# METHOD OF FINDING INCOMES IN HM-NETWORKS WITH TIME-DEPENDENT INTENSITY OF REQUESTS SERVICE

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**Abstract.** The article deals with an HM-network with time-dependent service rates of applications in the systems. The presented methods for finding the expected income systems: the direct method, Laplace transforms, the method of difference scheme, are implemented using the software package *Mathematica*. Expected earnings are important for solving problems of optimization and control of HM-networks, which are used in practice as stochastic models of various objects in computer technology, insurance, logistics, and medicine.

## Introduction

Consider a HM (Howard-Matalytski)-network of arbitrary structure with the same type of requests consisting of $n$ queuing systems (QS) $S_1, S_2, ..., S_n$. A request moving from one QS to another QS brings the last QS some income and respectively, the income of first system is reduced by that value [1]. Consider the case where incomes from the transitions between the states of the network are deterministic functions, depending on the network conditions and time, and the queuing network is a single line. It is assumed that the parameters of the requests service in the QS are time-dependent, therefore if at time $t$ service in the QS $S_i$ is requested, then in interval $[t; t+\Delta t]$ the service will end with the probability $\mu_i(t)\Delta t + o(\Delta t)$, $i = \overline{1,n}$.

The paper presents methods for finding the expected income systems in the network during time $t$ provided that we know of their income in the initial time. The examples of an HM-network in predicting incomes from inter-bank payment in a bank network, the Internet, insurance companies and logistic transport systems at the cost of flexible computing clusters are described in [1].

## 1. Expected incomes of networks systems

Let us denote by $v_i(k,t)$ - the full expected income which is received by system $S_i$ in a time $t$, if during the initial moment of time the network is in state $k$; $r_i(k)$ - the income of system $S_i$ in unit of time, when the network is in state $k$; $I_i$ - the vector with dimension $n$ with zero components, behind exception component $i$, which is equal to 1; $r_{0i}(k+I_i,t)$ - the income of system $S_i$, when the network makes a transition from state $(k,t)$ to state $(k+I_i,t+\Delta t)$ during time $\Delta t$; $-R_{i0}(k-I_i,t+\Delta t)$ - the income of this system if the network makes a transition from state $(k,t)$ to state $(k-I_i,t+\Delta t)$; $r_{ij}(k+I_i-I_j,t+\Delta t)$ - the income of system $S_i$ (the expense or loss of system $S_j$), when the network changes state from $(k,t)$ to $(k+I_i-I_j,t+\Delta t)$ during $\Delta t$, $i,j=\overline{1,n}$.

Suppose that the network is in state $(k,t)$. During time interval $\Delta t$, the network may either be in state $k$ or change its state to $(k-I_i),(k+I_i),(k+I_i-I_j)$, $i,j=\overline{1,n}$. If the network remains in state $(k,t+\Delta t)$, then the expected income of system $S_i$ is equal to $r_i(k)\Delta t$ plus the expected income $v_i(k,t)$, which it will receive at the remaining $t$ time units. The probability of such an event is equal to $1-(\lambda+\sum_{j=1}^{n}\mu_j(t)u(k_j))\Delta t+o(\Delta t)$, where $u(x)=\begin{cases}1, x\geq 0,\\ 0, x<0,\end{cases}$ - the Heaviside function.

If the network makes a transition to state $(k+I_i,t+\Delta t)$ with probability $\lambda p_{0i}\Delta t+o(\Delta t)$, then the income of system $S_i$ is equal to $[r_{0i}(k+I_i,t)+v_i(k+I_i,t)]$, and if to state $(k-I_i,t+\Delta t)$ with probability $\mu_i(t)u(k_i)p_{oi}\Delta t+o(\Delta t)$, then the income of this system equals $[-R_{0i}(k-I_i,t)+v_i(k-I_i,t)]$, $i=\overline{1,n}$. Similarly, if the network passes from state $(k,t)$ to state $(k+I_i-I_j,t+\Delta t)$ with probability $\mu_j(t)u(k_j)p_{ji}\Delta t+o(\Delta t)$, it brings to system $S_i$ the income $r_{ij}(k+I_i-I_j,t)$ plus the expected income of the network over the remaining time under the assumption that the initial network state was $(k+I_i-I_j,t)$.

