NOWHERE DENSE CHOICES AND $\pi$-WEIGHT

Abstract. The paper is devoted to inequalities between $\pi_0(X)$ and $\pi_d(X)$ where

$$\pi_0(X):=\min\{\pi(U): U \text{ open and non-empty subset of } X\},$$

$$\pi_d(X):=\min\{|\mathcal{B}|: \text{every open and dense subset of } X \text{ contains an element from } \mathcal{B}\}.$$  

From these definitions $\pi_d(X) \leq \pi_0(X)$ for every space $X$. In the paper we construct a space $X$ for which $\pi_d(X) = \omega_1$ and $\pi_0(X) = 2^{\aleph_0}$.

We shall define two cardinal functions $\pi_d$ and $\pi_0$. We shall give conditions which ensure that $\pi_d(X) = \pi_0(X)$ and give a consistent example of a space $X$ such that $\pi_d(X) < \pi_0(X)$.

Recall that for a topological space $X$, $\pi(X)$ denotes that least cardinal of a $\pi$-base for $X$, i.e.:

$$\min\{|\mathcal{B}|: \text{for each non-empty open } U \subseteq X, \text{ there is } B \in \mathcal{B} \text{ such that } B \subseteq U\}.$$  

For a space $X$ we denote by $\pi_0(X)$ the cardinal:

$$\min\{\pi(U): U \text{ is a non-empty open subset of } X\}.$$ 

And we denote by $\pi_d(X)$ the cardinal

$$\min\{|\mathcal{B}|: \text{for each dense open } U \subseteq X, \text{ there is } B \in \mathcal{B} \text{ such that } B \subseteq U\},$$

where such families $\mathcal{B}$ are called $\pi_d$-bases for $X$. All other terminology used in this article can be found in one of the standard textbooks [1], [3] or [5]. Furthermore we shall assume that all topological spaces under consideration are regular.

Motivation for the two new definitions comes from the following question.

QUESTION. Given a collection $\mathcal{U}$ of non-empty open subsets of a space $X$, can I pick a point $x(U) \in U$ for each $U \in \mathcal{U}$ such that $\{x(U): U \in \mathcal{U}\}$ is nowhere dense?

Note that $x: \mathcal{U} \to X$ can be considered as a choice function. The question asks for a choice function with nowhere dense image, hence the first half of our title. Now suppose that $\mathcal{U}$ is a $\pi$-base for an open subset $G \subseteq X$. Clearly, the image
any choice function on \( \mathcal{U} \) is dense in \( G \), giving a negative answer to the question and the second half of our title.

A moment’s thought will convince the reader that the question has a "NO" answer iff \( \mathcal{U} \) is a \( \tau_d \)-base for \( X \). Therefore the question has a "YES" answer for all collections \( \mathcal{U} \) such that \( |\mathcal{U}| \leq \kappa \) iff \( \tau_d(X) > \kappa \). The argument in the previous paragraph then shows that \( \tau_d(X) \leq \tau_0(X) \).

These considerations were first made by M. van de Vel. E. K. van Douwen observed that if \( \tau_d(X) = \omega \), so does \( \tau_0(X) \) and communicated the general problem of the relationship between \( \tau_d \) and \( \tau_0 \) to us.

We first draw some easy conclusions.

**THEOREM 1.** For each space \( X \) we have

(a) \( \tau_d(X) \leq \tau_0(X) \),
(b) \( \tau_0(X) \leq 2^{\tau_d(X)} \),
(c) \( \tau_0(X) \leq \tau_d(X) \cdot \tau_X(X) \).

**Proof.** (a) is above. To prove (b), let \( \mathcal{A} \) be a \( \tau_d \)-base of \( X \) of cardinality \( \tau_d(X) \). For each \( A \in \mathcal{A} \) pick \( X(A) \in A \). Then \( X(A) : A \in \mathcal{A} \) must be dense in some open set \( U \). Since \( X \) is regular we must have

\[ \tau_0(X) \leq \tau(U) \leq 2^{\tau(U)} \leq 2^{\tau_d(X)}. \]

Let us continue and prove (c). For each \( A \in \mathcal{A} \), pick a local \( \pi \)-base \( \mathcal{B}(A) \) for \( \tau(X) \) such that \( |\mathcal{B}(A)| \leq \tau_X(X) \). It is now easy to check that \( \bigcup \{ \mathcal{B}(A) : A \in \mathcal{A} \} \) forms a \( \pi \)-base for \( U \) and verifies (c).

