Abstract. In the New Scottish Book M. Katetov asked whether there exists a Hausdorff space $X$ without isolated points such that every real-valued function on $X$ is continuous at some point? In the paper it is shown that the existence of such a space is equiconsistent to the existence of measurable cardinal.

The "structure theory of ideals" is concerned with the combinatorial properties of ideals on cardinals. It has many connections with large cardinal theory and with infinitary combinatorics. It has recently come into prominence as a separate branch of set theory with the appearance of [1]. Its first non-trivial applications to topology can be found in [18].

Irresolvable spaces (spaces in which any two dense sets meet) were studied extensively first by Hewitt [4]. In the 60's and 70's, El'kin and Malyhin published a number of papers on this subject and its connections with various topological problems. One of the problems considered by Malyhin [14] concerns the existence of irresolvable spaces satisfying the Baire Category Theorem. He noted that there is such a space if and only if there is one on which every real-valued function is continuous at some point. The question about the existence of a Hausdorff space on which every real-valued function is continuous at some point was posed by M. Katetov in [9], and then repeated in 1958 in the New Scottish Book, Problem 109. Here we consider this question from the point of view of ideals theory. It will be convenient to assume (as we have tacitly done above) that all spaces are without isolated points. It will be convenient to assume also, that open sets in the sense of a topology on a cardinal $\kappa$ will have size $\kappa$.

1. Basic topological facts. A space is irresolvable if it does not admit disjoint dense sets. A space is strongly irresolvable if every open subspace is irresolvable.

Let $\kappa$ be a cardinal. A space is $\kappa$-Baire if the intersection of fewer than $\kappa$ dense open sets is dense. Thus Baire spaces are the $\omega_1$-Baire spaces. A ($\kappa$-)SIB is a strongly irresolvable ($\kappa$-)Baire space; $\omega$-SIB's coincide with strongly irresolvable spaces, which exist in ZFC [4].

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PROPOSITION 1.1 ([4]). Every irresolvable space includes a non-empty open strongly irresolvable subspace. In an irresolvable space, every dense set has non-empty interior.

Since $(x-) Baireness is inherited by open sets, existence questions may as well be confined to $(x-) SIB's.

PROPOSITION 1.2. For any space $X$ and a cardinal $\kappa$ the following conditions are equivalent

1) $X$ is $\kappa$-SIB,
2) the intersection of $<\kappa$ dense sets in $X$ is dense,
3) for any space $Y$ of weight $<\kappa$ and for any function $F: X \to Y$ the set of points of continuity of $F$ contains a dense and open set.

The easy proof is left to the reader; the equivalence (1)$\iff$(3) is, in fact, contained in [14].

In a strongly irresolvable space any set is the union of an open set and a nowhere dense set. In particular, any subset of a strongly irresolvable space has the Baire property. On the other hand we have

PROPOSITION 1.3. If $X$ is $\kappa$-Baire and each subset has the Baire property, then there is an extension $\tau$ of the topology on $X$ such that $(X, \tau)$ is $\kappa$-SIB.

Proof. Let us denote the topology on $X$ by $\Theta$. The topology $\tau$ in question is generated by sets of the form $U - E$, where $U \in \Theta$ and $E$ is of first category in $\Theta$. Indeed, the space $(X, \tau)$ has no isolated points; any set of first category in $\Theta$ becomes nowhere dense in $\tau$.

CLAIM. If $B$ is a boundary set in $\tau$, then $B$ is of first category in $\Theta$.

Indeed, since each subset of $X$ has the Baire property, $B = (U - E) \cup F$, where $U \in \Theta$ and $E, F$ are of first category in $\Theta$. Since $B$ is boundary in $\tau$, $U - E = \emptyset$. Hence $B = F$ and the claim is proved.

Now, to prove that $(X, \tau)$ is $\kappa$-SIB, take boundary sets $B_\alpha$, where $\alpha < \lambda$ and $\lambda < \kappa$, in $\tau$. By the claim, $\bigcup \{ B_\alpha : \alpha < \lambda \}$ is the union of $\leq \omega \cdot \lambda = \lambda$ nowhere dense sets in $\Theta$. Since $X$ is $\kappa$-Baire, $\bigcup \{ B_\alpha : \alpha < \lambda \}$ is boundary in $\Theta$. It is also boundary in $\tau$. So, again by the claim, $\bigcup \{ B_\alpha : \alpha < \lambda \}$ is of first category in $\Theta$ and therefore is nowhere dense in $\tau$.

NOTE. The proof of Proposition 1.3 for $\kappa = \omega_1$ was performed using only the countable axiom of choice.

2. Basic facts about ideals. Let $\kappa$ be an uncountable cardinal. A set $I \subset \mathcal{P}(\kappa)$ is an ideal over $\kappa$ if

1) $\alpha \in I$ for all $\alpha < \kappa$ and $\kappa \notin I$,
2) if $X \in I$ and $Y \subset X$, then $Y \in I$,
3) if $X \in I$ and $Y \in I$, then $X \cap Y \in I$.

Let $I$ be a given ideal over a cardinal $\kappa$. Then $I^+ = \{ X \subset \kappa : X \notin I \}$ and $I^* = \{ X \subset \kappa : \kappa - X \notin I \}$. A family $R \subset I^+$ is I-almost disjoint if for every distinct $S, S' \in R$, $S \cap S' \in I$; $R$ is I-dense if for every $X \in I^+$ there is an $S \in R$ such that $\forall S \in R$. 99
$\exists - \forall \in I$; $R$ is I-proper if for any finite subfamily $R' \subset R$ either $\cap R' = \emptyset$ or $\cap R' \in I^+$. 

