GYÖRGY TARGOŃSKI, MAREK CEZARY ZDUN

GENERATORS AND CO-GENERATORS OF SUBSTITUTION SEMIGROUPS

Abstract. In this note we give the form of generators and co-generators of semigroups of "substitution operators" in Banach space $C([a, b])$. We also establish some properties of these operators related to Schröder equation.

0. Introduction. Let us assume that:

(i) $f$ is defined and continuous in $[a, b]$ (*) (we admit $b = \infty$), strictly increasing and of class $C^1$ in $[a, b)$, $f'(a) \neq 0$, furthermore $a < x < f(x) < b$ for $x \in [a, b)$ ($b$ is therefore a fixed point of $f$).

THEOREM 0.1 ([5], [6]). Let $\{f^t, t > 0\}$ be an iteration semigroup of $f(\ast)$ such that all $f^t$ are continuous in $[a, b]$ and of class $C^1$ in $[a, b)$.

Then:

1° The following representation is valid

\[ f^t(x) = h(t + h^{-1}(x)), \quad t > 0, \quad x \in [a, b), \]

where $h$ maps $[0, \infty)$ onto $[a, b)$ in a strictly increasing way. Moreover, $h$ is of class $C^1$ in $[0, \infty)$ and $h^{-1}$ is of class $C^1$ in $[a, b)$.

2° All functions $f^t, t > 0$ satisfy (i).

3° The derivative

\[ (0.2) \quad g(x) := \frac{\partial f^t(x)}{\partial t} \bigg|_{t=0}, \quad x \in [a, b], \]

exists, $g(b) = 0$, $g$ is continuous in $[a, b]$ and $g > 0$ in $(a, b)$.

4° The integral

\[ (0.3) \quad \int_a^b \frac{du}{g(u)} \]

diverges.

Received November 06, 1979.


(*) continuity at $b = \infty$ is equivalent to saying that $\lim_{x \to b^-} f(x) = 0$.

(**) i.e. $f^t([a, b]) \subset [a, b)$, $f^t \circ f^s = f^{t+s}$ for $t, s > 0$ and $f^1 = f$. 169
All semigroups considered in this paper satisfy the assumptions of Theorem 0.1.

Let now

(0.4) \[ T^t \varphi : = \varphi \circ f^t, \quad t > 0 \]

be a semigroup of operators ("substitution operators") on \( C([a, b]) \) with the sup-norm (***)

In this paper, we shall determine first the form and some properties of the infinitesimal generator of the semigroup (0.4), the same investigation on the co-generator will be carried and next out.

1. Infinitesimal generators. We consider the infinitesimal generator of a semigroup (0.4)

(1.1) \[ A \varphi : = \lim_{t \to 0^+} \frac{T^t - I}{t} \varphi \]

(in the sense of the norm) defined in a domain \( D(A) \). As it is well known, \( D(A) \) is dense and \( A \) is closed (cf. [4]). Denoting the range of \( A \) by \( R(A) \), we prove:

**THEOREM 1.1.**

\[ D(A) = \{ \varphi \in C ([a, b]) \cap C^1([a, b]) : \lim_{x \to b^-} \varphi'(x) g(x) = 0 \}, \]

\[ R(A) = \{ \psi \in g \cdot C ([a, b]) \cap C([a, b]) : \]

\[ \psi(b) = 0 \text{ and the improper integral } \int_{a}^{b} \frac{u}{g(u)} \, du \text{ exists and is finite} \}

and

\[ (A \varphi) (x) = \begin{cases} g(x) \varphi'(x), & x \in [a, b) \\ 0, & x = b. \end{cases} \]

**Proof.** From Theorem 1 in [4, Ch. IX. § 4] it follows, that \( I-A \) has an inverse \( J = (I-A)^{-1} \) defined on \( C([a, b]) \) and continuous. Moreover

\[ J \varphi = \int_{0}^{\infty} e^{-t} (T^t \varphi) \, dt \]

in the sense of the Riemann integral in the Banach space \( C([a, b]) \). Furthermore we have ([4, Ch. IX, Cor. 2])

