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FINDING EXPECTED INCOMES IN HM-NETWORK WITH PRIORITY REQUESTS AND LINEAR TIME-DEPENDENT INTENSITY OF THEIR SERVICE

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Abstract. This article describes the system of difference-differential equations for the expected income in HM-network systems with priority requests and the method of its solution by means of difference schemes. Examples of finding the expected income for networks with different number of states are presented.

Introduction

Queueing networks with income or HM (Howard-Matalytski)-networks were introduced in [1, 2]. A survey of the results on Markov HM-networks are given in [3]. Such networks with different characteristics are currently under study. In this paper we study a Markov HM-network with priority requests in the case where the intensity of their service is time-dependent.

Consider a closed network in which K_1 requests of the first type and K_2 requests of the second type circulate, and requests cannot change their type. The matrix of transition probabilities between the systems network requests, $\| p_{ij} \|_{n \times n}$ is irreducible. System S_i contains m_i parallel lines of service, the service time requests of each type in every line of the system has an exponential distribution, $i = \overline{1,n}$. The probability of service requests type c in the lines of S_i during time interval $[t, t+\Delta t]$ is $\mu_{ic}(t)\Delta t + o(\Delta t)$, where $\mu_{ic}(t)$ - the intensity of service requests such as c in every line system S_i at time t, $i = \overline{1,n}$, $c = 1,2$. Functions $\mu_{ic}(t)$ will be considered restricted for each time interval, $i = \overline{1,n}$, $c = 1,2$.

The same type of request in the queue of a queuing system (QS) is selected for service in random order, for example, FIFO. Requests of the first type have absolute priority over the requests of the second type. In this case it means the fufillment of two conditions: a) if at the time of the release of a QS line after the service requests in its queue has priority requests, then any of them can occupies the vacant line, and b) if in the service system, all lines which are busy, but not only the priority of applications received priority application, it replaces the low-priority request from one of the lines and starts to service this line, the request is extruded into the consideration all the QS. When the request goes to the repressed service again, it is served over the remaining service time. Since the service time is exponentially distributed, we can assume that the displacement re-application will be maintained, i.e. we have so-called non-identical service.

Network states in this case given by vector $k(t)=(k,t)=$ $=(k_{11},k_{12};k_{21},k_{22};...;k_{n1},k_{n2};t)$, where k_{ic} - the number of requests of type c in QS i, $\sum_{i=1}^{n}k_{i1}=K_1$, $\sum_{i=1}^{n}k_{i2}=K_2$.

Let I_{i1} - the vector of dimension $2n$ with zero components, behind an exception component with number $2i-1$, which is equal to 1, I_{i2} - the vector of dimesion $2n$ with zero components, behind an exception component with number $2i$, which is equal to 1, I_0 - the vector of dimension $2n$ with zero components

$$\begin{aligned} k+I_{i1}-I_{j1}&=(k_{11},k_{12};...;k_{i1}+1,k_{i2};...;\ k_{j1}-1,k_{j2};...;k_{n1},k_{n2}) \\ k+I_{i2}-I_{j2}&=(k_{11},k_{12};...;k_{i1},k_{i2}+1;...;\ k_{j1},k_{j2}-1;...;k_{n1},k_{n2}) \end{aligned} \tag{1}$$

The system of equations for state probabilities $P(k,t)$ has the form:

$$\frac{dP(k,t)}{dt}=-\sum_{\substack{i,j=1\\ i\neq j}}^{n}[\mu_{i1}(t)\varepsilon_{i1}(k_{i1})+\mu_{i2}(t)\varepsilon_{i2}(k_{i1},k_{i2})]p_{ij}P(k,t)+$$

$$+\sum_{\substack{i,j=1\\ i\neq j}}^{n}[\beta_{i1}(k,t)P(k+I_{i1}-I_{j1},t)+\gamma_{ij}(k,t)P(k+I_{i2}-I_{j2},t)] \tag{2}$$

where:

$$\begin{gathered} \beta_{ij}(k,t)=\mu_{i1}(t)\varepsilon_{i1}(k_{i1})u(k_{i1})u(K_1-k_{j1})p_{ij} \\ \gamma_{ij}(k,t)=\mu_{i2}(t)\varepsilon_{i2}(k_{i1},k_{i2})u(k_{i2})u(K_2-k_{j2})p_{ij} \end{gathered} \tag{3}$$

$$\varepsilon_{i1}(k_{i1})=\min\{k_{i1},m_i\},\ i=\overline{1,n}$$

$$\varepsilon_{i2}(k_{i1},k_{i2})=\begin{cases} k_{i2},\ k_{i1}+k_{i2}<m_i, \\ m_i-k_{i1},\ k_{i1}<m_i,\ k_{i1}+k_{i2}\geq m_i,\ i=\overline{1,n}, \\ 0,k_{i1}\geq m_i. \end{cases} \tag{4}$$

It is displayed in the same way as in the case when the intensity of service requests in the lines of systems does not depend on time [4].

