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ON AXIOMS OF CONVERGENCE IN LINEAR SPACES

Abstract. By a (general) convergence in a given linear space $X$ we mean a mapping $G: X^N \to 2^X$, where $N$ denotes the set of all positive integers, and by a zero-convergence in $X$ we mean a convergence $G_0$ in $X$ for which $G_0(x) \neq \emptyset$ implies $0 \in G_0(x)$ for each $x \in X^N$. In the paper, the two operations are defined: 1° operation $C$, which to each zero-convergence $G_0$ in $X$ assigns some general convergence $G$ in $X$, and 2° operation $C_0$, which to each general convergence $G$ in $X$ assigns a zero-convergence $G_0$ in $X$. Various systems of axioms for general convergences and zero-convergences are considered and their connections with the operations $C$ and $C_0$ are studied. Also mutual independence of axioms is studied.

Very often convergences are defined by topology. However there exist important convergences which cannot be defined in this way, e.g., type I and type II convergences in the Mikusiński operational calculus (see [9], [2], [3]).

These and other examples show the need of development of a general theory, in which convergence in a given space is defined immediately by indicating convergent sequences and their limits and some general conditions (axioms) are supposed (see e.g. [10]).

Of course, such a convergence can be treated as a function, which to every sequence assigns a set of limits (the empty set if a sequence is divergent; a one-element set if a convergent sequence has a unit limit).

In particular, topological convergence (i.e. convergence defined by some topology) can be characterized in terms of conditions mentioned above. For Hausdorff convergences it is done in [7] and [8] and for multivalued convergences (i.e. without the assumption of uniqueness) in the paper [4] (see also [5] and [6]).

One can consider convergences in spaces equipped with some algebraic structure, e.g., in groups or in linear spaces (see e.g. [11]). In linear...
spaces only the sequences convergent to 0 are usually defined (zero-convergence). All the convergent sequences (general convergence) can be then defined by linearity.

In this note we present a general scheme how to pass in linear spaces from the definition of sequences convergent to 0 to the definition of sequences convergent to arbitrary elements and conversely.

More precisely, we first introduce axiomatically two kinds of convergence in a given linear space: zero-convergence and general convergence. Next we define an operation $C$ assigning to every zero-convergence a general convergence and an operation $C_0$, which makes correspond to every general convergence a zero-convergence (section 1). We study what axioms are preserved when operations $C$ and $C_0$ are performed (section 2).

In turn, we discuss relations between operations $C$ and $C_0$ (section 3). In particular, we find conditions, under which the identities $CC_0G = G$ and $C_0CG_0 = G_0$ hold for any general convergence $G$ and zero-convergence $G_0$.

Finally, we discuss independence of axioms (section 4).

1. We shall denote: by $N$ — the set of all positive integers, by $R$ — the set of all real numbers, by $X$ — an arbitrary fixed set, by $E$ — a fixed linear space over the field $R$, by Greek letters $\xi, \eta, \ldots$ — elements of $X$ or $E$, by Latin letters $x, y, \ldots$ — elements of $X^N$ or $E^N$, i.e. sequences $\{x_n\}, \{y_n\}, \ldots$ of elements of $X$ or $E$ respectively.

If $y$ is a subsequence of a sequence $x$, then we shall write $y \sim x$; the constant sequence $\xi, \xi, \xi, \ldots$, where $\xi \in X$ (or $\xi \in E$), will be denoted by $\xi$ and the set $\{\xi_n : n \in N\}$ for $x = \{x_n\}$ — by $(x)$ (cf. notation in [1]).

If $A, B \subseteq E$ and $\lambda \in R$, then we shall use the standard notation: $A + B = \{x + y : x \in A, y \in B\}$, $\lambda A = \{\lambda x : x \in A\}$ and the convention: $A + 0 = 0 + A = 0$, $\lambda 0 = 0$.

By a general convergence (shortly: convergence) on a given set $X$, we mean a mapping from $X^N$ into $2^X$ (cf. [1]).

Let $G$ and $G'$ be two convergences on $X$. We write $G \subseteq G'$ if $G(x) \subseteq G'(x)$ for every $x \in X^N$. If $G \subseteq G'$ and $G' \subseteq G$, then we write $G = G'$.

The following axioms concerning a convergence $G$ on $X$ were considered in [7], [8], [1], [9], [4]—[6]:

F. If $y \sim x$, then $G(x) \subseteq G(y)$;

U. If $\xi \notin G(x)$, then there exists $y \sim x$ such that $\xi \notin G(z)$ for every $z \sim y$;

H. For every $x \in X^N$ the set $G(x)$ contains at most one element;

S. $\xi \in G(\xi)$ for every $\xi \in X$.

It is convenient to consider the following axiom, complementary with respect to axioms S and H:

S'. If $\eta \in G(\xi)$, then $\xi = \eta$. 
In $E$, it is natural to consider for general convergences besides $F$, $U$, $H$, $S$, $S'$ also the following axioms of linearity (cf. [10]):

A. $G(x) + G(y) \subseteq G(x + y)$, $(x, y \in E^N)$;

M. $\lambda G(x) \subseteq G(\lambda x)$, $(x \in E^N, \lambda \in \mathbb{R})$

or the following weaker versions of linearity:

T. If $\xi \in G(x)$, then $0 \in (x - \xi)$, $(x \in E^N)$;

T'. If $0 \in G(x - \xi)$, then $\xi \in G(x)$, $(x \in E^N)$.