We shall see that for many types of spaces we actually have \( \tau_d \) equal to \( \tau_0 \).

We begin with the following useful lemma.

**LEMMA 2.** Suppose \( \tau_0(X) = \kappa \), \( \mathcal{A} \) is a family of \( < \kappa \) non-empty open subsets of \( X \) and \( \mathcal{B} \) is a family of \( \leq \kappa \) non-empty open subsets of \( X \). Then there is a family \( \mathcal{C} \) of \( \lambda \) non-empty open sets precisely refining \( \mathcal{B} \) such that no \( A \in \mathcal{A} \) is covered by finitely many members of \( \mathcal{C} \). Furthermore \( \mathcal{C} \) can be chosen as a subfamily of any given \( \pi \)-base for \( X \).

**Proof.** Enumerate \( \mathcal{B} \) as \( \{ B_\alpha : \alpha < \lambda \} \). We construct \( \mathcal{C} = \{ C_\alpha : \alpha < \lambda \} \) by recursively constructing \( C_\alpha \) for \( \alpha < \lambda \) with the inductive hypothesis at \( \beta < \lambda \) that for each \( \alpha < \beta \), \( C_\alpha \) is a non-empty open subset of \( B_\alpha \) such that no \( A \in \mathcal{A} \) is covered by finitely many members of \( \{ C_\alpha : \alpha < \beta \} \).

At stage \( \beta \) note that \( \tau(B_\beta) \geq \kappa \), and

\[ |\{ A \setminus \bigcup_{\alpha \in F} C_\alpha : A \in \mathcal{A} \text{ and } F \in [\beta]^{< \omega} \}| < \kappa. \]

Hence there is some open \( G \subset B_\beta \) such that for all \( A \in \mathcal{A} \) and for all \( F \in [\beta]^{< \omega} \)

\[ (A \setminus \bigcup_{\alpha \in F} C_\alpha) \setminus G \neq \emptyset. \]

Now pick a non-empty open \( C \) in our given \( \pi \)-base such that \( C_\beta \subset G \). This completes the induction and the proof.

The next lemma reduces the problem to considerations involving \( \pi \)-weight.
LEMMA 3. If X is a space, then X has an open subspace Y such that
\[ \pi_d(Y) \leq \pi_d(X) \leq \pi_0(X) \leq \pi_0(Y) = \pi(Y). \]

Proof. Let \( \mathcal{A} \) be a \( \pi_d \)-base for X. It suffices to prove that there is an open \( Y \subseteq X \) such that \( \pi_0(Y) = \pi(Y) \) and \( \{ Y \cap A : A \in \mathcal{A} \} \) is a \( \pi_d \)-base for Y.

Suppose not. Consider a maximal pairwise disjoint family of open sets \( \mathcal{U} \) such that for all \( U \in \mathcal{U}, \pi_0(U) = \pi(U) \). For each \( U \in \mathcal{U}, \) pick \( G(U) \), an open dense subset of \( U \) such that for all \( A \in \mathcal{A}, \) if \( U \cap A \neq \emptyset \) then \( (U \cap A) \setminus G(U) \neq \emptyset. \) Since \( \bigcup \mathcal{U} \) is dense in \( X \), so is \( G = \bigcup \{ G(U) : U \in \mathcal{U} \} \) and no \( A \in \mathcal{A} \) is contained in \( G \), contradicting that \( \mathcal{A} \) is a \( \pi_d \)-base.

We can now state some theorems.

THEOREM 4. If X is a locally compact space, then \( \pi_d(X) = \pi_0(X) \).

Proof. By Lemma 3 we can assume \( \pi_0(X) = \pi(X) \) without loss of generality. Let \( \kappa = \pi_0(X) > \pi_d(X) = \lambda \) and show a contradiction.