1. Case when incomes of transitions between states depends on states and time

Let us designate through $v_i(k,t)$ - the full expected income which is received by system S_i at time t, if during the initial moment of time the network is in state k; $r_i(k,t)$ - the income of system S_i in unit of time, when the network is in state (k,t); $r^{(1)}(k+I_{i1}-I_{j1},t)$ - the income of system S_i (the expense or loss of system S_j), when the network changes state from (k,t) to $\left(k+I_{i1}-I_{j1},t+\Delta t\right)$ during time Δt, $r^{(2)}(k+I_{i2}-I_{j2},t)$ - the income of system S_i (the expense or loss of system S_j), when the network changes state from (k,t) to $\left(k+I_{i2}-I_{j2},t+\Delta t\right)$ during time Δt, $j=\overline{1,n}$, $j\neq i$.

During time interval Δt, the network may either be in state (k,t) or change its state to $(k+I_{i1}-I_{j1},t+\Delta t)$, $(k-I_{i1}+I_{j1},t+\Delta t)$, $(k+I_{i2}-I_{j2},t+\Delta t)$ or $(k-I_{i2}+I_{j2},t+\Delta t)$. If during Δt the network passes to state $(k+I_{i1}-I_{j1},t+\Delta t)$ with probability $\beta_{ji}(k+I_{i1}-I_{j1},t)=\mu_{j1}(t)\varepsilon_{j1}(k_{j1})u(k_{j1})u(K_1-k_{i1})p_{ji}\Delta t+o(\Delta t)$, the income of system S_i is equal to $r^{(1)}(k+I_{i1}-I_{j1},t)$ plus the expected income $v_i(k+I_{i1}-I_{j1},t)$ of the network over the remaining time under the assumption that the initial the network state was $(k+I_{i1}-I_{j1})$, $j=\overline{1,n}$. If during Δt the network passes to state $(k+I_{i2}-I_{j2},t+\Delta t)$ with probability $\gamma_{ji}(k+I_{i2}-I_{j2},t)=$ $=\mu_{j2}(t)\varepsilon_{j2}(k_{j1},k_{j2})u(k_{j2})u(K_2-k_{i2})p_{ji}\Delta t+o(\Delta t)$, the income it is equal to $r^{(2)}(k+I_{i2}-I_{j2},t)$ plus the expected income of the network over the remaining time under the assumption that the initial network state was $(k+I_{i2}-I_{j2})$, $j\neq i$. If during Δt the network passes to state $(k-I_{i1}+I_{j1},t+\Delta t)$ with probability $\beta_{ij}(k-I_{i1}+I_{j1},t)=\mu_{i1}(t)\varepsilon_{i1}(k_{i1})u(k_{i1})u(K_1-k_{j1})p_{ij}\Delta t+o(\Delta t)$, the income of system S_i is equal to $r^{(1)}(k-I_{i1}+I_{j1},t)$ plus the expected income $v_i(k-I_{i1}+I_{j1},t)$ over the remaining time under the assumption that the initial network state was $(k-I_{i1}+I_{j1})$, $j=\overline{1,n}$. If during Δt the network passes to state $(k-I_{i2}+I_{j2},t+\Delta t)$ with probability $\gamma_{ji}(k-I_{i2}+I_{j2},t)=$ $=\mu_{i2}(t)\varepsilon_{i2}(k_{i1},k_{i2})u(k_{i2})u(K_2-k_{j2})p_{ij}\Delta t+o(\Delta t)$, the income equals $r^{(2)}(k-I_{i2}+I_{j2},t)$ plus the expected income over the remaining time under the assumption that the initial network state was $(k-I_{i2}+I_{j2})$, $j\neq i$. Similarly, if the

network remains in state $(k,t+\Delta t)$ with probability $1-\sum_{\substack{j=1,\\ j\neq i}}^{n}[(\mu_{j1}(t)\varepsilon_{j1}(k_{j1})+$
$+\mu_{j2}(t)\varepsilon_{j2}(k_{j1},k_{j2}))p_{ji}+(\mu_{i1}(t)\varepsilon_{i1}(k_{i1})+\mu_{i2}(t)\varepsilon_{i2}(k_{i1},k_{i2}))p_{ij}]\Delta t+o(\Delta t)$, the income of system S_i is equal to $r_i(k,t)\Delta t+v_i(k,t)$.

