEFINITION 2.13.** *A multivalued map  $\varphi: \Omega \multimap Y$  with closed values is called *measurable map* if  $\varphi^{-1}(V) \in \mathcal{U}$ , for each open  $V \subset Y$ .*

In what follows, we shall use the following Kuratowski–Ryll–Nardzewski selection theorem.

**THEOREM 2.14** (Kuratowski–Ryll–Nardzewski). *Let  $Y$  be a separable complete space. Then every measurable  $\varphi: \Omega \multimap Y$  has a (single-valued) measurable selection.*

Let  $\Omega = [0, a]$  be equipped with the Lebesgue measure and  $Y = \mathbb{R}^n$ .

**DEFINITION 2.15.** *A map  $\varphi: [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$  with nonempty compact values is called *u-Carathéodory map* (resp., *l-Carathéodory map*; resp., *Carathéodory map*) if it satisfies:*

- (1)  $t \multimap \varphi(t, x)$  is measurable, for every  $x \in \mathbb{R}^n$ ,
- (2)  $x \multimap \varphi(t, x)$  is u.s.c. (resp., l.s.c.; resp., continuous), for almost all  $t \in [0, a]$ ,
- (3)  $|y| \leq \mu(t)(1 + |x|)$ , for every  $(t, x) \in [0, a] \times \mathbb{R}^n$ ,  $y \in \varphi(t, x)$ , where  $\mu: [0, a] \rightarrow [0, +\infty)$  is an integrable function.

Let  $\varphi: [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$  be a fixed multivalued map. We are interested in the existence of Carathéodory selections, i.e., Carathéodory functions  $f: [0, a] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $f(t, u) \in \varphi(t, u)$ , for almost all  $t \in [0, a]$  and all  $u \in \mathbb{R}^n$ . It is evident that, in the case when  $\varphi$  is  $u$ -Carathéodory, this selection problem does not have a solution in general (the reason is exactly the same as in Michael's selection principle). For  $l$ -Carathéodory multivalued maps  $\varphi$ , however, this is an interesting problem.

We are now going to study this problem. We use the following notation:

$$C(\mathbb{R}^n, \mathbb{R}^n) = \{f: \mathbb{R}^n \rightarrow \mathbb{R}^n \mid f \text{ is continuous}\}.$$

We shall understand that  $C(\mathbb{R}^n, \mathbb{R}^n)$  is equipped with the topology of uniform convergence on compact subsets of  $\mathbb{R}^n$ . This topology is metrizable. Moreover, as usually, by  $L_1([0, a], \mathbb{R}^n)$  we shall denote the Banach space of Lebesgue integrable functions.

There are, essentially, two ways to deal with the above selection problem. Let  $\varphi: [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$  be an  $l$ -Carathéodory mapping. We can show that the multivalued map:

$$\begin{aligned} \Phi: [0, a] &\multimap C(\mathbb{R}^n, \mathbb{R}^n), \\ \Phi(t) &= \{u \in C(\mathbb{R}^n, \mathbb{R}^n) \mid u(x) \in \varphi(t, u(x)) \text{ and } u \text{ is continuous}\} \end{aligned}$$

is measurable. Then, if we assume that  $\varphi$  has convex values, in view of the Michael selection theorem, we obtain that  $\Phi(t) \neq \emptyset$ , for every  $t$ . Moreover, let us observe that every measurable selection of  $\Phi$  will then give rise to a Carathéodory selection of  $\varphi$ .

On the other hand, we can show that the multivalued map:

$$\begin{aligned} \Psi: \mathbb{R}^n &\multimap L_1([0, a], \mathbb{R}^n), \\ \Psi(x) &= \{u \in L_1([0, a], \mathbb{R}^n) \mid u(t) \in \varphi(t, u(t)), \text{ for almost all } t \in [0, a]\} \end{aligned}$$

is a l.s.c. mapping.

Consequently, continuous selections of  $\Psi$  will then give rise to Carathéodory selections of  $\varphi$ .

Hence, our problem can be solved by using the Michael and the Kuratowski–Ryll–Nardzewski selection theorems.