Let \( \mathcal{A} \) be a collection of open sets, of size \( \lambda \), such that for each dense open \( V \subseteq X \) there is some \( A \in \mathcal{A} \) such that \( A \) is compact and \( A \subseteq V \). Let \( \mathcal{B} \) be a \( \pi \)-base for \( X \) of size \( \kappa \). By Lemma 2, we can choose \( \mathcal{C} \) as in the statement of the lemma. Since \( \mathcal{C} \) refines \( \mathcal{B} \), \( \bigcup \mathcal{C} \) is dense, hence there is a compact \( \overline{A} \subseteq \bigcup \mathcal{C} \) which contradicts the other property of \( \mathcal{C} \).

THEOREM 5. \( \pi_d(X) = \pi_0(X) \) if either (i) \( X \) is locally connected, or (ii) \( X \) is a linearly ordered topological space.

Proof. We first show that in each case (i) and (ii) there is a \( \pi \)-base \( \mathcal{U} \) for \( X \) such that if \( U \in \mathcal{U} \) and \( \mathcal{V} \) is a pairwise disjoint subcollection of \( \mathcal{U} \) such that \( U \subseteq \bigcup \mathcal{V} \) then there is some \( V \in \mathcal{V} \) such that \( U \subseteq V \). For case (i) this is immediate. For case (ii), let \( \mathcal{U} \) be the collection: \( \{ \{ P \} : p \text{ is isolated} \} \cup \{ (a, b) : a \text{ has no immediate successor and } b \text{ has no immediate predecessor} \} \).

We now use this property of \( \mathcal{U} \) to complete the proof that \( \pi_d(X) = \pi_0(X) \). Let \( \mathcal{A} \subseteq \mathcal{U} \) be a subcollection of size \( < \pi_0(X) \). For each open \( V \) there is some \( U(V) \in \mathcal{U} \) such that no element of \( \mathcal{A} \) is contained in \( U(V) \). Let \( \mathcal{V} \) be a maximal pairwise disjoint subcollection of \( \{ U(V) : V \text{ is open in } X \} \); \( \bigcup \mathcal{V} \) is dense, showing that \( \mathcal{A} \) is not a \( \pi_d \)-base for \( X \).

It is not true that \( \pi_d(X) = \pi_0(X) \) for every \( X \), but we only have consistent counterexamples. These use the following lemma.

LEMMA 6. If \( X \) is a non-separable Lusin space of cardinality \( \omega_1 \), then \( \pi_d(X) \leq \omega_1 \).

Proof. Enumerate \( X \) as \( \{ x_\alpha : \alpha \in \omega_1 \} \). Since every nowhere dense subset of \( X \) is countable, the following collection forms a \( \pi_d \)-base:

\[ \{ X \setminus \text{cl}(\{ x_\beta : \beta \in \alpha \}) : \alpha \in \omega_1 \}. \]

In [6], there is constructed a dense Lusin subspace \( Y \) of \( 2^\kappa \), under the assumption BACH plus \( \omega_1 < \kappa < 2^{\omega_1} \). For this space we have \( \pi_0(Y) = \kappa \) and \( \pi_d(Y) = \omega_1 \).

We shall show that the inequality \( \pi_0(X) \leq 2^{\pi_d(X)} \) is sharp by showing that it is relatively consistent that \( 2^{\omega_1} \) is "anything reasonable" and there is a space \( X \) with \( \pi_d(X) = \omega_1 \) and \( \pi_0(X) = 2^{\omega_1} \). This is accomplished by Lemma 6 and the following theorem.
THEOREM 7. CON (ZFC plus $2^{\omega_1} = \kappa$) implies CON (ZFC plus $2^{\omega_1} = \kappa$ plus there is a dense Lusin subspace of $2^\kappa$ of cardinality $\omega_1$).