Then, using the total probability formula for conditional expectation, we can obtain a system of difference-differential equations (DDE):

$$\frac{dv_i(k,t)}{dt}=-(\sum_{\substack{j=1,\\ j\neq i}}^{n}[\mu_{j1}(t)\varepsilon_{j1}(k_{j1})+\mu_{j2}(t)\varepsilon_{j2}(k_{j1},k_{j2})p_{ji}+$$
$$+(\mu_{i1}(t)\varepsilon_{i1}(k_{i1})+\mu_{i2}(t)\varepsilon_{i2}(k_{i1},k_{i2}))p_{ij}]v_i(k,t)+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_{j1}(t)\varepsilon_{j1}(k_{j1})u(k_{j1})u(K_1-k_{i1})p_{ji}[r^{(1)}(k+I_{i1}-I_{j1},t)+v_i(k+I_{i1}-I_{j1},t)]+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_{j2}(t)\varepsilon_{j2}(k_{j1},k_{j2})u(k_{j2})u(K_2-k_{i2})p_{ji}[r^{(2)}(k+I_{i2}-I_{j2},t)+v_i(k+I_{i2}-I_{j2},t)]+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_{i1}(t)\varepsilon_{i1}(k_{i1})u(k_{i1})u(K_1-k_{j1})p_{ij}[-r^{(1)}(k-I_{i1}+I_{j1},t)+v_i(k-I_{i1}+I_{j1},t)]+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_{i2}(t)\varepsilon_{i2}(k_{i1},k_{i2})u(k_{i2})u(K_2-k_{j2})p_{ij}\times$$
$$\times[-r^{(2)}(k-I_{i2}+I_{j2},t)+v_i(k-I_{i2}+I_{j2},t)]+r_i(k,t),\ i=\overline{1,n} \tag{5}$$

System (1) in the matrix form can be rewritten as

$$\frac{dV_i(t)}{dt}=Q_i(t)+A_i(t)V_i(t),\ i=\overline{1,n} \tag{6}$$

where $V_i^T(t)=(v_i(1,t),\ldots,v_i(L,t))$ - the vector of incomes of system S_i, L - the number of network states.

The number of priority and non-priority requests do not depend on each other, so the number of states in the considered network equals the number of ways to place priority requests K_1 and non-priority requests K_2 in the service systems, that is $L=C_{n+K_1-1}^{n-1}C_{n+K_2-1}^{n-1}$.

Consider first the example of a network with a small number of states.

Example 1. Consider a closed network with the following parameters $n=2$. $K_1=3$, $K_2=2$, $m_1=m_2=1$. System (5) in this case is:

$$
\begin{cases}
\begin{aligned}
\frac{dv_1(k,t)}{dt} &= r_1(k,t)-[(\mu_{21}(t)\varepsilon_{21}(k_{21})+\mu_{22}(t)\varepsilon_{22}(k_{21},k_{22}))p_{21}+(\mu_{11}(t)\varepsilon_{11}(k_{11})+\\
&+\mu_{12}(t)\varepsilon_{12}(k_{11},k_{12})p_{12}]v_1(k,t)+\mu_{21}(t)\varepsilon_{21}(k_{21})u(k_{21})u(K_1-k_{11})p_{21}\times\\
&\times[r^{(1)}(k+I_{11}-I_{21},t)+v_1(k+I_{11}-I_{21},t)]+\mu_{22}(t)\varepsilon_{22}(k_{21},k_{22})u(k_{22})\times\\
&\times u(K_2-k_{12})p_{21}[r^{(2)}(k+I_{12}-I_{22},t)+v_1(k+I_{12}-I_{22},t)]+\mu_{11}(t)\varepsilon_{11}(k_{11})\times\\
&\times u(k_{11})u(K_1-k_{21})p_{12}[-r^{(1)}(k-I_{11}+I_{21},t)+v_1(k-I_{11}+I_{21},t)]+\mu_{12}(t)\times\\
\times\varepsilon_{12}(k_{11},k_{12})&u(k_{12})u(K_2-k_{22})p_{12}[-r^{(2)}(k-I_{12}+I_{22},t)+v_1(k-I_{12}+I_{22},t)]\\
\frac{dv_2(k,t)}{dt} &= r_2(k,t)-[(\mu_{11}(t)\varepsilon_{11}(k_{11})+\mu_{12}(t)\varepsilon_{12}(k_{11},k_{12}))p_{12}+(\mu_{21}(t)\varepsilon_{21}(k_{21})+\\
&+\mu_{22}(t)\varepsilon_{22}(k_{21},k_{22}))p_{21}]v_2(k,t)+\mu_{11}(t)\varepsilon_{11}(k_{11})u(k_{11})u(K_1-k_{21})p_{12}\times\\
&\times[r^{(1)}(k+I_{21}-I_{11},t)+v_2(k+I_{21}-I_{11},t)]+\mu_{12}(t)\varepsilon_{12}(k_{11},k_{12})u(k_{12})\times\\
&\times u(K_2-k_{22})p_{12}[r^{(2)}(k+I_{22}-I_{12},t)+v_2(k+I_{22}-I_{12},t)]+\mu_{21}(t)\varepsilon_{21}(k_{21})\times\\
&\times u(k_{21})u(K_1-k_{11})p_{21}[-r^{(1)}(k-I_{21}+I_{11},t)+v_2(k-I_{21}+I_{11},t)]+\mu_{22}(t)\times\\
\times\varepsilon_{22}(k_{21},k_{22})&u(k_{22})u(K_2-k_{12})p_{21}[-r^{(2)}(k-I_{22}+I_{12},t)+v_2(k-I_{22}+I_{12},t)].
\end{aligned}
\end{cases}
\tag{7}
$$

The initial conditions are $v_i(k,0)=0,\ i=\overline{1,2}$.

The number of network states are $L=C_{n\ K_1-1}^{n-1}C_{n\ K_2-1}^{n-1}=12$, they are (0,0;3,2), (0,1;3,1), (0,2;3,0), (1,0;2,2), (1,1;2,1), (1,2;2,0), (2,0;1,2), (2,1;1,1), (2,2;1,0), (3,0;0,2), (3,1;0,1), (3,2;0,0). We number them from 1 to 12.

Assume that the incomes from the transitions between states are linearly dependent on time. We define them by the means of matrices: $R^{(1)}(t)=\left\|r_{ij}^{(1)}(t)\right\|_{12\times12}$ and $R^{(2)}(t)=\left\|r_{ij}^{(2)}(t)\right\|_{12\times12}$. In matrix $R^{(1)}(t)$ at the intersection of line i and column j, ($i\neq j$) is the income from the transition from state i to state j for requests of the first type; if $i=j$, then on the main diagonal incomes $r_1(k,t)$. In matrix $R^{(2)}(t)$ at the intersection of line i and column j, ($i\neq j$) is the income from a transition from state i to state j for requests for the second type; if $i=j$ then, on the main diagonal incomes $r_2(k,t)$. Let incomes $r_1(k,t)$ and $r_2(k,t)$ depend linearly on time as well: $r_1(k,t)=r_1(k)t$ and $r_2(k,t)=r_2(k)t$. We define matrices $R^{(1)}(t)$ and $R^{(2)}(t)$ as:

$$R^{(1)}(t)=\begin{pmatrix}1&3&5&6&7&8&2&4&6&3&5&1\\3&3&8&1&3&6&4&5&7&9&6&1\\3&5&3&7&4&1&2&3&6&5&8&9\\1&4&5&1&6&7&8&2&0&4&2&6\\1&3&8&5&1&4&2&6&2&5&8&6\\4&5&6&9&8&4&3&6&5&8&6&2\\1&4&5&6&9&5&1&1&2&3&6&9\\7&8&2&6&3&4&6&7&7&1&2&3\\5&6&9&8&7&1&2&5&5&7&2&8\\1&2&3&4&9&8&7&6&5&1&3&1\\4&6&7&9&1&2&3&5&7&1&4&9\\9&7&6&4&3&1&1&5&3&5&7&9\end{pmatrix}t$$

$$R^{(2)}(t)=\begin{pmatrix}1&3&4&6&7&9&8&2&7&1&9&3\\3&3&6&7&9&1&5&1&6&2&3&6\\4&6&4&8&7&4&1&2&3&5&6&8\\1&2&3&1&8&7&4&5&6&9&8&5\\1&2&6&5&7&1&3&6&5&8&9&1\\2&3&6&9&7&5&2&3&6&4&7&5\\2&9&8&7&4&6&5&3&6&2&1&6\\2&8&6&4&1&2&5&9&8&7&1&3\\2&9&7&1&3&6&9&5&2&1&4&6\\1&3&6&2&6&5&4&1&3&6&9&1\\3&6&4&7&9&8&5&6&4&7&2&3\\1&3&6&9&5&2&3&6&5&4&7&4\end{pmatrix}t$$