Then, using the total probability formula for conditional expectation, we can obtain a system of difference-differential equations (DDE):

$$\frac{dv_i(k,t)}{dt}=r_i(k)-[\lambda+$$
$$+\sum_{j=1}^{n}\mu_j(t)u(k_j)]v_i(k,t)+\sum_{j=1}^{n}[\lambda p_{0j}v_i(k+I_j,t)+\mu_j(t)u(k_j)p_{j0}v_i(k-I_j,t)]+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}[\mu_j(t)u(k_j)p_{ji}v_i(k+I_i-I_j,t)+\mu_i(t)u(k_i)p_{ij}v_i(k-I_i+I_j,t)]+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}[\mu_j(t)u(k_j)p_{ji}r_{ij}(k+I_i-I_j,t)+\mu_i(t)u(k_i)p_{ij}r_{ij}(k-I_i+I_j,t)]+$$
$$+\sum_{\substack{c,s=1\\ c,s\neq i}}^{n}\mu_s(t)u(k_s)p_{sc}v_i(k+I_c-I_s,t)+\lambda p_{0i}r_{0i}(k+I_i,t)-\mu_i(t)u(k_i)p_{i0}R_{0i}(k-I_i,t) \quad (1)$$

The number of equations in this system equals the number of the network states.

For a closed networks system of equations (1) reduces to a system of finitely many linear non-uniform DDE which in the matrix form can be rewritten as

$$\frac{dV_i(t)}{dt} = Q_i(t) + A(t)V_i(t)\,,\ i = \overline{1,n} \tag{2}$$

where $V_i^T(t) = (v_i(1,t),\ldots,v_i(L,t))$ - the vector of incomes of system $S_i$, $L$ - the number of network states.

## 2. About solution methods of system (2)

The decision of system (2) can be found, using the Laplace transformation method. Let $U_i(s)$, $G_i(s)$, $W(s)$ - the vector of Laplace transformations of functions $v_i(j,t)$, $Q_i(t)$, $A(t)$, $i=\overline{1,L}$ respectively. Then $sU_i(s)-V_i(0)=G_i(s)+$ $+f_i(W(s),U_i(s))$. Solving this functional equation with respect to $U_i(s)$, we will receive:

$$U_i(s) = F_i\ (G_i(s), W(s)),\ i = \overline{1,n} \tag{3}$$

Taking the inverse Laplace transformations from both parts of equality (3), it is possible to find functions $v_i(j,t),\ i = \overline{1,L}$.

**Example 1.** We will consider the Laplace transformations method on an example of a closed network with small dimensions. Let there be a network with the following parameters: $n=2$, the number of requests in the network, and $K=2$, a the matrix of transitions probabilities of requests between QS networks

$$P = \left\| p_{ij} \right\|_{2\times 2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

The intensity of service is set by the functions $\mu_1(t) = at$, $\mu_2(t) = e^{t-b}$. As the network is closed, the number of its states is equal to $L = C_{n-K-1}^{n-1} = 3$, then will be states (0,2), (1,1), (2,0). Let us designate them respectively 1, 2, 3. Let us assume that incomes $r(k,t)$, $R(k,t)$ do not depend on time and are equal to $r(1)=6$, $r(2)=4$, $r(3)=3$, $R(1)=2$, $R(2)=1$, $R(3)=5$. System (2) in this case looks like:

$$\begin{cases} v_1'(1,t) = 3 - 4e^{t-b} - 2e^{t-b}tv_1(1,t) + 5e^{t-b}tv_1(2,t) \\ v_1'(2,t) = 1 - 2e^{t-b} + 6at + atv_1(1,t) + (-e^{t-b} - at)v_1(2,t) + 3e^{t-b}v_1(3,t) \\ v_1'(1,t) = 6 + 2at + atv_1(2,t) - 2atv_1(3,t) \end{cases} \tag{5}$$

Let us define for it a vector of entry conditions: $V(0) = (34, 25, 10)$.