A (general) convergence $G$ on a linear space $E$ will be called a zero-convergence if

$$G_0(x) \neq \emptyset \implies 0 \in G_0(x), (x \in E^N).$$

For zero-convergences, we consider the above axioms, too. However it seems to be more natural for those convergences to replace axioms $S$ and $S'$ by the weaker ones:

$S_0$. $0 \in G_0(0)$ or, equivalently, $G_0(0) \neq \emptyset$.

$S'_0$. If $G_0(\xi) \neq \emptyset$ (i.e. $0 \in G_0(\xi)$) for $\xi \in X$, then $\xi = 0$.

Relations between the above axioms will be studied later.

Now, we are going to introduce the following operations: $1^\circ$ operation $C$ assigning a general convergence $G$ on $E$ to every zero-convergence $G_0$ on $E$; $2^\circ$ operation $C_0$ assigning a zero-convergence $G_0$ on $E$ to every general convergence $G$ on $E$.

Namely, for a given zero-convergence $G_0$ on $E$ we define the general convergence $CG_0 = G$ as follows:

$$\xi \in G(x) \iff 0 \in G_0(x - \xi), (x \in E^N).$$

For a given general convergence $G$ on $E$ we construct the zero-convergence $C_0G = G_0$ on $E$ in the following way:

$$G_0(x) = G(x) \text{ if } 0 \in G(x) \text{ and } G(x) = \emptyset \text{ otherwise.}$$

Note that the above definition of the operation $C$ is based on linearity, but the definition of $C_0$ is not. Therefore it seems to be reasonable, for a given general convergence $G$, to treat a sequence $x$ as convergent to $0$ in the sense of a zero-convergence whenever for some $\eta \in R$ we have $\eta \in G(x + \eta)$. Accordingly, we define the second version of operation of type $2^\circ$ in the following way:

$$(1.1) \quad C_0G(x) = G_0(x) = \bigcup \{G(x + \eta) - \eta\}, (x \in E^N)$$

for the given general convergence $G$, where the union is taken over such $\eta \in X$ that $\eta \in G(x + \eta)$; if for some $x \in E^N$ such $\eta$ does not exist, then we adopt $G_0(x) = \emptyset$.

It is obvious that $G_0$ is a zero-convergence on $E$.

**PROPOSITION 1.1.** For every general convergence $G$, we have

$$(1.2) \quad C_0G \subseteq C_0G.$$
If \( G \) satisfies axioms \( S \) and \( A \), then \( C_0G = \overline{C_0G} \).

Proof. Let \( x \) be arbitrary. If \( \xi \in C_0G(x) \), then \( \xi, 0 \in C_0G(x) = G(x) \) and, by (1.1),

\[
\xi \in G(x+0) - 0 \subseteq C_0G(x),
\]

i.e., (1.2) holds.

Now, let \( \xi \in C_0G(x) \). Then there exists \( \eta \in X \) such that \( \eta, \xi + \eta \in G(x+\hat{\eta}) \). Hence, by \( S \) and \( A \), we have

\[
0 = \eta - \eta \in G(x+\hat{\eta}) + G(-\hat{\eta}) \subseteq G(x)
\]

and

\[
\xi = \xi + \eta - \eta \in G(x+\hat{\eta}) + G(-\hat{\eta}) \subseteq G(x).
\]

But this means that \( \xi \in G(x) = C_0G(x) \) and thus the second inclusion of (1.3) is true under \( S \) and \( A \).

EXAMPLE 1.1. Let \( E = \mathbb{R} \). We define \( G(x) = \emptyset \) if the sequence \( x = \{\xi_n\} \) is not convergent in the usual sense and \( G(x) = \{-\xi\} \) if \( \lim_{n \to \infty} \xi_n = \xi \) in the usual sense.

Note that \( G \) fulfills \( A \) (and \( U \), which will be needed later), but does not fulfill \( S \).

If \( \xi_n \to \xi \neq 0 \), we have \( C_0G(x) = \emptyset \). On the other hand, it is easy to see that

\[
\overline{C_0G(x)} = G \left( x - \frac{\xi}{2} \right) + \frac{\xi}{2} = \{0\},
\]

i.e., (1.3) does not hold.

EXAMPLE 1.2. Let \( E = \mathbb{R} \) and \( x = \{\xi_n\} \). If

\( \xi_n \searrow \xi > 0 \), i.e., \( \xi_n \to \xi \) and \( \xi_n \geq \xi \) for almost all \( n \),

or if

\( \xi_n \nearrow \xi < 0 \), i.e. \( \xi_n \to \xi \) and \( \xi_n \leq \xi \) for almost all \( n \),

then we adopt \( G(x) = \{\xi\} \). Moreover \( G(x) = \{0\} \) if \( x = \{\xi_n\} \) and \( \xi_n = 0 \) for almost all \( n \). In the remaining cases, we put \( G(x) = \emptyset \).

