Scientific Research of the Institute of Mathematics and Computer Science 4(11) 2012, 35-42

# DETERMINATION OF SURFACE LOCATION BETWEEN TWO SUB-DOMAINS USING THE TEMPERATURE DISTRIBUTION ON THE EXTERNAL SURFACE

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**Abstract.** In the paper two sub-domains which are in thermal contact are considered. The temperature field in these domains is described by the system of two Laplace equations supplemented by the boundary conditions. The position of surface between sub-domains is unknown. Additional information necessary to solve the identification problem results from a knowledge of external surface temperature distribution. The direct problem is solved using the boundary element method. To solve the inverse problem formulated the gradient method is applied. In the final part of the paper the results of computations are shown. The algorithm proposed here can be used, among others, in the medical practice (e.g. in burns therapy).

## 1. Direct problem

The temperature field in the domains is described by equations

$$(x,y)\in\Omega_e: \quad \lambda_e \frac{\partial^2 T_e(x,y)}{\partial x^2} + \lambda_e \frac{\partial^2 T_e(x,y)}{\partial y^2} = 0, \qquad e=1,\,2 \tag{1}$$

where $\lambda_e$ [W/(mK)] is the thermal conductivity of sub-domain $\Omega_e$, $T_e$, $x$, $y$ denote the temperature and the geometrical co-ordinates, respectively.

On the external surface (c.f. Fig. 1) the Robin condition is known

$$(x,y)\in\Gamma_{ex}: \quad -\lambda_1 \frac{\partial T_1(x,y)}{\partial n} = \alpha\left[T_1(x,y) - T_a\right] \tag{2}$$

where $T_a$ is the ambient temperature, $\alpha$ is the heat transfer coefficient, $\partial T_1/\partial n$ denotes the normal derivative. On the internal surface $\Gamma_{in}$ (c.f. Fig. 1) the Dirichlet condition can be taken into account

$$(x,y)\subset\Gamma_{in}: \quad T_2(x,y) = T_b \tag{3}$$

On the surface between sub-domains the ideal contact is assumed

$$(x,y)\in\Gamma_c: \quad \begin{cases} -\lambda_1 \dfrac{\partial T_1(x,y)}{\partial n} = \lambda_2 \dfrac{\partial T_2(x,y)}{\partial n} \\ T_1(x,y) = T_2(x,y) \end{cases} \tag{4}$$

On the remaining boundaries the no-flux condition is accepted.
The shape of internal surface $\Gamma_c$ is defined by the NURBS curve

$$\Gamma_c(t) = \frac{(1-t)^2 w_0 \mathbf{P}_0 + 2\,t(1-t)w_1\mathbf{P}_1 + t^2 w_2 \mathbf{P}_2}{(1-t)^2 w_0 + 2\,t(1-t)w_1 + t^2 w_2}\,, \qquad 0 \le t \le 1 \tag{5}$$

where $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ are the control points, $w_1$, $w_2$, $w_3$ are the weights [1].

In Figure 1 the position of internal surface corresponds to the control points $\mathbf{P}_0 = (0, 0.016)$, $\mathbf{P}_1 = (0.01, 0.009)$, $\mathbf{P}_2 = (0.04, 0.016)$, the weights $w_1 = w_2 = w_3 = 1$.

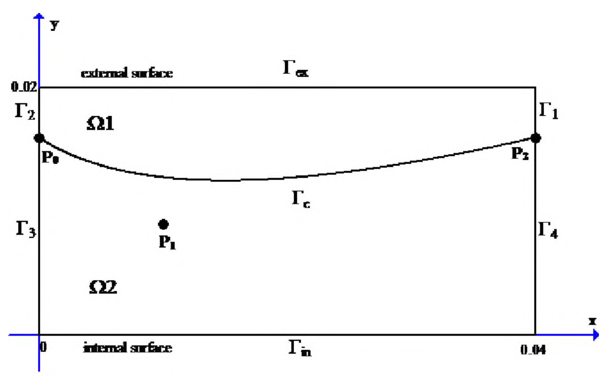


Fig. 1. Domain considered

The direct problem above formulated is solved by means of the BEM.