Let us formulate, only for informative purposes, the following result due to A. Cellina.

**THEOREM 2.16.** *Let  $\varphi: [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$  be a multivalued map with compact convex values. If  $\varphi(\cdot, x)$  is u.s.c., for all  $x \in \mathbb{R}^n$ , and  $\varphi(t, \cdot)$  is l.s.c., for all  $t \in [0, a]$ , then  $\varphi$  has a Carathéodory selection.*

A subset  $B \subset L_1([0, a], \mathbb{R}^n)$  is called *decomposable* provided for every two mappings  $u, v \in B$  and for every Lebesgue measurable subset  $J \subset [0, a]$  we have:

$$(\chi_J \cdot u + \chi_{[0, a] \setminus J} \cdot v) \in B,$$

where  $\chi_J$  and  $\chi_{[0, a] \setminus J}$  are characteristic functions of  $J$  and  $[0, a] \setminus J$ , respectively.

Observe that for any measurable and bounded  $\varphi: [0, a] \rightarrow \mathbb{R}^n$  the set of all measurable selections of  $\varphi$  is decomposable, i.e.,  $\{u: [0, a] \rightarrow \mathbb{R}^n \mid u \text{ is measurable and } u(t) \in \varphi(t) \text{ for every } t \in [0, a]\}$  is a decomposable subset of  $L_1([0, a], \mathbb{R}^n)$ .

We have the following Fryszkowski selection theorem.

**THEOREM 2.17.** *Let  $(X, d)$  be a separable metric space and  $\varphi: X \rightarrow L_1([0, a], \mathbb{R}^n)$  be an l.s.c. map with closed values. Then there exists a continuous map  $f: X \rightarrow L_1([0, a], \mathbb{R}^n)$  such that  $f(x) \in \varphi(x)$  for every  $x \in X$ , i.e.,  $f \subset \varphi$ .*

We shall end this section by introducing the class of admissible multivalued mappings. This class is very important in topological fixed point theory for multivalued mappings.

By  $H = \{H_u\}_{u \geq 0}$  we shall denote the Čech homology functor with compact carriers and coefficients in the field  $\mathbb{Q}$  of rational numbers.

A space  $X$  is called *acyclic* provided  $X \neq \emptyset$  and

$$H_q(X) = \begin{cases} 0, & q > 0, \\ \mathbb{Q}, & q = 0. \end{cases}$$

A space  $X$  is called *contractible* provided there exists a homotopy  $h: X \times [0, 1] \rightarrow X$  such that

$$h(x, 0) = x \quad \text{and} \quad h(x, 1) = x_0, \quad \text{for every } x \in X.$$

A compact space  $X$  is called an  $R_\delta$ -set provided there exists a decreasing sequence of compact contractible metric spaces  $\{X_n\}$  such that:

$$X = \bigcap_n X_n.$$

We have: if  $X$  is contractible or  $X$  is an  $R_\delta$ -set, then  $X$  is acyclic. We need the notion of Vietoris mappings.

DEFINITION 2.18. A continuous map  $f: X \rightarrow Y$  is called a *Vietoris map* provided  $f$  is proper, i.e., for every compact  $K \subset Y$  the set  $f^{-1}(K)$  is compact, and for every  $y \in Y$  the fibre  $f^{-1}(y)$  is acyclic.

In what follows we shall use the notation  $p: X \rightrightarrows Y$  for Vietoris mappings.

THEOREM 2.19 (Vietoris Mapping Theorem). *If  $p: X \rightrightarrows Y$  is a Vietoris map, then the induced linear map  $p_*: H(X) \xrightarrow{\sim} H(Y)$  is an isomorphism.*

DEFINITION 2.20. A multivalued map  $\varphi: X \multimap Y$  is called an *admissible map* provided there exists a space  $\Gamma$  and two continuous maps:  $p: \Gamma \rightrightarrows X$  and  $q: \Gamma \rightarrow Y$  such that

$$\varphi(x) = q(p^{-1}(x)), \quad \text{for every } x \in X.$$

Below, we shall list important properties of admissible mappings.