Note that \( G \) satisfies \( S \) (and \( U \), which will be needed later) but does not satisfy \( A \).

For \( x = \left\{ \frac{1}{n} \right\} \) and \( x' = \left\{ -\frac{1}{n} \right\} \) we have

\[
C_0G(x) = C_0G(x') = \emptyset
\]

and

\[
C_0G(x) = \bigcup_{n > 0} \left[ G \left( \left\{ n + \frac{1}{n} \right\} \right) - \eta \right] = \{0\},
\]
\[ C_0 G(x') = \bigcup_{\eta > 0} \left[ G\left( \left\{ \eta - \frac{1}{n} \right\} \right) - \eta \right] = \{0\}, \]

i.e., (1.3) does not hold.

2. In this section we study if individual axioms are preserved under the operations \( C, C_0 \) and \( C_0 \).

**Proposition 2.1.** If a zero-convergence \( G_0 \) satisfies \( F \), then the general convergence \( G = C G_0 \) satisfies \( F \). If a general convergence \( G \) satisfies \( F \), then the zero-convergences. \( G_0 = C_0 G \) and \( G_0 = C_0 G \) satisfy \( F \).

**Proof.** Suppose that \( G_0 \) satisfies \( F \) and let \( y \not\approx x \) and \( \xi \in G(x) \). By the definition of \( G \), we have \( 0 \in G_0(x-\xi) \). Since \( G_0 \) fulfills \( F \), we have \( 0 \in G_0(y-\xi) \) and hence \( \xi \in G(y) \). Thus we have proved that \( G(x) \subseteq G(y) \), i.e., \( G \) satisfies \( F \).

Suppose now that \( G \) fulfills \( F \) and let \( y \not\approx x \). If \( 0 \not\in G(y) \), then \( 0 \not\in G(x) \), by \( F \). Hence \( G_0(x) = \emptyset \subseteq G_0(y) = \emptyset \). If \( 0 \in G(y) \), then \( G_0(y) = G(y) \supseteq G(x) \supseteq G_0(x) \), because \( F \) holds for \( G \). Thus we have proved that \( G_0 \) fulfills \( F \).

Since \( F \) is assumed for \( G \), we have

\[
G(x+\eta) \subseteq G(y+\eta)
\]

and

\[
G(x+\eta)-\eta \subseteq G(y+\eta)-\eta
\]

for each \( \eta \in E \). In view of the definition of \( G_0 \), inclusions (2.1) and (2.2) yield \( G_0(x) \subseteq G_0(y) \), i.e., \( G_0 \) fulfills \( F \) and the proof is finished.

**Proposition 2.2.** If a zero-convergence \( G_0 \) satisfies \( U \), then the general convergence \( G = CG_0 \) satisfies \( U \). If a general convergence \( G \) satisfies \( U \), then the zero-convergence \( G_0 = C_0 G \) satisfies \( U \). If a general convergence \( G \) satisfies \( U \) and \( A \), then the zero-convergence \( G_0 = C_0 G \) satisfies \( U \).

**Proof.** Suppose \( U \) for \( G_0 \) and let \( \xi \not\in G(x) \), i.e., \( 0 \not\in G_0(x-\xi) \). Then there exists \( y \not\approx x \) such that \( 0 \not\in G_0(z-\xi) \) for each \( z \not\approx y \). This means, \( \xi \not\in G(z) \) for each \( z \not\approx y \), i.e., \( G \) fulfills \( U \).

Suppose now that \( G \) fulfills \( U \) and let for each subsequence \( y \) of a given sequence \( x \) exist \( z \not\approx y \) such that \( \xi \in G_0(z) \). But then \( G_0(z) = G(z) \) and \( 0, \xi \in G(z) \). Hence, by \( U \), we have \( 0, \xi \in G(x) = G_0(x) \). This proves the second part of the proposition.

The last part follows from the second one, by Proposition 1.1.

The assumption in the third part of Proposition 2.2 that \( G \) satisfies axioms \( S \) and \( A \) cannot be omitted.

In fact, the convergence \( G \) from Example 1.1 fulfills \( U \) and \( A \), but does not fulfill \( S \). We have in this case \( G_0(x) = C_0 G(x) = \{0\} \), provided there is a \( \xi \in R \) such that \( \xi_n \to \xi \), and \( G_0(x) = \emptyset \) otherwise. The zero-
-convergence $C_0$ does not fulfill $U$, because $0 \not\in C_0(x)$ for $x = (1, -1, 1, -1, ...)$, but for each subsequence $y$ of $x$ there exists subsequence $z$ of $y$, which is of the form $z = (1, 1, ...)$ or $z = (-1, -1, ...)$, i.e., $0 \in C_0(z)$.

On the other hand, the convergence $G$ from Example 1.2 satisfies $U$ and $S$, but does not fulfill $A$. The zero-convergence $C_0 = C_0G$ does not fulfill axiom $U$, because $0 \not\in C_0(x)$ for $x = \left(1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, ...ight)$, but for each subsequence $y$ of $x$ there exists a subsequence $z$ of $y$, which is either a subsequence of the sequence $\left(1, \frac{1}{2}, \frac{1}{3}, ...ight)$ or of the sequence $\left(-1, -\frac{1}{2}, -\frac{1}{3}, ...ight)$, i.e., $0 \in C_0(z)$.