## 2. Boundary element method

At first the homogeneous domain $\Omega$ is considered. In this case the boundary element method for the Laplace equation leads to the system of equations which for linear boundary elements is the following ($i = 1,2,\ldots,R$) [2, 3]

$$\sum_{r=1}^{R} G_{i\,r} q_r = \sum_{r=1}^{R} \hat{H}_{i\,r} T_r \tag{6}$$

where for the single node $r$ being the end of the boundary element $\Gamma_j$ and being the beginning of the boundary element $\Gamma_{j+1}$ one has

$$G_{i\,r} = G^{k}_{i\,j} + G^{p}_{i\,j-1}, \qquad \hat{H}_{i\,r} = \hat{H}^{k}_{i\,j} + \hat{H}^{p}_{i\,j-1} \tag{7}$$

while for double node $r$, $r+1$

$$\begin{gathered} G_{i\,r} = G^{k}_{i\,j}, \quad G_{i\,r+1} = G^{p}_{i\,j-1} \\ \hat{H}_{i\,r} = \hat{H}^{k}_{i\,j}, \quad \hat{H}_{i\,r+1} = \hat{H}^{p}_{i\,j-1} \end{gathered} \tag{8}$$

where

$$G^{p}_{i\,j} = \frac{l_j}{4\pi\lambda} \int_{-1}^{1} N_p \ln \frac{1}{r_{i\,j}} \, \mathrm{d}\theta \tag{9}$$

$$G^{k}_{i\,j} = \frac{l_j}{4\pi\lambda} \int_{-1}^{1} N_k \ln \frac{1}{r_{i\,j}} \, \mathrm{d}\theta \tag{10}$$

and

$$\hat{H}^{p}_{i\,j} = \frac{1}{4\pi} \int_{-1}^{1} N_p \frac{r^{j}_{x} l^{j}_{y} - r^{j}_{y} l^{j}_{x}}{r^{2}_{i\,j}} \mathrm{d}\theta \tag{11}$$

$$\hat{H}^{k}_{i\,j} = \frac{1}{4\pi} \int_{-1}^{1} N_k \frac{r^{j}_{x} l^{j}_{y} - r^{j}_{y} l^{j}_{x}}{r^{2}_{i\,j}} \mathrm{d}\theta \tag{12}$$

where

$$r_{i\,j} = \sqrt{(N_p x^{p}_{j} + N_k x^{k}_{j} - \xi_i)^2 + (N_p y^{p}_{j} + N_k y^{k}_{j} - \eta_i)^2} = \sqrt{(r^{j}_{x})^2 + (r^{j}_{y})^2} \tag{13}$$

$$l_j = \sqrt{(x^{k}_{j} - x^{p}_{j})^2 + (y^{k}_{j} - y^{p}_{j})^2} = \sqrt{(l^{j}_{x})^2 + (l^{j}_{y})^2} \tag{14}$$

In formula (13) $N_p = (1-\theta)/2$, $N_k = (1+\theta)/2$, $\theta \in [-1,1]$ are the shape functions, $(\xi_i, \eta_i)$ is the observation point, $(x_j^p, y_j^p)$, $(x_j^k, y_j^k)$ are the co-ordinates of beginning and end of element $\Gamma_j$.

It should be pointed out that if $(\xi_i, \eta_i)$ is the beginning of boundary element $\Gamma_j$, this means $(\xi_i, \eta_i) = (x_j^p, y_j^p)$ then

$$G^{p}_{i\,j} = \frac{l_j (3 - 2\ln l_j)}{8\pi\lambda}, \quad G^{k}_{i\,j} = \frac{l_j (1 - 2\ln l_j)}{8\pi\lambda}, \quad \hat{H}^{p}_{i\,j} = \hat{H}^{k}_{i\,j} = 0 \tag{15}$$

while if $(\xi_i, \eta_i)$ is the end of boundary element $\Gamma_j$: $(\xi_i, \eta_i) = (x_j^k, y_j^k)$ then

$$G^{p}_{i\,j} = \frac{l_j (1 - 2\ln l_j)}{8\pi\lambda}, \quad G^{k}_{i\,j} = \frac{l_j (3 - 2\ln l_j)}{8\pi\lambda}, \quad \hat{H}^{p}_{i\,j} = \hat{H}^{k}_{i\,j} = 0 \tag{16}$$

The system of equations (6) can be written in the form

$$\mathbf{G}\mathbf{q} = \mathbf{H}\mathbf{T} \tag{17}$$

where

$$H_{i\,r} = \begin{cases} \hat{H}_{i\,r} & i \neq r \\ -\sum\limits_{\substack{r=1 \\ r \neq i}}^{R} \hat{H}_{i\,r} & i = r \end{cases} \tag{18}$$

In the case of non-homogeneous domain $\Omega = \Omega_1 \cup \Omega_2$ two systems of equations for each sub-domain, should be taken into account separately.
So, the following denotations are introduced (c.f. Fig. 1):

- $\mathbf{T}_1^1$, $\mathbf{T}_1^2$, $\mathbf{T}_1^{ex}$, $\mathbf{q}_1^1$, $\mathbf{q}_1^2$, $\mathbf{q}_1^{ex}$ are the vectors of functions $T$ and $q$ at the boundary $\Gamma_1 \cup \Gamma_2 \cup \Gamma_{ex}$ of domain $\Omega_1$,
- $\mathbf{T}_{c1}$, $\mathbf{T}_{c2}$, $\mathbf{q}_{c1}$, $\mathbf{q}_{c2}$ are the vectors of functions $T$ and $q$ on the contact surface $\Gamma_c$ between sub-domains $\Omega_1$ and $\Omega_2$,
- $\mathbf{T}_2^3$, $\mathbf{T}_2^4$, $\mathbf{T}_2^m$, $\mathbf{q}_2^3$, $\mathbf{q}_2^4$, $\mathbf{q}_2^m$ are the vectors of functions $T$ and $q$ at the boundary $\Gamma_3 \cup \Gamma_4 \cup \Gamma_m$ of domain $\Omega_2$.