The matrix of transition request probabilities between QS networks has the form

$$P=\|p_{ij}\|_{2\times 2}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

and let $\mu_{11}(t)=\alpha_{11}t,\ \mu_{21}(t)=\alpha_{21}t,\ \mu_{12}(t)=\alpha_{12}t,\ \mu_{22}(t)=\alpha_{22}t,$ $\alpha_{11}=11,$ $\alpha_{21}=13,\ \alpha_{12}=7,\ \alpha_{22}=9.$ Substituting the expressions for the intensities and transition probabilities between the requests in QS system (3), we obtain

$$\begin{cases} \frac{dv_1(k,t)}{dt} = r_1(k)t - \left[(13\varepsilon_{21}(k_{21}) - 9\varepsilon_{22}(k_{21},k_{22}) - 11\varepsilon_{11}(k_{11}) - 7\varepsilon_{12}(k_{11},k_{12}))\right]tv_1(k,t) + \\ \quad + 13\varepsilon_{21}(k_{21})u(k_{21})u(K_1 - k_{11})t[r^{(1)}(k + I_{11} - I_{21})t + v_1(k + I_{11} - I_{21},t)] + \\ \quad + 9\varepsilon_{22}(k_{21},k_{22})u(k_{22})u(K_2 - k_{12})t[r^{(2)}(k + I_{12} - I_{22})t + v_1(k + I_{12} - I_{22},t)] + \\ \quad + 11\varepsilon_{11}(k_{11})u(k_{11})u(K_1 - k_{21})t\left[-r^{(1)}(k - I_{11} + I_{21})t + v_1(k - I_{11} + I_{21},t)\right] + \\ + 7\varepsilon_{12}(k_{11},k_{12})u(k_{12})u(K_2 - k_{22})t[-r^{(2)}(k - I_{12} + I_{22})t + v_1(k - I_{12} + I_{22},t)] \\ \frac{dv_2(k,t)}{dt} = r_2(k)t - \left[(13\varepsilon_{21}(k_{21}) - 9\varepsilon_{22}(k_{21},k_{22}) - 11\varepsilon_{11}(k_{11}) - 7\varepsilon_{12}(k_{11},k_{12}))\right]tv_2(k,t) + \\ \quad + 11\varepsilon_{11}(k_{11})u(k_{11})u(K_1 - k_{21})t[r^{(1)}(k + I_{21} - I_{11})t + v_2(k + I_{21} - I_{11},t)] + \\ \quad + 7\varepsilon_{12}(k_{11},k_{12})u(k_{12})u(K_2 - k_{22})t[r^{(2)}(k + I_{22} - I_{12})t + v_2(k + I_{22} - I_{12},t)] + \\ \quad + 13\varepsilon_{21}(k_{21})u(k_{21})u(K_1 - k_{11})t[-r^{(1)}(k - I_{21} + I_{11})t + v_2(k - I_{21} + I_{11},t)] + \\ \quad + 9\varepsilon_{22}(k_{21},k_{22})u(k_{22})u(K_2 - k_{12})t[-r^{(2)}(k - I_{22} + I_{12})t + v_2(k - I_{22} + I_{12},t)] \end{cases} \tag{8}$$

For various states of the network, this system can be rewritten in the form of two systems of differential equations:

$$\begin{cases} \frac{dv_1(1,t)}{dt} = -20tv_1(1,t) + 13t^2v_1(4,t) + 78t^2 + t, \\ \frac{dv_1(2,t)}{dt} = -20tv_1(2,t) + 7t^2v_1(1,t) + 13t^2v_1(5,t) + 18t^2 + 3t, \\ \frac{dv_1(3,t)}{dt} = -20tv_1(3,t) + 7t^2v_1(2,t) + 13t^2v_1(6,t) + 13t^2 + 3t, \\ \frac{dv_1(4,t)}{dt} = -24tv_1(4,t) + 11t^2v_1(1,t) + 13t^2v_1(7,t) + 104t^2 + t, \\ \frac{dv_1(5,t)}{dt} = -24tv_1(5,t) + 11t^2v_1(2,t) + 13t^2v_1(8,t) + 78t^2 + t, \\ \frac{dv_1(6,t)}{dt} = -24tv_1(6,t) + 11t^2v_1(3,t) + 13t^2v_1(9,t) + 65t^2 + 4t, \\ \frac{dv_1(7,t)}{dt} = -24tv_1(7,t) + 11t^2v_1(4,t) + 13t^2v_1(10,t) + 39t^2 + t, \\ \frac{dv_1(8,t)}{dt} = -24tv_1(8,t) + 11t^2v_1(5,t) + 13t^2v_1(11,t) + 26t^2 + 7t, \end{cases}$$

$$\begin{cases} \dfrac{dv_1(9,t)}{dt} = -20tv_1(9,t) + 11t^2v_1(6,t) + 13t^2v_1(12,t) + 104t^2 + 5t, \\ \dfrac{dv_1(10,t)}{dt} = -20tv_1(10,t) + 11t^2v_1(7,t) + 9t^2v_1(11,t) + 81t^2 + t, \\ \dfrac{dv_1(11,t)}{dt} = -20tv_1(11,t) + 11t^2v_1(8,t) + 9t^2v_1(12,t) + 27t^2 + 4t, \\ \dfrac{dv_1(12,t)}{dt} = 11t^2v_1(9,t) - 3t^2 + 9t, \\ v_1(j,0) = 0,\ j = \overline{1,12}; \end{cases}$$

$$\begin{cases} \dfrac{dv_2(1,t)}{dt} = 13t^2v_2(4,t) + t, \\ \dfrac{dv_2(2,t)}{dt} = -20tv_2(2,t) + 7t^2v_2(1,t) + 13t^2v_2(5,t) + 21t^2 + 3t, \\ \dfrac{dv_2(3,t)}{dt} = -20tv_2(3,t) + 7t^2v_2(2,t) + 13t^2v_2(6,t) + 42t^2 + 4t, \\ \dfrac{dv_2(4,t)}{dt} = -24tv_2(4,t) + 11t^2v_2(1,t) + 13t^2v_2(7,t) + 13t^2 + t, \\ \dfrac{dv_2(5,t)}{dt} = -24tv_2(5,t) + 11t^2v_2(2,t) + 13t^2v_2(8,t) + 33t^2 + 7t, \\ \dfrac{dv_2(6,t)}{dt} = -24tv_2(6,t) + 11t^2v_2(3,t) + 13t^2v_2(9,t) + 66t^2 + 5t, \\ \dfrac{dv_2(7,t)}{dt} = -24tv_2(7,t) + 11t^2v_2(4,t) + 13t^2v_2(10,t) + 66t^2 + 5t, \\ \dfrac{dv_2(8,t)}{dt} = -24tv_2(8,t) + 11t^2v_2(5,t) + 13t^2v_2(11,t) + 33t^2 + 9t, \\ \dfrac{dv_2(9,t)}{dt} = -24tv_2(9,t) + 11t^2v_2(6,t) + 13t^2v_2(12,t) + 22t^2 + 2t, \\ \dfrac{dv_2(10,t)}{dt} = -18tv_2(10,t) + 11t^2v_2(7,t) + 9t^2v_2(11,t) + 77t^2 + 6t, \\ \dfrac{dv_2(11,t)}{dt} = -18tv_2(11,t) + 11t^2v_2(8,t) + 9t^2v_2(12,t) + 77t^2 + 2t, \\ \dfrac{dv_2(12,t)}{dt} = -11tv_2(12,t) - 11t^2v_2(9,t) + 33t^2 + 4t, \\ v_2(j,0) = 0,\ j = \overline{1,12}. \end{cases}$$

We solve this system by means of difference schemes, for each node makes up a system of linear algebraic equations

$$\begin{cases} V_i(t_{j-1}) = V_i(t_j) + Q_i(t_j)h + A_i(t_j)V_i(t_j)h, \\ t_j = t_0 + jh, \ t_0 = 0, \ h = \dfrac{T}{N}, \ i = \overline{1,n}. \end{cases}$$

where $V_i^T(t) = (v_i(1,t),\ldots,v_i(12,t))$, $n=2$, $h=0.01$, $T=30$, $N=3000$, deciding to obtain the values of incomes at these points. Figure 1 shows a graph of the income changing of system S_1 if the initial state of the network was k = (3,0;0,2).