(1.3) \[ AJ = J-I. \]

As seen above \( D(J) = C([a, b]) \) and \( R(J) = D(A) \). Put \( \psi : = J \varphi \), where \( \varphi \in C ([a, b]) \). From (1.2) and (0.1) we have

\[ (***) \text{ if } b = \infty, \text{ then } C ([a, b]) \text{ denotes the space of all continuous and bounded functions } h \text{ in } [a, \infty) \text{ such that the finite limit } \lim_{x \to b^-} h(x) \text{ exists, } \|h\| = \sup_{x \in [a, b)} |h(x)|. \]
\[ \psi(x) = \int_0^\infty e^{-t} \varphi(f^t(x)) \, dt = \int_0^\infty e^{-t} \varphi(h(t + h^{-1}(x))) \, dt = \int_{h^{-1}(x)}^\infty e^{h^{-1}(x)-u} \varphi(h(u)) \, du = e^{h^{-1}(x)} \int_{h^{-1}(x)}^\infty e^{-u} \varphi(h(u)) \, du. \]

From this it follows that \( \psi \in C([a, b]) \cap C^1([a, b]) \). Further differentiating both sides of the last equality we get \( \psi'(x) = \frac{\psi(x) - \varphi(x)}{g(x)} \) for \( x \in [a, b] \), since \((h^{-1})' = 1/g\). Hence \( g(x) \psi'(x) = \psi(x) - \varphi(x) \) for \( x \in [a, b] \) (\( \varphi, \psi \in C([a, b]) \)). From the definition of \( \psi \) it follows that \( \psi(b) = \varphi(b) \). \( \varphi \) and \( \psi \) being continuous at \( b \), it follows that the limit of \( g(x) \psi'(x) \) at \( b \) exists and equals zero. Furthermore (see (1.3)) \( AJ \varphi = J \varphi - \varphi \) and this implies \( A \psi = \psi - \varphi = g \psi' \) in \([a, b]\). From this

\[ A \psi = \begin{cases} g \psi' \text{ in } [a, b] \\ 0 \text{ in } b \end{cases}, \]

for \( \psi \in R(J) = D(A) \).

From our discussion it follows that

\[ (1.4) \quad D(A) \subseteq \{ \varphi \in C([a, b]) \cap C^1([a, b]) : \lim_{x \to b-} \varphi'(x) g(x) = 0 \}. \]

We denote the set on the right-hand side of (1.4) by \( K \). Let \( \psi \in K \).
Then

\[ (1.5) \quad \varphi(x) : = \begin{cases} \psi(x) - \psi'(x) g(x), & x \in [a, b] \\ \psi(b), & x = b \end{cases} \]

belongs to \( C([a, b]) \).

Put

\[ (1.6) \quad \bar{\psi} : = J \varphi \in D(A) \subseteq K. \]

By (1.3), \( A \bar{\psi} = \bar{\psi} - \varphi \), therefore \( g(x) \bar{\psi}'(x) = \bar{\psi}(x) - \varphi(x) \) for \( x \in [a, b] \) and \( 0 = \bar{\psi}(b) - \varphi(b) \). Hence

\[ (1.7) \quad \varphi(x) = \begin{cases} \bar{\psi}(x) - \bar{\psi}'(x) g(x), & x \in [a, b] \\ \bar{\psi}(b), & x = b \end{cases}. \]

Put \( \omega = \psi - \bar{\psi} \), then by (1.5) and (1.7),

\[ \omega(x) = \begin{cases} \omega'(x) g(x), & x \in [a, b] \\ 0, & x = b \end{cases}. \]

and \( \omega \in K \). So \( \omega(x) = \exp \int_a^x \frac{du}{g(u)}, \ x \in [a, b] \). If \( c \neq 0 \), then from (0.3) it follows that \( \lim_{x \to b-} \omega(x) = \pm \infty \), but \( \omega \in K \), so it is bounded in \([a, b]\), therefore \( c = 0 \) and \( \omega(x) = 0 \); hence \( \bar{\psi} = \psi \). Now from (1.6) it follows that \( \psi \in D(A) \), thus \( K = D(A) \).