Using above notations, one obtains the following systems of equations

- for sub-domain $\Omega_1$

$$\begin{bmatrix} \mathbf{G}_1^1 & \mathbf{G}_1^{ex} & \mathbf{G}_1^2 & \mathbf{G}_{c1} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^1 \\ \mathbf{q}_1^{ex} \\ \mathbf{q}_1^2 \\ \mathbf{q}_{c1} \end{bmatrix} \quad \begin{bmatrix} \mathbf{H}_1^1 & \mathbf{H}_1^{ex} & \mathbf{H}_1^2 & \mathbf{H}_{c1} \end{bmatrix} \begin{bmatrix} \mathbf{T}_1^1 \\ \mathbf{T}_1^{ex} \\ \mathbf{T}_1^2 \\ \mathbf{T}_{c1} \end{bmatrix} \tag{19}$$

- for sub-domain $\Omega_2$

$$\begin{bmatrix} \mathbf{G}_{c2} & \mathbf{G}_2^3 & \mathbf{G}_2^m & \mathbf{G}_2^4 \end{bmatrix} \begin{bmatrix} \mathbf{q}_{c2} \\ \mathbf{q}_2^3 \\ \mathbf{q}_2^m \\ \mathbf{q}_2^4 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{c2} & \mathbf{H}_2^3 & \mathbf{H}_2^m & \mathbf{H}_2^4 \end{bmatrix} \begin{bmatrix} \mathbf{T}_{c2} \\ \mathbf{T}_2^3 \\ \mathbf{T}_2^m \\ \mathbf{T}_2^4 \end{bmatrix} \tag{20}$$

The condition (4) written in the form

$$\begin{cases} \mathbf{q}_{c1} = -\mathbf{q}_{c2} = \mathbf{q} \\ \mathbf{T}_{c1} = \mathbf{T}_{c2} = \mathbf{T} \end{cases} \tag{21}$$

should be introduced to the equations (19), (20).

Next, coupling these systems of equations and taking into account the remaining boundary conditions one has

$$\begin{bmatrix} -\mathbf{H} & \alpha\mathbf{G}-\mathbf{H} & -\mathbf{H} & -\mathbf{H} & \mathbf{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{H} & -\mathbf{G} & -\mathbf{H} & \mathbf{G} & -\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{T} \\ \mathbf{T} \\ \mathbf{T} \\ \mathbf{q} \\ \mathbf{T} \\ \mathbf{q} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \alpha\mathbf{G}\,T \\ \mathbf{G}\,T \end{bmatrix} \tag{22}$$

Finally, the system of equations (22) can be written in the form

$$\mathbf{AZ} = \mathbf{B} \tag{23}$$

where **A** is the main matrix, **Z** is the unknown vector and **B** is the vector of the right-hand side. The system of equations (23) allows one to find the "missing" boundary values. Knowledge of nodal boundary temperatures and heat fluxes constitutes a basis for determination of internal temperatures at the optional set of points selected from the domain considered [2, 3].

## 3. Inverse problem

The inverse problem considered here is based on the assumption that the temperature distribution at the boundary $\Gamma_{ex}$ is known, while the position of $\Gamma_c$ is unknown. This problem is solved using the gradient method [4-7].

As mentioned above, the surface $\Gamma_c$ is described by the NURBS curve (5). The aim of investigations is to determine the value of co-ordinate $y_1$ of control point $\mathbf{P}_1 = (0.01, y_1)$. Let $b = y_1$ is the shape parameter.

The criterion which should be minimized is of the form

$$S(b) = \frac{1}{M}\sum_{i=1}^{M}(T_i - T_{di})^2 \tag{24}$$

where $T_{di}$, $T_i$ are the known from the measurements and calculated temperatures, respectively.