Proof. We can suppose that we have

$$V \models \text{"ZFC plus CH plus } 2^{\omega_1} = \kappa".$$  

We shall construct a generic extension of $V$ in order to prove the theorem. We first describe a partial order $\mathcal{P}$ in the model $V$. Using CH, let $X$ be a dense Baire (for example, countably compact) subspace of $2^\kappa$ of size $\omega_1$. Enumerate $X$ as $\{x_\alpha : \alpha \in \omega_1\}$. Let $H(\kappa)$ be the collection of all finite partial functions from $\kappa$ into $2$. For each $e \in H(\kappa)$, denote by $[e]$ the set $\{f \in 2^\kappa : e \subseteq f\}$ which is an elementary open subset of $2^\kappa$. Let $\mathcal{D}$ denote the set

$$\{D \in [H(\kappa)]^{<\omega} : \bigcup \{[e] : e \in D\} \text{ is dense in } 2^\kappa\}.$$  

Finally, let $\mathcal{G}$ be the set

$$\{\langle Y, \mathcal{V} \rangle : Y \in [X]^{<\omega} \text{ and } \mathcal{V} \in [\mathcal{D}]^{<\omega}\}$$  

with the ordering $\langle Y_1, \mathcal{V}_1 \rangle \leq \langle Y_2, \mathcal{V}_2 \rangle$ iff $Y_2 \subseteq Y_1$, $\mathcal{V}_2 \subseteq \mathcal{V}_1$ and for each $D \in \mathcal{V}_2$,

$$Y_1 \setminus Y_2 \subseteq \bigcup \{[e] : e \in D\}.$$  

Let $\mathcal{G}$ be $\mathcal{P}$-generic over $V$. We claim that $V[\mathcal{G}] \models "2^{\omega_1} = \kappa$ and there is a dense Lusin subspace of $2^\kappa$ of size $\omega_1."$.

Let $X^* = \bigcup \{Y : \text{for some } \mathcal{V}, \langle Y, \mathcal{V} \rangle \in \mathcal{G}\}$. Observe that $P$ is countably closed and hence $V[\mathcal{G}]$ contains no new countable subsets of $V$. Since $V \models \text{CH}$ and $P$ is $2^{\omega_1}$-centered, all cardinals are preserved. We know that $|X^*| = \omega_1$ by considering the following dense sets:

$$\{\langle Y, \mathcal{V} \rangle : \text{for some } \alpha > \beta, x_\alpha \in Y, \beta \in \omega_1\}.$$  

It remains to show that $X^*$ is a dense Lusin subspace of $2^\kappa$. $X^*$ is dense in $2^\kappa$ because the following sets are dense in $P$:

$$\{\langle Y, \mathcal{V} \rangle : Y \cap [\varepsilon] \neq \emptyset, \varepsilon \in H(\kappa)\}.$$  

Note that for each $D \in \mathcal{D}$, the set $\{\langle Y, \mathcal{V} \rangle : D \in \mathcal{V}\}$ is dense. We will show that this implies that every dense open subset of $X^*$ is co-countable. Let $U$ be a dense open subset of $2^\kappa$. Let $E$ be a maximal pairwise disjoint collection of elementary open subsets of $U$. $|E| \leq \omega$ and hence $D = \{e : [e] \in E\} \in \mathcal{D}$. For some $\langle Y, \mathcal{V} \rangle$, $\langle Y, \{D\} \rangle \in \mathcal{G}$ and since elements of $\mathcal{G}$ are compatible we have that $X^* \setminus U \subseteq Y$ and is hence countable.

We note that this proof can be generalized to obtain the following corollary.

COROLLARY 8. CON (ZFC plus $2^{(\omega_1)^+} = \kappa$) implies CON (ZFC plus there is a space $X$ with $\pi_4(X) = \lambda^+$ and $\pi_0(X) = \kappa$).