Vectors $Q_1(t)$, $Q_2(t)$, matrix $A(t)$ in this case look like:

$$Q_1(t) = \begin{pmatrix} 3-4e^{t-b} \\ 1-2e^{t-b}+6at \\ 6+2at \end{pmatrix}, \quad Q_2(t) = \begin{pmatrix} 5-6e^{t-b} \\ 1-6e^{t-b}+8at \\ 3+1at \end{pmatrix}$$

$$A(t) = \begin{pmatrix} -2e^{t-b} & 5e^{t-b} & 0 \\ at & -at-e^{t-b} & 3e^{t-b} \\ 0 & at & -2at \end{pmatrix}$$

Integrating the left and right parts of equality (2) from 0 to $t$, we receive:

$$\int_0^t dV_i(u) = \int_0^t Q_i(u)du + \int_0^t A(u)V_i(u)du \tag{6}$$

We take the Laplace transformation from the left and right parts. From the properties of Laplace the transformation [2], it is said that expressions $U_i(s) - \frac{V_i(0)}{s}$, $\frac{G_i(s)}{s}$, $e^{-b}\frac{U_i(s)}{s-1}$, $-\frac{aU_i'(s)}{s}$ are transformations of Laplace of functions $\int_0^t dV_i(u)$, $\int_0^t Q_i(u)du$, $\int_0^t e^{u-b}V_i(u)du$, $\int_0^t auV_i(u)du$ respectively. Using these relations, from (6), we received the following system of equations:

$$\begin{cases} U_1(s) - \frac{34}{s} = \frac{2}{s} - \frac{6e^{-b}}{s-1} - \frac{5e^{-b}U_1(s)}{s-1} + \frac{4e^{-b}U_2(s)}{s-1} \\ U_2(s) - \frac{25}{s} = \frac{6a}{s^2} - \frac{3e^{-b}}{s-1} + \frac{2}{s} - \frac{a}{s}U_1'(s) - \frac{2e^{-b}U_2(s)}{s-1} + \frac{a}{s}U_2'(s) + \frac{6e^{-b}U_3(s)}{s-1} \\ U_3(s) - \frac{10}{s} = \frac{5a^2}{s^2} + \frac{4}{s} - \frac{a}{s}U_2'(s) + \frac{3a}{s}U_3'(s) \end{cases} \tag{7}$$

From the first equation having received $U_1(s)$ through $U_2(s)$: $U_1(s)=\frac{23(s-1)}{s(s-1+e^{-b})}-\frac{e^{-b}(7+U_2(s))}{(s-1+e^{-b})}$ and having substituted it into the second and third equations, we will lead the received system to the following system of differential equations:

$$\begin{cases} \frac{ae^{-b}-a(s-1+e^{-b})}{s(s-1+e^{-b})}U_2'(s)-\left(\frac{2as}{(s-1+e^{-b})^2}+\frac{e^{-b}}{s-1}\right)U_2(s)-\frac{e^{-b}}{s-1}U_3(s)= \\ =\frac{7a+27s}{s^2}-\frac{4e^{-b}}{s-1}-\frac{23ae^{-b}s}{s^3(s-1+e^{-b})^2}-\frac{23a(s-1)}{s^3(s-1+e^{-b})} \\ \frac{a}{s}U_2'(s)+3\frac{a}{s}U_3'(s)-U_3(s)=\frac{7a^2}{s^2}+\frac{14}{s} \end{cases} \quad (8)$$

Solving the given system by means of the *Mathematica* package, we receive expressions for functions $U_1(s),\ U_2(s),\ U_3(s)$. For example, the expression for $U_2(s)$ looks likes:

$$U_2(s)=\frac{3e^{-b}}{(s-1)^2}+\frac{2as^2}{(s-1)^2(s^2-1)(s-1+e^{-b})}+\frac{e^{-b}(s-3)}{s(s+1)(s+2)^3}+\frac{2(s-3)e^{-b}}{(s+1)^2}+$$

$$\frac{as}{(s+2)(s+4)}+\frac{e^{-b}}{s^{1}}+\frac{5}{s(s-3)^3}+\frac{s-1+e^{-b}}{a(s-3)^3}+4+3e^{-\sqrt{1s}}+$$

$$+2e^{-b}\frac{1}{(s-2)^2}\ln\frac{s-1}{s-5}+\frac{ae^{\ b}}{(s-1)^2}+\frac{3e^{-b}s}{s(s+1)(s^2-1)}+\frac{2e^{-b}(s-3)(s-4)}{(s+2)^3(s+3)^2}+$$

$$+\frac{4as}{(s+2)(s+4)}+\frac{3e^{-b}}{s^5(s-1)^2}+\frac{11}{a\sqrt{s+1}(s-3)^2}+e^{-2t} \quad (9)$$