Using the necessary condition of optimum, one obtains

$$\sum_{i=1}^{M}(T_i - T_{di})\frac{\partial T_i}{\partial b} = 0 \tag{25}$$

The function $T_i$ is expanded into the Taylor series taking into account the first derivatives

$$T_i = T_i^k + U_i^k \left( b^{k-1} - b^k \right) \tag{26}$$

where

$$U_i^k = \left( \frac{\partial T_i}{\partial b} \right)_{b=b^k} \tag{27}$$

are the sensitivity coefficients, $k$ is the number of iteration, for $k = 0$ $b^0$ is the arbitrary assumed value of parameter $b$, while for $k > 0$ it results from previous iteration.
Substituting (26) into (25) one has

$$\sum_{i=1}^{M} \left( T_i^k - T_{di} \right) U_i^k + (b^{k+1} - b^k) \sum_{i=1}^{M} \left( U_i^k \right)^2 = 0 \tag{28}$$

hence

$$b^{k+1} = b^k + \frac{\sum_{i=1}^{M} \left( T_i^k - T_{di} \right) U_i^k}{\sum_{i=1}^{M} \left( U_i^k \right)^2}, \qquad k = 0, 1, 2, \ldots, K \tag{29}$$

where $K$ is the number of iterations.
To determine the sensitivity coefficients (27) the following formula is used [3]

$$U_i^k = U(x_i, y_i, b^k) = \frac{T(x_i, y_i, b^k + \Delta b^k) - T(x, y, b^k)}{\Delta b^k} \tag{30}$$

where $\Delta b^k = 10^{-1} b^k$ is a small increase in parameter $b^k$.
This approach requires in each iteration to solve two direct problems with the parameters $b^k$ and $b^k + \Delta b^k$, respectively.

## 4. Results of computations

The rectangular domain of dimensions $2L \times L$ ($L = 0.02$ m) shown in Figure 1 has been considered. The following input data have been assumed: thermal conductivities $\lambda_1 = 2.5$ W/(mK), $\lambda_2 = 1$ W/(mK), heat transfer coefficient

$\alpha$ = 50 W/(m$^2$K), ambient temperature $T_a$ = 20°C (c.f. condition (2)), boundary temperature $T_b$ = 100°C (c.f. condition (3)).
At first, the direct problem described in the Section 1 is solved. In Figure 2 the temperature distribution on the external surface of the domain is presented.

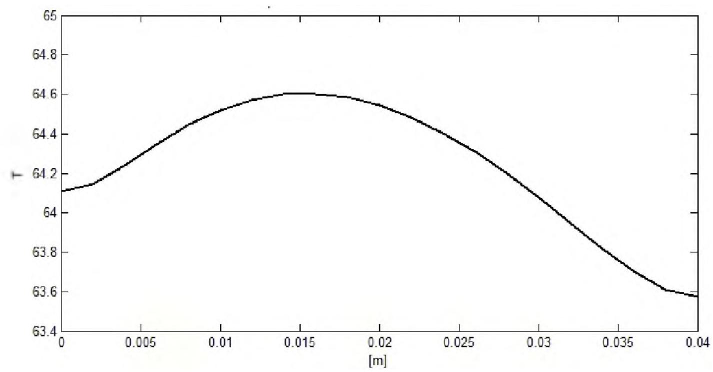


Fig. 2. Temperature distribution on the external surface

Next, the inverse problem is considered. It is assumed that the values of temperatures on the external surface (Fig. 2) resulting from the solution of direct problem are given. The inverse problem is solved under the assumption that the initial position of internal boundary is described by the NURBS curve (5) marked in Figure 3 as 'initial'.

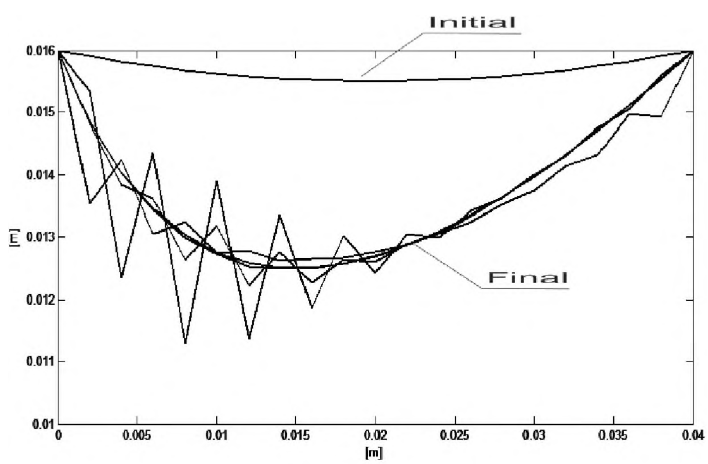


Fig. 3. Results of identification

It is visible that the iteration process is convergent and after several iterations the real position of the internal boundary is obtained.

## Conclusions

The non-homogeneous domain from two sub-domains compound has been considered. The temperature distribution has been described by the system of two Laplace equations. The inverse problem has been solved by means of BEM and the gradient method. The algorithm proposed allows one to identify the unknown position of surface between two sub-domains.

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