- (1) Any acyclic map  $\varphi: X \multimap Y$  is admissible, where  $\varphi$  is called *acyclic* provided  $\varphi$  is u.s.c. and for every  $x \in X$  the set  $\varphi(x)$  is acyclic.
- (2) Any admissible map  $\varphi: X \multimap Y$  is u.s.c. with compact values.
- (3) If  $\varphi: X \multimap Y$  and  $\psi: Y \multimap Z$  are two admissible mappings then the composition  $\psi \circ \varphi: X \multimap Z$  of two acyclic mappings is also an acyclic map.

Observe that the composition of two acyclic mappings is not necessary an acyclic map.

### 3. Main problems considered in the topological fixed point theory of multivalued mappings

Recall that a metric space  $X$  is an ANR-space (AR-space) provided it is homeomorphic to a retract of an open set in a normed space (a retract of a convex set in a normed space). Evidently every AR-space is an ANR-space. Consider the diagram:

$$(3.1) \quad X \xleftarrow{p} \Gamma \xrightarrow{q} Y,$$

with such a diagram we associate the multivalued map  $\varphi = \varphi(p, q): X \multimap Y$  by putting:

$$(3.2) \quad \varphi(x) = q(p^{-1}(x)), \quad \text{for every } x \in X.$$



Note that  $\varphi(p, q)$  is always a u.s.c. map. Moreover,  $\varphi(p, q)$  is compact provided  $q$  is a compact map, i.e.,  $\overline{\varphi(X)} = \overline{q(\Gamma)}$  is a compact subset of  $Y$ . Finally, let us add that  $\varphi = \varphi(p, q)$  is an admissible map.

If  $X$  is a subset of  $Y$  we let:

$$\begin{aligned} \text{Fix}(\varphi(p, q)) &= \{x \in X \mid x \in \varphi(p, q)(x)\}, \\ C(p, q) &= \{y \in \Gamma \mid p(y) = q(y)\}. \end{aligned}$$

The we have:

$$\text{Fix}(\varphi(p, q)) \neq \emptyset \iff C(p, q) \neq \emptyset.$$

In this section, for simplicity, we shall consider only multivalued mappings defined by the formula (3.2). For each mapping  $\varphi: X \multimap Y$  we define the induced linear map  $\varphi_*: H(X) \rightarrow H(Y)$  by letting:

$$\varphi_* = q_* \circ p_*^{-1}.$$

In particular for  $X = Y$  we define the Lefschetz number  $\Lambda(\varphi)$  of  $\varphi$  by the formula:

$$\Lambda(\varphi) = \Lambda(\varphi_*).$$

Note that if  $X$  is a contractible space, in particular  $X \in \text{AR}$ , then  $\Lambda(\varphi) = 1$ .

Now, we will formulate the fundamental result of the topological fixed point theory.

**THEOREM 3.1 (Lefschetz Fixed Point Theorem).** *Let  $X \in \text{AR}$  and  $\varphi: X \multimap X$  be a compact admissible map. Then:*

- (1) *the Lefschetz number  $\Lambda(\varphi)$  of  $\varphi$  is well defined and*
- (2) *if  $\Lambda(\varphi) \neq 0$ , then  $\text{Fix}(\varphi) \neq \emptyset$ .*

As a simple consequence of Theorem 3.1 we get:

**COROLLARY 3.2.** *If  $X \in \text{AR}$  and  $\varphi: X \multimap X$  is a compact admissible map, then  $\text{Fix}(\varphi) \neq \emptyset$ .*

Let  $X$  be an ANR-space and let  $U$  be an open subset of  $X$ . We let:

$$\begin{aligned} \mathcal{K}(U, X) &= \{\varphi: U \multimap X \mid \varphi \text{ is compact admissible map} \\ &\quad \text{such that } \text{Fix}(\varphi) \text{ is compact}\}. \end{aligned}$$

THEOREM 3.3 (Existence of the fixed point index). *There exists a function:*

$$\text{ind}: \mathcal{K}(U, X) \rightarrow \mathbb{Z},$$

where  $\mathbb{Z}$  is the set of integers, which satisfies the following properties:

- (1) (Existence) *If  $\text{ind}(\varphi) \neq 0$ , then  $\text{Fix}(\varphi) \neq \emptyset$ ;*
- (2) (Additivity) *Let  $U_1, U_2$  be open in  $X$  such that  $U = U_1 \cup U_2$  and  $U_1 \cap U_2 = \emptyset$ . Assume further that  $\varphi_1: U_1 \rightarrow X$  and  $\varphi_2: U_2 \rightarrow X$  are restrictions of  $\varphi: U \rightarrow X$  to  $U_1$  and  $U_2$ , respectively. Then*

$$\text{ind}(\varphi) = \text{ind}(\varphi_1) + \text{ind}(\varphi_2);$$

- (3) (Homotopy) *If  $\varphi, \psi \in \mathcal{K}(U, X)$  are homotopic (we have assumed that the admissible homotopy  $\chi: U \times [0, 1] \rightarrow X$  joining  $\varphi$  and  $\psi$  has a compact set of fixed points, i.e.,*

$$\{x \in U \mid x \in \chi(x, t) \text{ for some } t \in [0, 1]\}$$

*is compact*). Then

$$\text{ind}(\varphi) = \text{ind}(\psi);$$

- (4) (Normalization) *If  $\varphi \in \mathcal{K}(X, X)$ , then  $\text{ind}(\varphi) = \Lambda(\varphi)$ .*

Let  $\varphi \in \mathcal{K}(U, X)$  and let  $x_0 \in \text{Fix}(\varphi)$ . We shall say that  $x_0$  is an essential fixed point iff there exists an open set  $U_0 \subset U$  such that  $\varphi|_{U_0} \in \mathcal{K}(U_0, X)$  and  $\text{ind}(\varphi|_{U_0}) \neq 0$ . We put:

$$\text{Ess Fix}(\varphi) = \{x \in \text{Fix}(\varphi) \mid x \text{ is essential}\}.$$

In what follows we shall use the following proposition.

PROPOSITION 3.4. *Let  $\varphi \in \mathcal{K}(U, X)$  be a map such that the topological dimension  $\dim \text{Fix}(\varphi)$  of  $\text{Fix}(\varphi)$  is equal zero. Then  $\text{Ess Fix}(\varphi) \neq \emptyset$ .*

We shall point out that all results presented in this section are useful in the theory of differential inclusions.

#### 4. Differential inclusions

In what follows we shall deal with the existence and topological properties of solutions to the Cauchy problem for differential inclusions of the form:

$$(4.1) \quad \begin{cases} x'(t) \in \varphi(t, x(t)), \\ x(0) = x_0, \end{cases}$$

where  $\varphi: [0, a] \times C \multimap \mathbb{R}^n$  is multivalued map and  $C$  is a closed subset of  $\mathbb{R}^n$ .

EXAMPLE (Control problems). Consider the following control problem:

$$(4.2) \quad \begin{cases} x'(t) = f(t, x(t), u(t)), \\ x(0) = x_0, \end{cases}$$

controlled by parameters  $u(t)$  (the controls), where  $f: [0, a] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $u \in U$ .

In order to solve (4.2), we define a multivalued map  $F: [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$  as follows:

$$F(t, x) = \{f(t, x, u)\}_{u \in U}.$$

Then solutions of (4.2) are those of the following differential inclusions:

$$(4.3) \quad \begin{cases} x'(t) \in F(t, x(t)), \\ x(0) = x_0. \end{cases}$$

Thus any control problem (4.2) can be transformed, in view of multivalued maps, into problem (4.3).

Note that many other examples come from the game theory, mathematical economics, convex analysis and nonlinear analysis.

Fundamental results concerning the problem (4.1) are summarized below. We let:

$$S(\varphi, x_0) = \{x: [0, a] \rightarrow C \mid x \text{ is a solution of (4.1)}\},$$

where  $x$  is an absolutely continuous function as solutions are understood in the sense of almost everywhere.