The formula for \( R(A) \) can be verified directly.
2. Co-generators. The notion of co-generator of a semigroup of operators (in Hilbert space) has been introduced by B. Sz. Nagy and C. Foiaș (see [2, Ch. III, 8] and [3]). We generalize this notion to Banach space.

Let \( X \) be a Banach space, \( \{T^t, t \geq 0\} \) be a continuous semigroup with infinitesimal generator \( A \). From the properties of the resolvent it follows, that \( A-I \) has an inverse \( -J = (A-I)^{-1} \) defined in all \( X \) and continuous. Moreover, (1.3) holds. Therefore we can define the co-generator

\[
T := (A + I) (A - I)^{-1}
\]

defined in all \( X \). From (1.3) it follows that \( T = -(I + A)J = -J - AJ = I - 2J \). \( T \) is continuous. Let \( T^t \) be given by (0.4). We now determine the co-generator for \( T^t \). Let \( \phi \in C([a, b]) \). Put \( \psi := (A-I)^{-1} \phi \), so \( \psi \in K \). From (\( A-I \)) \( \psi = \phi \) follows, that

\[
\phi = -\psi + \psi' \psi
\]
in \([a, b]\) has exactly one solution because of 4° in Theorem 0.1. This solution is

\[
\psi(x) = \int_a^x \frac{\phi(u)}{g(u)} \left( \exp - \int_a^u \frac{1}{g(t)} \, dt \right) \, du \exp \int_a^x \frac{1}{g(u)} \, du.
\]

Then, \( T\psi = (A+I)\psi = g\psi' + \psi = 2\psi + \phi \), so

\[
(T\phi)(x) = \phi(x) + 2 \exp \int_a^x \frac{1}{g(u)} \, du \int_a^x \frac{\phi(u)}{g(u)} \left( \exp - \int_a^u \frac{1}{g(t)} \, dt \right) \, du.
\]

Remembering (0.1), we can write

\[
(T\phi)(x) = \phi(x) + 2 e^{h^{-1}(x)} \int_0^{h^{-1}(x)} \phi(h(t)) e^{-t} \, dt = \phi(x) - 2e^{h^{-1}(x)} \int_a^x \phi(s) [e^{-h^{-1}(s)}]' \, ds.
\]

Put \( u(x) = e^{-h^{-1}(x)} \) in (2.5) and (0.1). Then \( f'(x) = u^{-1} [e^{-t} u(x)] \), for \( t > 0 \) and

\[
(T\phi)(x) = \phi(x) - \frac{2}{u(x)} \int_0^x \phi(s) u'(s) \, ds,
\]

where \( u \) satisfies the Schröder equation \( u(f(x)) = e^{-1} u(x) \).

3. Some properties of the infinitesimal generator. Consider \( A \), the infinitesimal generator of our semigroup. According to the expression for \( A \) given in Theorem 1.1, \( Af_1 = Af_2 \) implies \( f_1 - f_2 = \text{const} \). So, \( A \) is "invertible up to an additive constant". We introduce the linear operator

\[
(B\psi)(x) := \int_a^x \frac{\psi(u)}{g(u)} \, du.
\]
We restrict the domain of $B$ in such a way that $D(B) = \{ \varphi \in D(A) : \varphi(a) = 0 \}$, $B$ is invertible and \[ B^{-1} = A_{|R(B)} \cdot \]

THEOREM 3.1.

\[(B \varphi)(x) = \int_0^\infty \varphi(f^t(x)) \, dt + \int_a^b \frac{\varphi(u)}{g(u)} \, du, \quad \text{for } \varphi \in D(B). \]

**Proof.** Let $\varphi \in R(A)$. In the integral $\int_a^b \frac{\varphi(u)}{g(u)} \, du$ (it exists as an improper integral according to our assumption) we can put (because of (0.1) and (0.2)) $g = h \circ h^{-1}$; substituting $u = h(t+h^{-1}(x))$ we obtain

\[ \int_0^\infty \varphi(h(t+h^{-1}(x))) \, dt = \int_0^\infty \varphi(f^t(x)) \, dt. \]

Further we find (3.2) by definition of $B$. Introducing

\[ C \varphi := \int_0^\infty \varphi(f^t(\cdot)) \, dt \]

we can define (according to Theorem 3.1) $C$ on $D(B) = : D(C)$. From (3.1) it follows that $AC = I$ in $D(B)$. We have thus obtained.