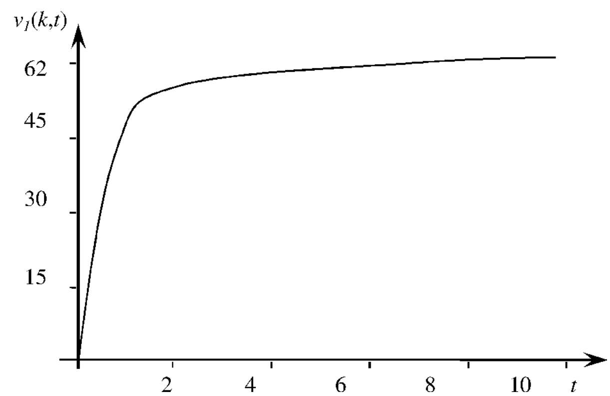

Fig. 1. Income of system S_1 for initial state of network k = (3,0;0,2)

Vectors $Q_1(t)$, $Q_2(t)$ and matrices $A_1(t)$ and $A_2(t)$ have a greater dimension equal to the number of states of a network. Their calculation takes a long time due to the fact that it is necessary to keep in memory a large number of elements (especially networks of a large dimension). Therefore, an algorithm for finding incomes significantly speeds up the process if one does not receive an explicit matrix in (2) and does not store any intermediate information, and immediately gets the final result for the expected incomes.

Example 2. Consider a closed network with the following parameters $n=20$, $K_1=60$, $K_2=40$, $m=(2,1,1,3,1,4,1,5,3,2,4,3,2,1,3,2,1,3,4,2)$, where component i of the vector is the number of service lines of QS number i, $i=\overline{1,20}$. Let the matrix of transitions probabilities of requests between QS networks:

$$P=\begin{pmatrix}
0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{7} & 0 & 0 & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

The intensity of service is set by such functions as $\mu_{i1}(t)=15t$, $\mu_{i2}(t)=10t$, if modulo i to 3 is 1; $\mu_{i1}(t)=13t$, $\mu_{i2}(t)=18t$, if modulo i to 3 is 2; $\mu_{i1}(t)=17t$, $\mu_{i2}(t)=16t$, if modulo i to 3 is 0.

Assume that the incomes from the transitions between the states of the network are linear functions of time. In this case we define the coefficients of random sofware with built-in functions Delphi 7.0, which are based on a linear congruent random number generator. It is necessary to compute the members of a linear recurring sequence modulo with a positive integer m given by the following formula:

$$X_{k+1}=(aX_k+c)\bmod m$$

where a and c - the integer coefficients. The resulting sequence depends on the choice of starting X_0 and its different values are obtained by different sequences of random numbers.

We present the systems of differential equations for the expected income of some QS:

$$\frac{dv_2(k,t)}{dt}=r_2(k)t-\left[(15\varepsilon_{11}(k_{11})-10\varepsilon_{12}(k_{11},k_{12})-13\varepsilon_{21}(k_{21})-18\varepsilon_{22}(k_{21},k_{22}))\right]v_2(k,t)+$$
$$+15\varepsilon_{11}(k_{11})u(k_{11})u(K_1-k_{21})t[r^{(1)}(k+I_{21}-I_{11})t+v_2(k+I_{21}-I_{11},t)]+$$
$$+10\varepsilon_{12}(k_{11},k_{12})u(k_{12})u(K_2-k_{22})t[r^{(2)}(k+I_{22}-I_{12})t+v_2(k+I_{22}-I_{12},t)]+$$
$$+13\varepsilon_{21}(k_{21})u(k_{21})u(K_1-k_{11})t[-r^{(1)}(k-I_{21}+I_{11})t+v_2(k-I_{21}+I_{11},t)]+$$
$$+18\varepsilon_{22}(k_{21},k_{22})u(k_{22})u(K_2-k_{12})t[-r^{(2)}(k-I_{22}+I_{12})t+v_2(k-I_{22}+I_{12},t)],$$