We also note that this theorem gives a consistent example of an $L$-space of weight $2^{\omega_1}$ where $2^{\omega_1}$ is arbitrarily large. See [2], [4] and [6]. Now we will show that the existence of a dense subspace $X$ of $2^{(2^{\omega_1})}$ such that $\pi_4(X) < \pi_0(X) = 2^{\omega_1}$ is denied by Martin's Axiom and is hence independent of ZFC.
Let $X$ be a space and $\mathcal{U}$ be a collection of subsets of $X$. We denote by $\mathcal{P}(X, \mathcal{U})$ the set

$$\{\langle S, \mathcal{V} \rangle : S \in [X]^{<\omega} \setminus \{\emptyset\}, \mathcal{V} \in [\mathcal{Y}]^{<\omega} \text{ and } S \cap \cup \mathcal{V} = \emptyset\}$$

with the partial ordering $\langle S_1, \mathcal{V}_1 \rangle \leq \langle S_2, \mathcal{V}_2 \rangle$ iff $S_2 \subseteq S_1$ and $\mathcal{V}_2 \subseteq \mathcal{V}_1$.

**Theorem 9.** Assume MA. If $\kappa \leq 2^{\omega}$ and $X$ is a dense subspace of $2^\kappa$, then $\pi_d(X) = \pi_0(X) = \kappa$.

**Proof.** We show that if $\lambda < \kappa$, then $\pi_d(X) > \lambda$. Suppose not and derive a contradiction by assuming that $\mathcal{A}$ is a $\pi_\tau$-base of size $\lambda$. Without loss of generality assume that each $A \in \mathcal{A}$ is an elementary open set. Since $\lambda < \kappa$ we can find $Y \in [\kappa]^{<\omega}$ such that the support of any $A \in \mathcal{A}$ is disjoint from $Y$. Let $\mathcal{U}$ be the collection of all elementary open sets with support contained in $Y$.

Let us notice the following facts. $\mathcal{U}$ is countable. If $A \in \mathcal{A}$ and $\mathcal{V} \in [\mathcal{U}]^{<\omega}$, then either $\cup \mathcal{V} = 2^\kappa$ or $X \cap (A \setminus \cup \mathcal{V}) \neq \emptyset$. If $\mathcal{U}' \subseteq \mathcal{U}$ such that for each $U \in \cup \mathcal{U}'$, $U \cap \cup \mathcal{U}' = \emptyset$ then $\cup \mathcal{U}'$ is a dense open subset of $2^\kappa$.

Now consider $\mathcal{P}(X, \mathcal{U})$. From the above facts, we have that $\mathcal{P}(X, \mathcal{U})$ is $\sigma$-centered and that for each $A \in \mathcal{A}$, the set

$$\{\langle S, \mathcal{V} \rangle : S \cap (A \setminus \cup \mathcal{V}) \neq \emptyset\}$$

is dense in $\mathcal{P}(X, \mathcal{U})$. Furthermore, for each $U \in \mathcal{U}$ the set

$$\{\langle S, \mathcal{V} \rangle : U \cap \cup \mathcal{V} \neq \emptyset\}$$

is also dense in $\mathcal{P}(X, \mathcal{U})$.

Let $\mathcal{G} \subseteq \mathcal{P}(X, \mathcal{U})$ be a filter which meets each of the dense sets above; and let $G = \cup \{\cup \mathcal{V} : \text{for some } S, \langle S, \mathcal{V} \rangle \in \mathcal{G}\}$. Then $G$ is a dense open set contradicting that $\mathcal{A}$ is a $\pi_\tau$-base for $X$.

Only MA for $\sigma$-centered posets was used above. In the following theorem we use only MA for a countable poset.

**Theorem 10.** Assume MA. If $X$ is separable then $\pi_d(X) = \pi_0(X)$.

**Proof.** Since $\pi(X) \leq 2^{\text{card}(X)} \leq c$, it suffices to show that if $\pi_d(X) = \lambda < c$ then $\pi_0(X) = \lambda$. We suppose $\pi_0(X) = \kappa > \lambda$ and derive a contradiction. By Lemma 3 we can assume that $\pi(X) = \kappa$. We can also assume, without loss of generality, that $X$ is countable and has no isolated points.

Let $\mathcal{A}$ be a $\pi_d$-base for $X$ of cardinality $\lambda$, and let $\mathcal{B}$ be a $\pi$-base for $X$ of cardinality $\kappa$. Let $\mathcal{C}$ be the family obtained from Lemma 2. Let $\mathcal{D}$ be a complete pairwise disjoint subfamily of $\mathcal{C}$ such that $\cup \mathcal{D}$ is dense in $X$. Let $\mathcal{U} = \{D \setminus F : D \in \mathcal{D}$ and $F \in [X]^{<\omega}\}$.