**THEOREM 4.1.** *Let  $\varphi: [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$  be a  $u$ -Carathéodory map with convex compact values, then  $S(\varphi, x_0)$  is an  $R_\delta$ -set.*

**THEOREM 4.2.** *Let  $\varphi: [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$  be an  $l$ -Carathéodory map with closed values, then  $S(\varphi, x_0) \neq \emptyset$ .*

**THEOREM 4.3.** *Let  $C$  be a closed subset of  $\mathbb{R}^n$  and  $\varphi: [0, a] \times C \multimap \mathbb{R}^n$  be a Carathéodory map with closed values such that:*

$$\forall t \in [0, a] \forall x \in C \varphi(t, x) \cap T_C(x) \neq \emptyset,$$

*then  $S(\varphi, x_0) \neq \emptyset$ .*

**THEOREM 4.4.** *If all assumptions of Theorem 4.3 are satisfied for  $C$  being a proximate retract and for  $\varphi$  having convex compact values, then  $S(\varphi, x_0)$  is an  $R_\delta$ -set.*

Instead of the Cauchy problem it is possible to study the rolling boundary value problem:

$$\begin{cases} x'(t) \in \varphi(t, x(t)), \\ L(x) = x_0, \end{cases}$$

where  $L: C([0, a], \mathbb{R}^n) \rightarrow \mathbb{R}^n$  is a linear continuous map.

**REMARK.** Note that the above boundary value problem was started by A. Lasota.

There are many results concerning boundary value problems for differential inclusions (see [1]). Below we recall one of them due to A. Lasota and Z. Opial ([18]).

Let  $\varphi, \psi: [0, 1] \times \mathbb{R}^n \multimap \mathbb{R}^n$  be two  $u$ -Carathéodory mappings with closed convex values. Let  $\chi: [0, 1] \multimap \mathbb{R}^n$  be the map given as follows:

$$\chi(t) = \{x \in \mathbb{R}^n \mid \|x\| \leq t\}.$$

Assume further that for every  $t \in [0, 1]$  and for every  $x \in \mathbb{R}^n$  we have:

$$\psi(t, x) \subset \varphi(t, x) + \chi(t).$$

Let  $L: C([0, a], \mathbb{R}^n) \rightarrow \mathbb{R}^n$  be a linear continuous map.

Consider the following two boundary value problems:

$$(4.4) \quad \begin{cases} x'(t) \in \varphi(t, x(t)), \\ L(x) = 0 \end{cases}$$

and

$$(4.5) \quad \begin{cases} x'(t) \in \psi(t, x(t)), \\ L(x) = x_0. \end{cases}$$

Then we have the following result.

**THEOREM 4.5.** *Assume that the above conditions are satisfied and moreover  $\varphi$  is homogeneous, i.e.,*

$$\varphi(t, \lambda x) = \lambda \cdot \varphi(t, x),$$

for every  $t \in [0, 1]$ ,  $\lambda \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ .

*If the problem (4.4) has the unique solution  $x = 0$ , then the problem (4.5) has at least one solution.*

## 5. Two applications to the ordinary differential equations

In this section we would like to present two direct applications of the topological fixed point theory for multivalued mappings to the theory of ordinary differential equations. Of course the same is possible for differential inclusions.

### 5.1. The Poincaré translation operator

Let  $f: [0, a] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous map with linear growth. Then, in view of Theorem 4.1, we get that the set of solutions  $S(f, x_0)$  of (4.1) is an  $R_\delta$ -set.

Consider the following periodic problem:

$$(5.1) \quad \begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x(a). \end{cases}$$

Define the following map:

$$P: \mathbb{R}^n \multimap C([0, a], \mathbb{R}^n)$$

defined by the formula:

$$P(x) = S(f, x).$$

It is well known that  $P$  is an acyclic map.

We let  $e_a: C([0, a], \mathbb{R}^n) \rightarrow \mathbb{R}^n$  by putting:

$$e_a(x) = x(a).$$

Then we have the following diagram:

$$\mathbb{R}^n \xrightarrow{P} C([0, a], \mathbb{R}^n) \xrightarrow{e_a} \mathbb{R}^n.$$

We let:  $P_a = e_a \circ P$ . Observe that  $P_a$  is an admissible map.