$$\frac{dv_6(k,t)}{dt}=r_6(k)t-\left[(15\varepsilon_{51}(k_{51})-10\varepsilon_{52}(k_{51},k_{52})-17\varepsilon_{61}(k_{61})-16\varepsilon_{62}(k_{61},k_{62}))\right]v_6(k,t)+$$
$$+15\varepsilon_{51}(k_{51})u(k_{51})u(K_1-k_{61})t[r^{(1)}(k+I_{61}-I_{51})t+v_6(k+I_{61}-I_{51},t)]+$$
$$+10\varepsilon_{52}(k_{51},k_{52})u(k_{52})u(K_2-k_{62})t[r^{(2)}(k+I_{62}-I_{52})t+v_6(k+I_{62}-I_{52},t)]+$$
$$+17\varepsilon_{61}(k_{61})u(k_{61})u(K_1-k_{51})t[-r^{(1)}(k-I_{61}+I_{51})t+v_6(k-I_{61}+I_{51},t)]+$$
$$+16\varepsilon_{22}(k_{61},k_{62})u(k_{62})u(K_2-k_{52})t[-r^{(2)}(k-I_{62}+I_{52})t+v_6(k-I_{62}+I_{52},t)]$$

$$\frac{dv_{14}(k,t)}{dt}=r_{12}(k)t-\left[(15\varepsilon_{11,1}(k_{11,1})-10\varepsilon_{11,2}(k_{11,1},k_{11,2})-\right.$$
$$\left.-13\varepsilon_{14,1}(k_{14,1})-18\varepsilon_{14,2}(k_{14,1},k_{14,2}))\right]v_{14}(k,t)+$$
$$+15\varepsilon_{11,1}(k_{11,1})u(k_{11,1})u(K_1-k_{14,1})t[r^{(1)}(k+I_{14,1}-I_{11,1})t+v_{14}(k+I_{14,1}-I_{11,1})]+$$
$$+10\varepsilon_{11,2}(k_{11,1},k_{11,2})u(k_{11,2})u(K_2-k_{14,2})t[r^{(2)}(k+I_{14,2}-I_{11,2})t+v_{14}(k+I_{14,2}-I_{11,2})]+$$
$$+13\varepsilon_{14,1}(k_{14,1})u(k_{14,1})u(K_1-k_{11,1})t[-r^{(1)}(k-I_{14,1}+I_{11,1})t+v_{14}(k-I_{14,1}+I_{11,1},t)]+$$
$$+18\varepsilon_{22}(k_{21},k_{22})u(k_{22})u(K_2-k_{11,2})t[-r^{(2)}(k-I_{14,2}+I_{11,2})t+v_{14}(k-I_{14,2}+I_{11,2},t)]$$

The initial conditions are given in the form $v_i(k,0)=0,\ i=\overline{1,20}$.

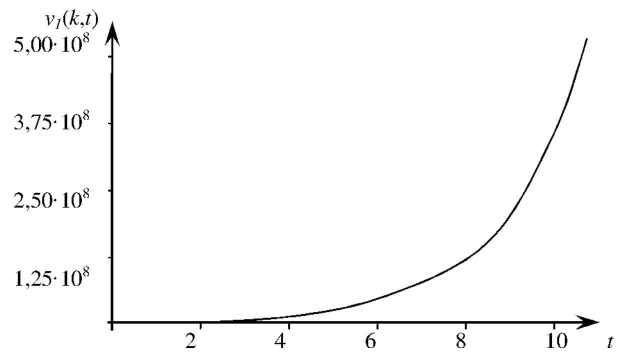

Fig. 2. Income of system S_1 for initial state of network
$k=(3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2;3.2)$

The number of states of the network is equal to $L = C_{n-K_1-1}^{n-1} C_{n+K_2-1}^{n-1} =$ $= 1234957962251465078613775579490700$.

These systems are solved by means of difference schemes. The schedule of income in the first system for initial state $k = (3,2;3,2;3,2;3,2;3,2;3,2;$ $3,2;3,2;3,2;3,2;3,2;3,2;3,2;3,2;3,2;3,2;3,2;3,2;3,2;3,2)$ is shown in Figure 2.

References

[1] Matalytski M., Pankov A., Incomes probabilistic models of the banking network, Scientific Research of the Institute of Mathematics and Computer Science 2003, 1(2), 99-104.

[2] Matalytski M., Pankov A., Probability analysis of incomes in banking networks, Vestnik BSU, Minsk, Belorussian University 2004, 2, 86-91 (in Russian).

[3] Matalytski M., On some results in analysis of optimization of Markov networks with incomes and their applications, Automation and Remote Control 2009, 70(10), 1683-1697.

[4] Matalytski M., Queueing Networks in Stationary and Transient Regime, Grodno 2001 (in Russian).