Now, note that axiom $H$ is not preserved in general when the operation $C$ is applied, as examples below will show. However we have the following statement.

**Proposition 2.3.** If a zero-convergence $G_0$ satisfies axioms $A$, $M$ and $S'_0$, then the general convergence $G = CG_0$ satisfies $H$. If a general convergence satisfies $H$, then the zero-convergences $G_0 = C_0G$ and $G_0 = C_0G$ satisfy $H$.

**Proof.** Suppose that a zero-convergence $G_0$ fulfills $A$, $M$ and $S'_0$, and let $\xi, \eta \in G(x)$. By the definition of $G$, we have $0 \in G_0(x-\xi)$ and $0 \in G_0(x-\eta)$. Hence, by $M$ and $A$, we get

$$0 \in G_0(x-\xi) + G(\xi-x) \subseteq G(\xi-\eta)$$

whence $\xi = \eta$ results, by virtue of $S'_0$. The first part of the proposition is shown.

Assume now that a general convergence $G$ fulfills $H$. If $\xi \in G_0(x)$, then $0, \xi \in G(x) = G_0(x)$ and, since $G$ satisfies $H$, we get $\xi = 0$, i.e., the zero-convergence $G_0$ satisfies $H$. If $\xi \in G_0(x)$, then by the definition of $G$, there exists $\eta \in E$ such that $\eta, \xi + \eta \in G(x+\eta)$ and hence $\eta = \xi + \eta$, i.e., $\xi = 0$. Thus the zero-convergence $G_0$ satisfies $H$ too and the proof is completed.

**Example 2.1.** Let $E = R$ and let $G_0(x) = \{0\}$ if $x = \xi$ for $\xi \in R$ and $G_0(x) = \emptyset$ otherwise. Of course, the zero-convergence $G$ fulfills $H$, $A$, $M$, and axiom $S'_0$ is not fulfilled. The general convergence $G = CG_0$ does not satisfy $H$, because $G(\xi) = R$ for every $\xi \in R$.

**Example 2.2.** Let $E = R$. Define the zero-convergence $G_0$ as follows: $G_0(0) = \{0\}$, $G_0(\{\xi + a_n\}) = \{0\}$ for any $\xi \in R$ and $a_n \to 0$ with $a_n \not\in \emptyset (n \in N)$; in remain cases let $G_0(x) = \emptyset$.

Evidently, the zero-convergence $G_0$ fulfils axioms $H$, $M$, $S'_0$. Axiom $A$ is not satisfied, because

$$G_0\left(\left\{\xi + \frac{1}{n}\right\}\right) + G_0\left(\left\{\eta - \frac{1}{n}\right\}\right) = \{0\}$$

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and

$$G_0(\xi + \eta) = \emptyset$$

for any $\xi, \eta \in R$ such that $\xi + \eta \neq 0$. Note that $G\left(\left\{ \xi + \frac{1}{n} \right\} \right) = R$, i.e., axiom H does not hold for $G = CG_0$.

**EXAMPLE 2.3.** Let $E = R$. We adopt $G_0(x) = \{0\}$ if $x = 0$ or if $x = \{\xi_n\}$, where $\xi_n \to \infty$; otherwise let $G_0(x) = \emptyset$.

It is clear that the zero-convergence $G_0$ satisfies axioms H, A and $S_0$, and does not fulfill axiom M. Let us note that $G(x) = CG_0(x) = R$, i.e. the general convergence $G$ has not property H.

**PROPOSITION 2.4.** If a zero-convergence $G_0$ satisfies $S_0$, then the general convergence $G = CG_0$ satisfies $S$. If a general convergence $G$ satisfies $S$, then the zero-convergences $G_0 = C_0G$ and $G_0 = \tilde{C}_0G$ satisfy $S_0$.

**Proof.** The first part follows from the fact that the condition $\xi \in G(\xi)$ is equivalent, by the definition of $G$, to the condition $0 \in G_0(0)$.

If $G$ satisfies $S$, then we have in particular $0 \in G(0) = G_0(0)$, that means $G_0$ satisfies $S_0$. On the other hand, we have then $\eta \in G(\eta)$ and $0 \in G(\eta) - \eta$ for every $\eta \in E$, i.e., $0 \in G_0(0)$.

**PROPOSITION 2.5.** If a zero-convergence $G_0$ satisfies $S'_0$, then the general convergence $G = CG_0$ satisfies $S'$. If a general convergence $G$ satisfies $S'$, then the zero-convergences $G_0 = C_0G$ and $G_0 = \tilde{C}_0G$ satisfy $S'_0$.

**Proof.** Assume that $G_0$ fulfills $S'_0$ and let $\eta \in G(\xi) = CG_0(\xi)$. This means $0 \in G_0(\xi - \eta)$ and thus $\xi = \eta$, by $S'_0$, which shows the first part of our assertion.

Now, let $G$ fulfill $S'$ and let $0 \in G_0(\xi) = C_0G(\xi)$. This means $0 \in G(\xi) = G_0(\xi)$ and $\xi = 0$, by $S'$.