REMARK.  $P_a: \mathbb{R}^n \multimap \mathbb{R}^n$  is called the multivalued Poincaré translation operator.

We have: Problem (5.1) has a solution iff  $\text{Fix}(P_a) \neq \emptyset$ .

So we have the following proposition.

PROPOSITION 5.1. *Assume that  $\text{Fix}(P_a) \subset B(0, r)$ , where  $B(0, r) = \{x \in \mathbb{R}^n \mid \|x\| < r\}$ .*

*If the fixed point index  $i(P_a) \neq 0$ , for the map  $P_a: B(0, r) \rightarrow \mathbb{R}^n$ , then problem (5.1) has a solution.*

Now, we are going to calculate  $i(P_a)$  in terms of the right hand side  $f$ . In order to do this, firstly we recall the notion of the direct potential.

A  $C^1$ -function  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be a direct potential if the following condition is satisfied:

$$(5.2) \quad \exists_{r_0 > 0} \text{grad } V(x) \neq 0, \quad \text{for every } x \in \mathbb{R}^n \text{ such that } \|x\| \geq r_0,$$

where  $\text{grad } V = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n}\right)$  denotes the gradient of the function  $V$ .

For any  $r \geq r_0$  we can consider the gradient vector field  $(j - \text{grad } V): B(0, r) \rightarrow \mathbb{R}^n$ , defined by  $(j - \text{grad } V)(x) = x - \text{grad } V(x)$ . It follows from (5.2) that the fixed point index  $i(j - \text{grad } V)$  is well defined and does not depend on the choice of  $r \geq r_0$ .

We let:

$$(5.3) \quad \text{Ind } V = i(j - \text{grad } V).$$

A direct potential  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  is called a *guiding function* for  $f: [0, a] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  provided the following condition is satisfied:

$$(5.4) \quad \exists_{r_0 > 0} \forall_{\|x\| \geq r_0} \forall_{t \in [0, a]} \langle f(t, x), \text{grad } V(x) \rangle \geq 0,$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^n$ .

Finally, in view of Proposition 5.1, we are able to prove the following theorem:

**THEOREM 5.2.** *Let  $f: [0, a] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous map with linear growth. If there exists a guiding function  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  for  $f$  such that  $\text{Ind } V \neq 0$ , then problem (5.1) has a solution.*

**REMARK.** Note that Theorem 5.2 remains true for  $f = \varphi$  being a  $u$ -Carathéodory map with compact convex values and for  $V$  being a locally Lipschitz map. Then  $\text{grad } V$  is a multivalued mapping with compact convex values.

## 5.2. Implicit differential equations

The aim of this section is to show that, using topological fixed point theory as a tool, many types of implicit differential equations can be reduced very easily to differential inclusions with right hand sides not depending on the derivative.

Let  $f: [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a compact map. We shall consider the following equation:

$$(5.5) \quad x'(t) = f(t, x(t), x'(t)),$$

where the solution is understood in the sense of almost everywhere in  $[0, 1]$ . We define a multivalued map  $\varphi: [0, 1] \times \mathbb{R}^n \multimap \mathbb{R}^n$  by putting:

$$\varphi(t, x) = \text{Fix } f(t, x, \cdot),$$

where  $f(t, x, \cdot): \mathbb{R}^n \rightarrow \mathbb{R}^n$  is induced by  $f$ .

Consider the following differential inclusion:

$$(5.6) \quad x'(t) \in \varphi(t, x(t)).$$

Evidently, problem (5.5) is equivalent to (5.6). Unfortunately  $\varphi$  is u.s.c. with compact but not necessary convex values. Consequently, we are unable to solve (5.6). Therefore we put the following assumption:

$$(5.7) \quad \forall_{t \in [0,1]} \forall_{x \in \mathbb{R}^n} \dim \text{Fix } f(t, x, \cdot) = 0,$$

where  $\dim \text{Fix } f(t, x, \cdot)$  denotes the topological dimension of  $\text{Fix } f(t, x, \cdot)$ .