Now consider $\mathcal{P}(X, \mathcal{U})$. This is countable, and each set $\{\langle S, \mathcal{V} \rangle \in \mathcal{P}(X, \mathcal{U}) : S \cap (A \setminus \cup \mathcal{V}) \neq \emptyset\}$, where $A \in \mathcal{A}$, is dense in $\mathcal{P}(X, \mathcal{U})$. Also each set $\{\langle S, \mathcal{V} \rangle \in \mathcal{P}(X, \mathcal{U}) : D \cap \cup \mathcal{V} \neq \emptyset\}$, where $D \in \mathcal{D}$, is also dense. MA allows us to find a filter $\mathcal{G} \subseteq \mathcal{P}(X, \mathcal{U})$ which meets each of the above dense sets.

Let $G = \cup \{\cup \mathcal{V} : \langle S, \mathcal{V} \rangle \in \mathcal{G}$ for some $S\}$. Since $\mathcal{G}$ meets each of the first type of dense set, no $A \in \mathcal{A}$ is contained in $G$. Since $\mathcal{G}$ meets each of the second type of dense set and $X$ has no isolated points, $G$ is a dense open subset of $X$. This contradicts that $\mathcal{A}$ is a $\pi_d$-base for $X$. 89
The result of van Douven that $\pi_d(X) = \omega$ implies $\pi_0(X) = \omega$ can be gleaned from the proof of this last theorem. If $\mathcal{A}$ is a countable $\pi_d$-base for $X$, let $Y$ be a countable subset of $X$ meeting each set in $\mathcal{A}$. Now follow the proof of Theorem 10 for the subspace $\text{Int}(Y)$ of $X$. MA is not needed since only countable many dense sets need to be met. However, van Douwen’s original proof is easier and more straightforward.

We have one more result about $\pi_d$ and $\pi_0$. It uses the following lemma, which is of independent interest.

**Lemma 11.** If $X$ has no isolated points and $c(X) = \omega$, then either there is a Suslin tree of open subsets of $X$ or there is a countable collection of open subsets of $X$ such that for each $F \in [X]^{<\omega}$, $\cup \{C \in \mathcal{C} : C \cap F = \emptyset\}$ is dense.

**Proof.** We build a tree of open subsets of $X$, by recursion on the levels of the tree, starting with $T_0 = \{X\}$. If level $T_\alpha$ has been defined and $t \in T_\alpha$ we define the node of $t$, $N(t)$, to be a maximal non-trivial collection of open subsets of $t$ such that for all $U, V \in N(t)$, $U \cap V = \emptyset$. Let $T_{\alpha+1} = \cup \{N(t) : t \in T_\alpha\}$.

If $\text{lim}(\lambda)$ and we have $T_\alpha$ for all $\alpha < \lambda$, consider the tree $\cup \{T_\alpha : \alpha \in \text{Lim} \}$. For each branch $b$ of this tree consider $\text{Int}(\cap b)$. Let $T_\alpha = \{\text{Int}(\cap b) : b \text{ is a branch of } \cup \{T_\alpha : \alpha < \lambda \} \}$.

Note that since $c(X) = \omega$ this recursion stops after at most $\omega_1$ steps and that the resulting tree $T$ has no uncountable chains or antichains.

If $T$ is not a Suslin tree, then $|T| = \omega$. In this case, let $\mathcal{G} = T$. Since $\mathcal{G}$ is closed under finite intersections, it only remains to prove that for any $x \in X \cup \{C \in \mathcal{G} : x \in C\}$ is dense. To this end let $p \in X$ and show that $p$ is in the closure of $\cup \{C \in \mathcal{G} : x \in C\}$. However, this result is obtained by a straightforward consideration of the ways in which $p$ and $x$ can ”leave” the tree construction and is therefore left for the reader (i.e. it is messy to write out).