In turn, if $0 \in G_0(\xi) = C_0G(\xi)$, then by the definition of $G_0$ there exists $\eta \in E$ such that $\eta \in G(\xi + \eta)$. Hence, by virtue of $S'$, we get $\xi + \eta = 0$, which yields the desired assertion.

**PROPOSITION 2.6.** If a zero-convergence $G_0$ satisfies $A$, then the general convergence $G = CG_0$ satisfies $A$. If a general convergence $G$ satisfies $A$, then the zero-convergences $G_0 = C_0G$ and $G_0 = \tilde{C}_0G$ satisfy $A$.

**Proof.** Assume that a zero-convergence $G_0$ fulfills $A$ and that $\xi \in G(x) + G(y)$, i.e. $\xi = \xi + \eta$ with $\xi \in G(x)$ and $\eta \in G(y)$. By the definition of $G$, we have $0 \in G_0(x - \xi)$ and $0 \in G_0(y - \eta)$. Since $A$ holds for $G_0$, we obtain

$$G_0(x - \xi) + G_0(y - \eta) \subseteq G(x + y - (\xi + \eta)),$$

i.e.,

$$\xi + \eta \in G(x + y),$$

which completes the proof of the first part.
Suppose now that a general convergence \( G \) fulfils A. If \( 0 \not\in G(x) \) or \( 0 \not\in G(y) \), then \( G_0(x) = \emptyset \) or \( G_0(y) = \emptyset \) respectively and, consequently,
\[
G_0(x) + G_0(y) = \emptyset \subseteq G_0(x+y).
\]
If \( 0 \in G(x) \) and \( 0 \in G(y) \), then
\[
0 \in G(x) + G(y) \subseteq G(x+y),
\]
since A holds for \( G \), and thus
\[
G_0(x) + G_0(y) = G(x) + G(y) \subseteq G(x+y) = G_0(x+y).
\]

To prove the last assertion suppose that
\[
(2.3) \quad \eta_1 \in G(x + i_1) \quad \text{and} \quad \eta_2 \in G(x + i_2).
\]
Of course we have
\[
\eta_1 + \eta_2 \in G(x + i_1) + G(y + i_2) \subseteq G(x+y + i_1 + i_2)
\]
and
\[
[G(x + i_1) - \eta_1] + [G(y + i_2) - \eta_2] \subseteq G(x+y + i_1 + i_2) - (\eta_1 + \eta_2),
\]
because A is satisfied by \( G \). From this results the inclusion
\[
G_0(x) + G_0(y) \subseteq G_0(x+y)
\]
in the case, when there exist \( \eta_1, \eta_2 \in E \) satisfying (2.3).

In the opposite case we have
\[
G_0(x) + G_0(y) = \emptyset \subseteq G_0(x+y).
\]
Thus the proof is complete.

**Proposition 2.7.** If a zero-convergence \( G_0 \) satisfies M, then the general convergence \( G = CG_0 \) satisfies M. If a general convergence satisfies M, then the zero-convergences \( G_0 = C_0G \) and \( G_0 = C_0G \) satisfy M.

**Proof.** Assume that \( G_0 \) fulfils M and that \( \zeta \in \lambda G(x) \) for \( G = CG_0, \lambda \in \mathbb{R} \) and \( x \in E^N \). That means, we have \( \zeta = \lambda \xi \) with \( \xi \in G(x) \). Hence \( 0 \in G_0(x - \xi) \) and
\[
0 \in G_0(x - \xi) = \emptyset \subseteq G_0(\lambda(x - \xi)).
\]
This yields, by the definition of \( G \), the relation \( \zeta = \lambda \xi \in G(\lambda x) \), which finishes the proof of the first part of the proposition.

In turn, assume that a general convergence \( G \) fulfils axiom M. If \( 0 \not\in G(x) \), then
\[
\lambda G_0(x) = \emptyset \subseteq G_0(\lambda x)
\]
for every \( \lambda \in \mathbb{R} \) and \( G_0 = C_0G \).

If \( 0 \in G(x) \), then \( 0 \in \lambda G(x) \subseteq G(\lambda x) \) and we get
\[
\lambda G_0(x) = \lambda G(x) \subseteq G(\lambda x) = G_0(\lambda x),
\]
owing to M holding for \( G \).
It remains to prove that $G_0 = C_0G$ fulfils M. If
\[(2.4) \quad \eta \in G(x + \hat{\eta}),\]
then we have, by virtue of M,
\[\mu = \lambda \eta \in \lambda G(x + \hat{\eta}) \subseteq G(\lambda x + \hat{\mu})\]
and
\[\lambda[G(x + \hat{\eta}) - \eta] \subseteq G(\lambda x + \hat{\mu}) - \mu\]
for any $\lambda \in R$. This yields
\[\lambda G_0(x) \subseteq G_0(\lambda x), \quad (\lambda \in R)\]
in the case, when there exists $\eta \in \mathbb{E}$ such that (2.4) holds.

In the converse case, we have
\[G_0(x) = \emptyset \subseteq G_0(\lambda x), \quad (\lambda \in R)\]
and the proof is over.