Now, we have:

LEMMA 5.3. *Under all of the above assumption we have:*

$$\forall_{t \in [0,1]} \forall_{x \in \mathbb{R}^n} \exists y \in \text{Fix } f(t, x, \cdot) \text{ such that } i(f|_U) \neq 0,$$

for some open  $U \subset \mathbb{R}^n$  such that  $y \in U$  (compare Proposition 3.4).

REMARK. A point  $y \in \text{Fix } f(t, x, \cdot)$  as described in Lemma 5.3 shall be called an essential fixed point.

We let:

$$\text{Ess Fix } f(t, x, \cdot) = \{y \in \text{Fix } f(t, x, \cdot) \mid y \text{ is essential}\}.$$

Using Lemma 5.3 we define a map  $\psi: [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by letting:

$$(5.8) \quad \psi(t, x) = \text{Ess Fix } f(t, x, \cdot).$$

Of course for every  $t \in [0, 1]$  and  $x \in \mathbb{R}^n$  we have:

$$\psi(t, x) \subset \varphi(t, x).$$

Moreover, we are able to prove the following:

THEOREM 5.4. *The map  $\psi$  defined in (5.8) is l.s.c.*

Now, we remark that if a map  $\psi$  is l.s.c. then  $\text{cl } \psi$  is l.s.c. too, where  $\text{cl } \psi(t, x) = \overline{\psi(t, x)}$  denotes the closure of  $\psi(t, x)$  in  $\mathbb{R}^n$ . Since  $\text{Fix } f(t, x, \cdot)$  is a compact nonempty set for every  $t \in [0, 1]$  and  $x \in \mathbb{R}^n$  we get:

$$(5.9) \quad \text{cl } \psi(t, x) \subset \varphi(t, x),$$

and any solution of the differential inclusion:

$$(5.10) \quad x'(t) \in \text{cl } \psi(t, x)$$

is a solution of (5.6) and hence (5.5).



Therefore in view of Theorem 4.2 we have proved the following theorem:

**THEOREM 5.5.** *If  $f: [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a compact map and the condition (5.7) is satisfied, then problem (5.5) has a solution.*

Observe that the condition (5.7) is restrictive. Below, we shall describe how large is the set of all compact mappings satisfying (5.7).

Let  $C([0, 1] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$  be the set of all compact maps from  $[0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$  into  $\mathbb{R}^n$  with the usual maximum norm.

We let

$$\mathcal{A} = \{f \in C([0, 1] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n) \mid \text{such that } f \text{ satisfies (5.7)}\}.$$

**THEOREM 5.6.** *The set  $\mathcal{A}$  is dense in  $C([0, 1] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ .*

Finally, we would like to add that the method presented in this section can be applied also to the following problems:

Ordinary differential equations of higher order

$$(5.11) \quad x^{(k)}(t) = f(t, x(t), x'(t), \dots, x^{(k)}(t)).$$

Hyperbolic equations

$$(5.12) \quad u_{(t,s)} = f(t, s, u(t, s), u_t(t, s), u_s(t, s), u_{(t,s)}(t, s)).$$

Elliptic differential equations

$$(5.13) \quad \Delta(u)(z) = f(z, u(z), D(u)(z)),$$

where  $\Delta$  denotes the Laplace operator and

$$D(u)(z) = u_{z_1}(z) + \dots + u_{z_n}(z),$$

$$z = (z_1, \dots, z_n) \in K_r^n,$$

$$K_r^n = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}.$$

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SCHAUDER CENTER FOR NONLINEAR STUDIES  
NICOLAUS COPERNICUS UNIVERSITY  
CHOPINA 12/18  
87-100 TORUŃ  
POLAND

AND  
INSTITUTE OF MATHEMATICS  
KAZIMIERZ WIELKI UNIVERSITY IN BYDGOSZCZ  
WYESSENHOFFA 11  
85-072 BYDGOSZCZ  
POLAND  
e-mail: gorn@mat.uni.torun.pl