COROLLARY 3.1. $C$ defined by (3.3) has the property $AC = I$ in $D(C)$.

We are now going to consider some properties of $C$.

THEOREM 3.2. $C(\varphi \circ f^s) = (C \varphi) \circ f^s$, $s > 0$.

**Proof.**

\[ C(\varphi \circ f^s) = \int_0^\infty \varphi(f^s(f^t(x))) \, dt = \int_0^\infty \varphi(f^t(f^s(x))) \, dt = (C \varphi) \circ f^s. \]

An immediate consequence is.

COROLLARY 3.2: If $\varphi \in D(C)$ satisfies a Schröder equation

\[ \varphi(f(x)) = s \varphi(x), \text{ for an } s > 0, \]

then $C \varphi$ satisfies the same equation.

Another observation concerning the Schröder equation is the following.

THEOREM 3.3. Let $\varphi \in D(C)$, then the following two statements are equivalent:

(a) $\varphi$ is eigenfunction of $C$.

(b) For every $t > 0$ there exists a $\lambda_t$ such that

\[ \varphi(f^t(x)) = \lambda_t \varphi(x). \]
Proof. (a) $\Rightarrow$ (b): Let $\varphi \in D(C)$ and $C\varphi = \mu \varphi (\mu \neq 0)$ then $A\varphi = \mu A\varphi$, but $A = I$ in $D(C)$, so $\varphi = \mu A\varphi = g \varphi'$ in $[a, b)$ (see Theorem 1.1). This differential equation has exactly one-parameter family of solutions:

$$\varphi(x) = k \exp \frac{1}{\mu} \int_a^x \frac{1}{g(u)} du.$$ 

One finds (cf. (0.1) and (0.2)) $\varphi(x) = k \exp \frac{1}{\mu} h^{-1}(x)$. From (0.1) we have $h^{-1}(f^t(x)) = t + h^{-1}(x) t > 0, x \in [a, b))$. From this it is easily verifiable that $\varphi$ satisfies (b).

(b) $\Rightarrow$ (a). From the definition, $\lambda_t$ is determined uniquely. For all $t, s > 0$ we have $\varphi(f^{t+s}(x)) = \lambda_{t+s} \varphi(x)$, moreover from (b) follows

$$\varphi(f^{t+s}(x)) = \varphi(f^t(f^s(x))) = \lambda_t \varphi(f^s(x)) = \lambda_t \lambda_s \varphi(x).$$

Thus $\lambda_{t+s} = \lambda_t \lambda_s (t, s > 0)$.

The continuity of $t \mapsto \lambda_t$ now follows from Theorem 0.1 and from the continuity of $\varphi$. Therefore (cf. [1] p. 38) since $\lambda_t \neq 0 (***)$, $\lambda_t = \gamma^t$, for some fixed $\gamma > 0$. So $\varphi(f^t(x)) = \gamma^t \varphi(x), t > 0, x \in [a, b)$, Now $C\varphi = = \int_0^f \varphi(f^t(x)) dt = \int_0^\infty \gamma^t \varphi(x) dt$. Thus $C\varphi = \mu \varphi$ and (a) is satisfied with $\mu = \int_0^\infty \gamma^t dt$. Since $\varphi$ is continuous in $b$, the inequality $0 < \gamma < 1$ must be satisfied, thus the integral defining $\mu$ exists.

In closing we remark: it follows from our above considerations that the spectrum of $C$ is the interval $(-\infty, 0)$.

(***) $\lambda_t = 0$ would imply $\varphi = 0$ (cf. (b) in Theorem 3.3), but this is impossible.

REFERENCES