Taking the inverse Laplace transformation from them, we will receive expressions for expected incomes $v_i(j,t),\ i=\overline{1,n}.$ For example, for income $v_2(2,t)$ when $a=1,\ b=1$ we receive:

$$v_2(2,t)=-\frac{1}{9}+\frac{32}{8e^2}-105e^{-1-3t}+\frac{126}{8}e^{-1-2t}+\frac{13e^{-1+t}}{4}-\frac{e^{-t}}{8}-\frac{e^{2t-\frac{t}{e}}}{4}-\frac{2e^{-t}}{e}+$$

$$+\frac{5e^{2t+1-\frac{t}{e}}}{3e}+\frac{e^{3t}}{27}-2\frac{e^{5t}}{3e}+3ate^{-4t}+\frac{e^{-t}}{8\sqrt{\pi t}^{-\frac{3}{2}}}+\frac{2e^{-\frac{1}{t}}}{\sqrt{\pi t}^{\frac{3}{2}}}-\frac{2e^{1t-\frac{t}{e}}}{t}+ \quad (10)$$

$$+\frac{3e^{-8t}}{\sqrt{\pi t}}+\frac{2t}{e}-23te^{-1-3t}+\frac{2e^{3t}}{t}-\frac{112}{4}te^{-1-2t}-\frac{11}{2}te^{-1-t}+$$

$$+te^{t-1}+\frac{5te^{t}}{4}+\frac{(t+e)e^{t}}{e}-\frac{2}{9}te^{3t}+\frac{3t^{2}}{e}+\frac{53}{4}t^{2}+\frac{t^{3}}{2e}+\frac{t^{4}}{13e}+\frac{e^{3t}}{18}$$

In Figure 1 the expected income of system $S_2$ for an initial condition of network $k=(1,1)$.

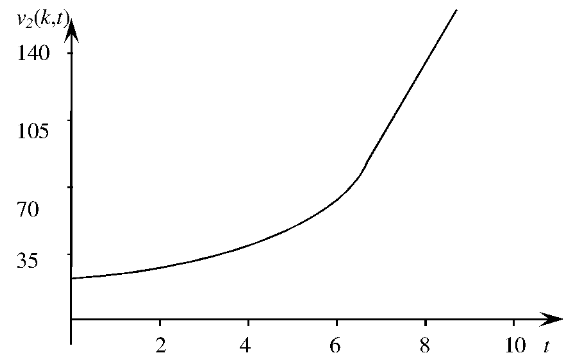


Fig. 1. Income of system in state $k$ = (1,1)

Consider the case when the intensity of request service applications in the network systems are step functions of time with several intervals of constancy that are the same for all systems on the network:

$$\mu_i(t)=\begin{cases}\mu_i^{(1)},\ t\in[0,t_1],\\ \mu_i^{(2)},\ t\in[t_1,t_2],\\ \ldots\\ \mu_i^{(m)},\ t\in[t_{m-1},T]\end{cases},\ i=\overline{1,n} \tag{11}$$

Then the system of equations (2) may be represented at different intervals in a matrix form:

$$\frac{dV_i^{(q)}(t)}{dt}=Q_i^{(q)}(t)+A^{(q)}V_i^{(q)}(t) \tag{12}$$

where $V_i^{(q)}(t)=(v_i^{(q)}(1,t),\ldots,v_i^{(q)}(l,t))^T$ - the vector of income $S_i$ on the $q$ time interval. We define the vector of initial conditions: $V_i(0)$, $V_i^{(q)}(t_{q-1})=V_i^{(q-1)}(t_{q-1})$, $q=\overline{1,m}$. We describe how to find the solution of (4) at various time intervals. Multiplying both sides of (6) by $e^{-A^{(q)}t}$, we get

$$e^{-A^{(q)}t}\left(V_i'(t)-A^{(q)}V_i(t)\right)=\frac{d}{dt}\left(e^{-A^{(q)}t}V_i(t)\right)=e^{-A^{(q)}t}Q_i(t) \qquad (13)$$

which implies

$$e^{-A^{(q)}t}V_i(t)=V_i(0)+\int_0^t e^{-A^{(q)}\tau}Q_i(\tau)d\tau, \qquad (14)$$

that is

$$V_i(t)=e^{A^{(q)}t}V_i(0)+\int_0^t e^{A^{(q)}(t-\tau)}Q_i(\tau)d\tau \qquad (15)$$

where $e^{A^{(q)}t}=I+A^{(q)}t+\frac{{A^{(q)}}^2t^2}{2!}+...+\frac{{A^{(q)}}^nt^n}{n!}+...$ - the matrix exponential.