**PROPOSITION 2.8.** The general convergence $G = C_0G$ satisfies axioms T and $T'$ for every zero-convergence $G_0$. If a general convergence $G$ satisfies T, then the zero-convergences $G_0 = C_0G$ and $G_0 = C_0G$ satisfy T.

**Proof.** The first assertion follows from the following equivalences:
\[\xi \in G(x) \iff G_0(x - \hat{\xi}) \iff 0 \in G(x - \hat{\xi}),\]
which are consequences of the definition of $G$.

To prove the second one suppose that $G$ satisfies T. First let $\xi \in G_0(x) = C_0G(x)$. Then $\xi, 0 \in G_0(x) = G(x)$ and, by T, we obtain $0 \in G(x - \hat{\xi})$, i.e.,
\[0 \in G_0(x - \hat{\xi}) = G(x - \hat{\xi}).\]

Finally note that the usual convergence $G$ in $\mathbb{R}$ satisfies axiom $T'$ (and all others) and the zero-convergences $G_0 = C_0G$ and $G_0 = C_0G$ (which coincide with the usual convergence to 0 in $\mathbb{R}$) do not fulfil it.

3. Now we are going to study connections between the operations $C, C_0$ and $C_0$.

**PROPOSITION 3.1.** If $G, G'$ are general convergences on $\mathbb{E}$ and $G \subseteq G'$ then $C_0G \subseteq C_0G'$ and $C_0G \subseteq C_0G'$.

**Proof.** First let $\xi \in C_0G(x)$. By the definition of $C_0$ this means that $\eta \in G(x)$ and $0 \in G(x)$. Hence, by the assumption, $\xi, 0 \in G'(x)$ and, consequently, $\xi \in C_0G'(x) = G'(x)$.

Now let $\xi \in C_0G(x)$. This means that there exists an $\eta \in \mathbb{E}$ such that $\eta, \xi + \eta \in G(x + \hat{\eta})$. By the assumption we have $\eta, \xi + \eta \in G'(x + \hat{\eta})$ and thus $\xi = \xi + \eta - \eta \in C_0G'(x)$. The proof is complete.

**PROPOSITION 3.2.** If $G_0, G'_0$ are zero-convergences and $G_0 \subseteq G'_0$, then $CG_0 \subseteq CG'_0$.

**Proof.** If $\xi \in CG_0(x)$, then $0 \in G_0(x - \hat{\xi})$ and, by the assumption, $0 \in G'_0(x - \hat{\xi})$. But this means that $\xi \in CG'_0(x)$, which completes the proof.
PROPOSITION 3.3. If a general convergence $G$ on $E$ satisfies $S$ and $A$, then

$$CC_0G = G$$

and

$$CC_0G = G.$$  

Proof. In view of Proposition 1.1, it suffices to prove (3.1). Let $\xi \in G(x)$. By $S$, we have

$$0 = \xi - \xi \in G(x) + G(-\xi).$$

Hence, by $A$, we get $0 \in G(x - \xi)$, i.e.,

$$0 \in C_0G(x - \xi),$$

which means that $\xi \in CC_0G(x)$.

Now let $\xi \in CC_0G(x)$. This implies in the sequel $0 \in C_0G(x - \xi)$ and

$$0 \in G(x - \xi).$$

Since $\xi \in G(\xi)$, in view of $S$, we obtain from (3.3) the relation

$$\xi = 0 + \xi \in G(x - \xi) + G(\xi) \subset G(x),$$

by virtue of $A$. Thus identity (3.1) is proved.

Note that identity (3.2) requires assuming axioms $S$ and $A$ for $G$, but the inclusion

$$G \subset CC_0G$$

holds generally.

PROPOSITION 3.4. Relation (3.4) holds for every general convergence.

Proof. Let $\xi \in G(x)$. Of course,

$$0 = \xi - \xi \in G(x - \xi + \xi) - \xi$$

and, by the definition of $C_0G$, we have

$$0 \in C_0G(x - \xi),$$

i.e., $\xi \in CC_0G(x)$. The proof is finished.

Now we shall show that identity (3.1) and the inclusion

$$CC_0G \subset G$$

are false, if one axioms $S$, $A$ is not satisfied by $G$.

EXAMPLE 3.1. Let $G$ be as in Example 1.1. As we have noticed, $G$ satisfies $A$, but not $S$. We have $C_0G(x) = \{0\}$ if $x = \{\xi_n\}$, $\xi_n \to 0$ and $C_0G(x) = \emptyset$ for other sequences $x$. Further, $CC_0G(x) = \{\xi\}$ if $x = \{\xi_n\}$, $\xi_n \to \xi$ and $CC_0G(x) = \emptyset$ otherwise. Therefore
\{-\xi\} = G(x) \subseteq CC_0 G(x) = \{\xi\},
\{\xi\} = CC_0 G(x) \subseteq G(x) = \{-\xi\}

for \(x = \{\xi_n\}\), \(\xi_n \to \xi \neq 0\).

Now, it is easy to see that \(C_0 G(x) = \{0\}\) if \(\xi_n \to \xi\) for some \(\xi \in \mathbb{R}\) and \(C_0 G(x) = \emptyset\) otherwise. Hence \(CC_0 G(x) = \mathbb{R}\) if \(\xi_n \to \xi\) (\(\xi \in \mathbb{R}\)), i.e., for such a sequence \(x = \{\xi_n\}\) we have

\(CC_0 G(x) \subseteq G(x)\).