Recall that the *Novak number* of a space $X$ is

$$n(X) = \min \{\kappa : X \text{ can be covered by } \kappa \text{ nowhere dense sets}\}.$$

**Corollary 12.** If $X$ has no isolated points, then $n(X) \leq 2^{c(X)}$.

**Proof.** This follows from the proof of the lemma since each element of $T$ and each branch of $T$ determine a nowhere dense set, and their union is all of $X$. The tree $T$ has at most $(c(X))^+ \text{ elements and } 2^{c(X)} \text{ branches.}$

We use Lemma 11 in the following theorem.

**Theorem 13.** Assume MA. If $c(X) = \omega$ and $\pi(X) < c$, then $\pi_d(X) = \pi_0(X)$.

**Proof.** Suppose $\pi_d(X) < \pi_0(X)$. By Lemma 3 we can assume $\pi_0(X) = \pi(X)$. Let $\mathcal{A}$ be a $\pi_d$-base for $X$ of cardinality $\pi_d(X)$. By Lemma 2 there is a $\pi$-base $\mathcal{C}$ such that no finite subcollection of $\mathcal{C}$ covers any element of $\mathcal{A}$. Let $\mathcal{C}_1$ be a maximal pairwise disjoint subcollection of $\mathcal{C}$. By Lemma 11 obtain a countable collection $\mathcal{C}_2$ of open subsets of $X$ such that for each $F \in [X]^{<\omega}$, $\cup \{C \in \mathcal{C}_2 : C \cap F = \emptyset\}$ is dense.

Let $\mathcal{U} = \{C_1 \cap C_2 : C_1 \in \mathcal{C}_1 \text{ and } C_2 \in \mathcal{C}_2\}$. Then $\mathcal{U}$ has the following properties:

(i) no finite subcollection of $\mathcal{U}$ covers an element of $\mathcal{A}$;

(ii) for each $F \in [X]^{<\omega}$, $\cup \{U \in \mathcal{U} : U \cap F = \emptyset\}$ is dense.
Now, consider $\mathcal{P}(X, \mathcal{U})$. Since $\mathcal{U}$ is countable, $\mathcal{P}(X, \mathcal{U})$ is $\sigma$-centered. By property (i) for each $A \in \mathcal{A}$ the set
$$\{\langle S, \mathcal{V} \rangle : S \cap (A \setminus \bigcup \mathcal{V}) \neq \emptyset\}$$
is dense in $\mathcal{P}(X, \mathcal{U})$. Fix a $\pi$-base $\mathcal{B}$ of size $< C$. By property (ii), for each $B \in \mathcal{B}$ the set
$$\{\langle S, \mathcal{V} \rangle : \bigcup \mathcal{V} \cap B \neq \emptyset\}$$
is dense in $\mathcal{P}(X, \mathcal{U})$. Let $\mathcal{G} \subseteq \mathcal{P}(X, \mathcal{U})$ be a filter meeting each of the above dense sets. Let
$$G = \{\bigcup \mathcal{V} : \langle S, \mathcal{V} \rangle \in \mathcal{G} \text{ for some } S\}.$$
Then $G$ is a dense open subset of $X$ witnessing that $\mathcal{A}$ is not a $\pi_\delta$-base for $X$.

We could have eliminated the hypothesis "$\pi(X) < C$" from Theorem 13 if we could have constructed $\mathcal{U}$ in the proof such that it "self-witnessed denseness" as in the proofs of Theorems 9 and 10. We need an extension of Lemma 10, which, in conclusion, we ask as a question.

**QUESTION 14.** Assume MA. Suppose $X$ is a space with $c(X) = \omega$ and no isolated points. Does there exist a countable family $\mathcal{U}$ of open subsets of $X$ with the following two properties:
1. for each finite $F \subseteq X$, $\bigcup \{U \in \mathcal{U} : U \cap F \neq \emptyset\}$ is dense;
2. if $\mathcal{V} \subseteq \mathcal{U}$ such that for each $U \in \mathcal{U}$, $(\bigcup \mathcal{V}) \cap U \neq \emptyset$, then $\bigcup \mathcal{V}$ is dense in $X$?

**REFERENCES**