The number of states $L=C_{n-K-1}^{n-1}$ is large enough even for a relatively small $n$ and $K$, and therefore the number of equations in (11) will also be large enough. The direct method involves the problem of finding the exponent of a matrix with constant elements. Implementing these algorithms in mathematical packages requires huge expenditures of computer memory and CPU time. For more rapid and efficient computation of matrix exponents, one can use a special algorithm for "fast" computations [3], based on the formula:

$$e^{A^{(q)}t}\approx\left(\sum_{s=0}^{N}\frac{(A^{(q)})^s}{s!}\left(\frac{t}{2^M}\right)^s\right)^{2^M} \qquad (16)$$

where $N\geq 1$, $M\geq 0$ - are integers. The algorithm for finding matrix exponent $e^{A^{(q)}t}$ with this accuracy is as follows.

1. Find $\left\|A^{(q)}\right\|t$, where $\left\|A^{(q)}\right\|=\max\limits_{1\leq i\leq n}\sum_{j=1}^{n}\left|(a^{(q)})_{ij}\right|$, where $(a^{(q)})_{ij}$ - a component of matrix $A^{(q)}$.

2. For a given $\varepsilon$, selected values $M$ and $N$ of the following relations:

$$M=\begin{cases}[z]+1, & \left\|A^{(q)}\right\|t>v\\ 0, & \left\|A^{(q)}\right\|t\leq v\end{cases}, \quad z=\frac{1}{\ln 2}\left(\ln\frac{\left\|A^{(q)}\right\|t}{N+1}-\frac{1}{2(N+1)}\right)+N, \quad v=\frac{1}{2^N}(N+1)e^{\frac{1}{2(N+1)}},$$

where $N$ may be found from inequality $\frac{e^{\left\|A^{(q)}\right\|t}\left(\left\|A^{(q)}\right\|t\right)^{N+1}}{2^{MN}(N+1)!}\leq\varepsilon$.

3. We assume $i:=0$ and organize a cycle.

4. We assume, that $G:=\sum_{s=0}^{N}\frac{(A^{(q)})^s}{s!}\left(\frac{t}{2^M}\right)^s$.

5. Increment counter $i$ per unit $i := i + 1$.
6. In 4 instead of $G$ write $G^2$.
7. If $i < M$, then go to step 4.
8. If $i = M$, then assume $e^{A(t)t} \approx G$.
9. End of algorithm.

**Example 2**. In this example, we consider the solution of equations (2) by the direct method. Consider a closed network with the following parameters: $n = 4$, $K = 4$, $T = 10$, $m = 3$, $t_1 = 4$, $t_2 = 6$ and a matrix of transition probabilities between the QS network applications

$$P = \left\| p_{ij} \right\|_{4\times 4} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The intensity of requests service is defined in the form:

$$\mu_1(t) = \begin{cases} 20, t \in [0,4], \\ 10, t \in (4,6], \\ 11, t \in (6,10], \end{cases} \quad \mu_2(t) = \begin{cases} 12, t \in [0,4], \\ 15, t \in (4,6], \\ 9, t \in (6,10], \end{cases} \mu_3(t) = \begin{cases} 12, t \in [0,4], \\ 17, t \in (4,6], \\ 14, t \in (6,10], \end{cases} \mu_4(t) = \begin{cases} 15, t \in [0,4], \\ 18, t \in (4,6], \\ 13, t \in (6,10]. \end{cases}$$

In this case, the number of states is $L = C_{n\ K-1}^{n-1} = 35$, it is (0,0 ,0,4), (0,0,1,3), (0,0,2,2) and etc. We rename them respectively from 1 to 35. Let incomes $r(k)$, $R(k)$ have the values:

$$r(1) = r(23) = 5,\ \ r(4) = r(17) = r(19) = r(34) = 4,\ \ r(5) = r(15) = r(26) = r(32) = 6,$$
$$r(9) = 8,\ \ r(6) = r(29) = 3,\ \ r(2) = r(18) = r(31) = 9,$$
$$r(7) = r(11) = r(12) = r(21) = r(27) = 7,\ r(3) = 10,\ r(8) = r(16) = r(28) = r(33) = 2,$$
$$r(13) = r(20) = r(22) = r(24) = r(25) = r(30) = r(35) = 1;\ R(13) = R(24) = 1,\ R(33) = 3,$$
$$R(17) = R(27) = 2,\ R(1) = R(7) = R(8) = R(14) = R(18) = R(23) = 5,\ R(28) = R(29) = 8,$$
$$R(2) = R(10) = R(15) = R(16) = R(25) = R(34) = 10,\ R(9) = R(31) = 9,$$
$$R(3) = R(6) = R(11) = R(30) = R(32) = 7,\ R(5) = R(12) = R(21) = 4,$$
$$R(4) = R(19) = R(20) = R(22) = R(26) = R(35) = 6.$$

We define the vector of initial conditions:

$$V(0) = (7,4,9,10,7,8,5,9,5,10,2,7,1,1,5,5,1,2,3,1,2,4,6,7,8,2,3,4,1,2,3,1,2,1,1)$$

Performing all the necessary calculations, we obtain the following system of differential equations for the incomes:

$$
\begin{cases}
\dfrac{dv_1(t)}{dt} = -6{,}7v_2(t) + 6{,}7v_3(t) + 72.6; \ \dfrac{dv_2(t)}{dt} = -6{,}7v_2(t) + 6{,}7v_6(t) + 16{,}3; \\
\dfrac{dv_3(t)}{dt} = 20v_1(t) - 16v_3(t) + 6{,}7v_8(t) - 183{,}3; \ \dfrac{dv_4(t)}{dt} = -6{,}7v_4(t) + 6{,}7v_9(t) + 44; \\
\dfrac{dv_5(t)}{dt} = -6{,}7v_5(t) + 6v_{12}(t) + 60{,}3; \ \dfrac{dv_6(t)}{dt} = 20v_2(t) - 16v_6(t) + 6{,}7v_{14}(t) - 51{,}3; \\
\dfrac{dv_7(t)}{dt} = -6{,}7v_7(t) + 6v_{15}(t) + 48 \ ; \dfrac{dv_8(t)}{dt} = 20v_3(t) - 16{,}7v_8(t) + 6v_{17}(t) - 188{,}3; \\
\dfrac{dv_9(t)}{dt} = 20v_4(t) - 16v_9(t) + 6{,}7v_{18}(t) + 13{,}6; \ \dfrac{dv_{10}(t)}{dt} = -6{,}7v_{10}(t) + 6{,}7v_{19}(t) + 42{,}3; \\
\dfrac{dv_{11}(t)}{dt} = -6.7v_{11}(t) + 6v_{22}(t) + 42{,}3; \ \dfrac{dv_{12}(t)}{dt} = 20v_5(t) - 16v_{12}(t) + 16v_{24}(t) + 28{,}6; \\
\dfrac{dv_{13}(t)}{dt} = -6v_{13}(t) + 6{,}7v_{25}(t) + 41{,}3; \ \dfrac{dv_{14}(t)}{dt} = 20v_6(t) - 16{,}7v_{14}(t) + 6v_{27}(t) + 22; \\
\dfrac{dv_{15}(t)}{dt} = 20v_7(t) + 6{,}7v_{28}(t) - 142{,}7; \ \dfrac{dv_{16}(t)}{dt} = -6{,}7v_{16}(t) + 6{,}7v_{29}(t) + 47; \\
\dfrac{dv_{17}(t)}{dt} = 20v_8(t) + 6{,}7v_{31}(t); \ \dfrac{dv_{18}(t)}{dt} = 20v_9(t) - 16{,}7v_{18}(t) + 6v_{32}(t) + 100{,}7; \\
\dfrac{dv_{19}(t)}{dt} = 20v_{10}(t) - 16v_{19}(t) + 6{,}7v_{33}(t) - 33; \ \dfrac{dv_{20}(t)}{dt} = -6{,}7v_{20}(t) + 6v_{34}(t) - 76{,}7; \\
\dfrac{dv_{21}(t)}{dt} = 2; \ \dfrac{dv_{22}(t)}{dt} = -10v_{22}(t) - 116; \ \dfrac{dv_{23}(t)}{dt} = 5; \ \dfrac{dv_{24}(t)}{dt} = -10v_{24}(t) - 151; \\
\dfrac{dv_{25}(t)}{dt} = -6{,}7v_{17}(t) + 6{,}7v_{33}(t); \ \dfrac{dv_{26}(t)}{dt} = 7; \ \dfrac{dv_{27}(t)}{dt} = -10v_{27}(t) - 19; \\
\dfrac{dv_{28}(t)}{dt} = -10v_{27}(t) - 112; \\
\dfrac{dv_{29}(t)}{dt} = -6{,}7v_{29}(t) + 6{,}7v_{34}(t) + 48{,}7; \ \dfrac{dv_{30}(t)}{dt} = -16{,}7v_{30}(t) + 6{,}7v_{35}(t) - 22; \\
\dfrac{dv_{31}(t)}{dt} = -6{,}7v_{31}(t) + 6{,}7v_{35}(t) + 22{,}7; \ \dfrac{dv_{32}(t)}{dt} = -10v_{32}(t) - 134; \\
\dfrac{dv_{33}(t)}{dt} = 5; \ \dfrac{dv_{34}(t)}{dt} = -10v_{34}(t) - 95; \ \dfrac{dv_{35}(t)}{dt} = 10.
\end{cases}
$$