**EXAMPLE 3.2.** Let \(G\) be as in Example 1.2. As we have seen, \(G\) satisfies \(S\) and not \(A\). We have

\(CC_0 G(x) = \begin{cases} \{\xi\} & \text{if } \xi_n = \xi \text{ for almost all } n \\ \emptyset & \text{otherwise} \end{cases}\)

and thus

\(\{\xi\} = G(x) \subseteq CC_0 G(x) = \emptyset\)

if \(\xi_n = 1 + \frac{1}{n}\), for instance.

**EXAMPLE 3.3.** Let \(E = \mathbb{R}\). We define a general convergence \(G\) as follows: if \(\xi_n \to 0\), then we adopt \(G(x) = \{0\}\); if \(\xi_n = \xi\) for almost all \(n\), then \(G(x) = \{\xi\}\); in the remaining cases \(G(x) = \emptyset\). Obviously, \(G\) satisfies \(S\) and does not satisfy \(A\).

Let \(x = \{\xi_n\}\), where \(\xi_n = \xi + \frac{1}{n}\) with \(\xi \in \mathbb{R}\), \(n \in \mathbb{N}\). Then we have \(CC_0 G(x) = \{\xi\} = CC_0 G(x)\), that is,

\(CC_0 G(x) \subseteq G(x)\)

and

\(CC_0 G(x) \subseteq G(x)\).

**PROPOSITION 3.5.** If a zero-convergence \(G_0\) on \(E\) satisfies axiom \(T\), the

\((3.5)\) \quad G_0 \subseteq C_0 CG_0

and

\((3.6)\) \quad G_0 \subseteq C_0 CG_0.

**Proof.** To prove \((3.5)\), suppose that \(\xi \in G_0(x)\). This means that

\((3.7)\) \quad 0 \in G_0(x)

and, by \(T\), that

\((3.8)\) \quad 0 \in G_0(x - \xi).

By the definition of \(C\), we get from \((3.7)\) and \((3.8)\)

\(\xi, 0 \in CG_0(x)\).
Hence
\[ (3.9) \quad \eta, \xi \in C_0 CG_0(x) = CG_0(x) \]
and
\[ (3.10) \quad \eta, \xi + \eta \in C_0 CG_0(x + \eta) \]
for every \( \eta \in E \).

From (3.10), we obtain
\[ (3.11) \quad \xi \in C_0 CG_0(x). \]

Relations (3.9) and (3.11) prove our assertion.

Since for zero-convergences
\[ (3.12) \quad \text{H implies T}, \]
we have immediately

**COROLLARY 3.1.** If a zero-convergence \( G_0 \) on \( E \) satisfies axiom H, then relations (3.5) and (3.6) hold.

Now, we shall show that relations (3.5) and (3.6) are not true generally (if \( G_0 \) does not satisfy H or T).

**EXAMPLE 3.4.** Let \( E \) be an arbitrary linear space and let \( G_0(x) = E \) if \( \xi_n = 0 \) for almost all \( n \). For remaining sequences \( x \), we put \( G_0(x) = \emptyset \). Note that H does not hold.

We have
\[ CG_0(x) = \begin{cases} \{\xi\} & \text{if } \xi_n = \xi \text{ for almost all } n \\ \emptyset & \text{otherwise} \end{cases} \]
and thus
\[ C_0 CG_0(x) = C_0 CG_0(x) = \{0\} \]
if \( \xi_n = \xi \) for almost all \( n \). This means that (3.5) and (3.6) are false in this case.

**PROPOSITION 3.6.** If a zero-convergence \( G_0 \) satisfies \( T' \), then
\[ (3.13) \quad C_0 CG_0 \subset G_0 \]
and
\[ (3.14) \quad C_0 CG_0 \subset G_0. \]

**Proof.** If \( \xi \in C_0 CG_0(x) \), then in turn
\[ \xi \in CG(x) \]
and
\[ 0 \in G_0(x - \xi). \]

The last relation, by \( T' \), implies that \( \xi \in G_0(x) \) and (3.13) holds.
If $\xi \in C_0 CG_0(x)$, then there exists an $\eta \in E$ such that $\eta, \xi + \eta \in CG_0(x + \eta)$ and hence $0 \in G_0(x + \eta)$. This implies $\xi \in G_0(x)$, by $T'$ and (3.14) is proved.

Since axiom $T'$ is somewhat artificial for zero-convergences (see section 4), we shall prove (3.13) and (3.14) also under other assumptions.

**PROPOSITION 3.7.** If a convergence $G_0$ satisfies $S'_0$, $A$ and $M$, then (3.13) and (3.14) hold.

**Proof.** By Proposition 2.3, the general convergence $CG_0$ and the zero-convergence $C_0 CG_0$ satisfy axiom $H$. To prove (3.13), it remains to note that $0 \in C_0 CG_0(x)$ implies $0 \in CG_0(x)$ and this implies $0 \in G_0(x)$.

Relation (3.14) is obvious if $G_0 = \emptyset$. Further, note that for non-empty zero-convergences condition $M$ implies $S'_0$. Hence, by Propositions 2.4 and 2.6, the general convergence $CG_0$ fulfils axioms $S$ and $A$. Consequently, we have $C_0 CG_0 = C_0 CG_0$, in view of the second part of Proposition 1.1. Hence (3.14) follows, by virtue of the first part of this proposition.

Note that for zero-convergences

$$T' \text{ implies } T.$$ (3.15)

In fact, assume that $\xi \in G_0(x)$. Then $0 \in G_0(x) = G_0(x - \xi + \xi)$ and we obtain

$$-\xi \in G_0(x - \xi),$$

in view of $T'$. But this means, by the definition of zero-convergences, that $0 \in G_0(x - \xi)$. Thus (3.15) holds.

By virtue of implications (3.12) and (3.15), we obtain from Propositions 3.5, 3.6 and 3.7 the following result.

**COROLLARY 3.2.** If a zero-convergence $G_0$ on $E$ satisfies $1^o T'$; or $2^o S'_0$, $A$, $M$ and $T$; or $3^o S'_0$, $H$, $A$ and $M$, then the identities

$$C_0 CG_0 = G_0$$ (3.16)

and

$$C_0 CG_0 = G_0$$ (3.17)

hold.

We shall show now that relations (3.13)—(3.14) and (3.16)—(3.17) are not true generally (if $T'$ and one of axioms $S'_0$, $A$, $M$ do not hold for $G_0$).

**EXAMPLE 3.5.** If $G_0$ is as in Example 2.1, then axioms $H$, $A$, $M$ are satisfied and axioms $S'_0$, $T'$ — not. We have

$$C_0 CG_0(\xi) = C_0 CG_0(\xi) = R \subseteq \{0\} = G_0(\xi)$$

for every $\xi \in X$. 

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If $G_0$ is taken as in Example 2.2, then $H, M, S'_0$ hold and, at the same time, $A$ as well as $T'$ do not hold. Moreover we have

$$C_0CG_0 \left( \left\{ \frac{\xi}{n} \right\} \right) = C_0CG_0 \left( \left\{ \frac{\xi + 1}{n} \right\} \right) = R \subseteq \{0\} = G_0 \left( \left\{ \frac{\xi + 1}{n} \right\} \right)$$

for every $\xi \in X$.

At last, if $G_0$ is taken from Example 2.3, then $H, A, S'_0$ are fulfilled, but any of axioms $M, T'$ is not. We have in this case

$$C_0CG_0(x) = C_0CG_0(\{x\}) = R \subseteq \{0\} = G_0(\{x\})$$

for $x = \{\xi_n\}$ with $\xi_n \rightarrow \infty$.

4. Finally, we would like to present, without proofs, mutual relations between axioms concerning general convergences.

First note that each of axioms $F, U, S, H, A, M$ is independent of others. Axioms $F, U, A, M$ do not depend on axioms $S', T, T'$ either. However we have the following implication:

$$(4.1) \quad S' \land A \land M \Rightarrow H.$$  

On the other hand, according to (4.1) it can be shown that $H$ does not depend: 1° on $F, U, S, A, M, T$ and $T'$; 2° on $F, U, S, S', M, T$ and $T'$.

Further note that $S$ does not depend on axioms $F, U, H, A, M, S', T$ and $T'$, because the trivial convergence (defined as $G(x) = \emptyset$ for all $x \in X^N$) satisfies all mentioned axioms except $S$. The situation is different when considering only non-trivial convergences. Then we have the implication

$$(4.2) \quad M \land T' \Rightarrow S.$$  

On the other hand, one can show that in the class of nontrivial convergences $S$ does not depend: 1° on $F, U, S', H, A, M$ and $T$; 2° on $F, U, S', H, A, T$ and $T'$.

We are passing now to axioms $S', T, T'$. The following relations hold:

$$(4.3) \quad S \land H \Rightarrow S',$$

$$(4.4) \quad H \land M \land T' \Rightarrow S',$$

$$(4.5) \quad H \land A \land T \land T' \Rightarrow S',$$

$$(4.6) \quad S \land A \Rightarrow T,$$

$$(4.7) \quad S \land A \Rightarrow T'.$$

However
1° axiom $S'$ does not depend: a) on $F$, $U$, $S$, $A$, $M$, $T$, $T'$; b) on $F$, $U$, $H$, $A$, $M$, $T$; c) on $F$, $U$, $H$, $A$, $T$; d) on $F$, $U$, $H$, $T$, $T'$ — see (4.3), (4.4), (4.5) and (4.2);

2° axiom $T$ does not depend: a) on $F$, $U$, $S$, $S'$, $H$, $M$, $T'$; b) on $F$, $U$, $S'$, $H$, $A$, $T'$; c) on $F$, $U$, $S'$, $H$, $A$, $M$ — see (4.6) and (4.2);


Finally note that axiom $T'$ is unnatural for zero-convergences. For we have the implication

$$S_0 \land S'_0 \Rightarrow \sim T',$$

provided $E \neq \{0\}$.

We omit here other relations between axioms $S_0$, $S'_0$ and the remaining ones for zero-convergences.

REFERENCES