It is a system of linear non-homogeneous differential equations with constant coefficients. Solving it with the Mathematica package, we can obtain expressions for the incomes, for example, $v_1(k,t) = 5{,}6 + 18t + 4{,}2t^{128} + 2{,}13t^{203} + 8.24t^{218}$.

We describe another way of finding the expected incomes. From the system of linear inhomogeneous differential equations (2), we can move to a system of difference equations. We divide time interval $[0,T]$ by net $\{t_j\}_{j=1}^{N}$ with step $h$, $t_j = t_0 + jh,\ t_0 = 0,\ h = \frac{T}{N}$. We assume $\frac{dV_i(t_j)}{dt} = \frac{V_i(t_{j-1}) - V_i(t_j)}{h}$. Then we obtain the difference scheme: $\begin{cases} V_i(t_{j-1}) = V_i(t_j) + Q_i(t_j)h + A(t_j)V_i(t_j)h, \\ t_j = t_0 + jh,\ t_0 = 0,\ h = \frac{T}{N},\ i = \overline{1,n}. \end{cases}$. The initial conditions are as follows: $V_i(0) = 0,\ Q_i(0) = 0$. Then for each $t_j = t_0 + jh$ we obtain a system of linear algebraic equations, solving them with the help of the Mathematica package, we obtain values $V_i(t_0), V_i(t_1), \ldots, V_i(t_N)$.

**Example 3.** We will consider a closed network with following parameters: $n$ = 5, $K$ = 10, $h$ = 0.01, $T$ = 10, $N$ = 1000. The number of its states equal $L = 1001$. Let us designate states from 1 to 1001. Let the intensity of service of requests for every QS look like:

$$\mu_1(t) = 1 + 2t,\ \mu_2(t) = 1 + t,\ \mu_3(t) = 1 + 5t,\ \mu_4(t) = 1 + 3t,\ \mu_5(t) = 1 + 8t$$

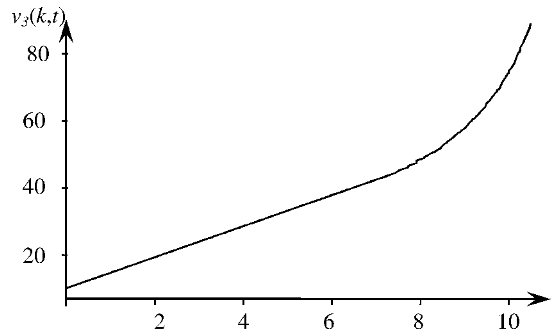


Fig. 2. Income of system in state $k$ = (1,2,2,1)

For each node of the net we make a system of linear algebraic equations and by solving we receive values of incomes at the indicated points. On them it is possible to construct a graph of functions $v_i(k,t)$. In Figure 2 the expected income of system $S_2$ for an initial condition of a network $k$ =(1,2,2,1).

## References


[1] Matalytski M., On some results in analysis and optimization of Markov networks with incomes and their application, Automation and Remote Control 2009, 70, 10, 1683-1697.

[2] Bakhvalov N., Numerical Methods, Nauka, Moscow 1987 (in Russian).

[3] Ershova V., Pulse Function. Functions of Complex Variables. Operational Calculus, Vyzshaja Shkola, Minsk 1976 (in Russian).