$I$ is $\lambda$-saturated if every $I$-almost disjoint collection has size $< \lambda$. $I$ is $\lambda$-complete if it is closed under unions of size $< \lambda$. Let $A \in I^+$. An I-partition of $A$ is a maximal $I$-almost disjoint family of subsets of $A$. An I-partition $P_2$ of $A$ is a refinement of an I-partition $P_1$ of $A$, $P_1 \leq P_2$, if every $X \in P_2$ is a subset of some $Y \in P_1$.

The ideal $I$ is weakly-precipitous iff it is $\omega_1$-complete and whenever $A$ is in $I^+$ and $\{P_n : n \in \omega\}$ are I-partitions of $A$ such that $P_0 \subseteq P_1 \subset \ldots \subseteq P_n \subseteq \ldots$, then there exists a sequence of sets $W_0 \supseteq W_1 \supseteq \ldots \supseteq W_n \supseteq \ldots$, such that $W_n \in P_n$ for each $n$, and $\cap \{W_n : n \in \omega\} \neq \emptyset$. It should be noted that our definition of weakly precipitous ideals differs from the notion of precipitous ideal used by other authors ([8], [20]) only by assumption of $\omega_1$-completeness instead of $\kappa$-completeness. For weakly precipitous ideals, the associated Boolean ultrapower is well-founded.

**THEOREM 2.0** (T. Jech, M. Magidor, W. Mitchell and K. Prikry [7]). If $\kappa$ is a regular cardinal that carries a weakly precipitous ideal, then there is a measurable cardinal in some transitive model of ZFC.

Originally, the theorem was formulated for precipitous ideals but the proof works for weakly precipitous ideals, as well.

The ideal $I$ has a lifting if there is a homomorphism $h : \mathcal{P}(\kappa)/I \to \mathcal{P}(\kappa)$ of the factor Boolean algebra $\mathcal{P}(\kappa)/I$ into the power algebra $\mathcal{P}(\kappa)$ such that $h(\xi) \in \xi$ for each $\xi \in \mathcal{P}(\kappa)/I$.

**THEOREM 2.1.** For any cardinal $\kappa$, the following conditions are equivalent

1. there is a topology on $\kappa$ which is $\lambda$-SIB,
2. there is a $\lambda$-complete ideal $I$ over $\kappa$ for which there is a family $R$ which is both $I$-dense and $I$-proper,
3. there is a $\lambda$-complete ideal over $\kappa$ which has a lifting.

**Proof.** We shall show that (1)$\iff$(2), (1)$\implies$(3) and (3)$\implies$(2).

Suppose $\tau$ is a topology on $\kappa$ which is $\lambda$-SIB. Then the ideal $I$ consisting of nowhere dense subsets of $\kappa$ with respect to $\tau$ is $\lambda$-complete while the family $\tau$ is both $I$-dense and $I$-proper.

To prove the converse, having a $\lambda$-complete ideal $I$ over $\kappa$ and a family $R$ which is both $I$-dense and $I$-proper, define the topology $\tau$ on $\kappa$ by declaring as open base sets of the form $\cap P - X$, where $P \in [R]^{<\omega}$ and $X \in I$. To prove that $\tau$ is $\lambda$-SIB it suffices only to check that dense sets in $\tau$ are in $I^*$. Thus (1)$\iff$(2) is proved.

To prove that (1)$\implies$(3), suppose again that $\tau$ is a topology on $\kappa$ which is $\lambda$-SIB and $I$ is the ideal consisting of nowhere dense sets in $\tau$. Then $I$ is $\lambda$-complete. Note that if $\tau'$ is any extension of the topology $\tau$ such that $\tau'$ has no isolated points, then there are no "new" boundary (= nowhere dense) sets in $\tau'$. By the Kuratowski-Zorn Lemma, there is a maximal dense-in-itself topology $\tilde{\tau}$ on $\kappa$ extending the topology $\tau$. By Proposition 1.2, this topology is also $\lambda$-SIB. Since $\tilde{\tau}$ is maximal, it is, in particular, extremally disconnected. Let us put $h(\xi)$ to be the closed-open set with respect to $\tilde{\tau}$ which is in $\xi \in \mathcal{P}(\kappa)/I$. Then $h$ is the required homomorphism for $I$ to have a lifting.
To prove \((3) \Rightarrow (2)\), observe that if \(h : \mathcal{P}(\kappa)/I \to \mathcal{P}(\kappa)\) is a lifting for \(I\), then \(\{h(\xi) : \xi \in \mathcal{P}(\kappa)/I\}\) is a family which is both \(I\)-dense and \(I\)-proper.

If \(X\) is a space, then \(F \subseteq X\) is a \(P(\lambda)\)-set iff it is closed and for any family \(R\) of neighborhoods of \(F\), \(|R| < \lambda\), there is \(R \cap F \subseteq \text{int} R\); \(P(\omega_1)\)-sets are called \(P\)-sets.

**Lemma 2.2.** Let \(x\) be regular and \(F \subseteq \mathcal{U}(x)\) a \(P(\lambda)\)-set which is a retract of \(\beta x\). Then \(I_F = \{A \subseteq x : \text{cl}_{P(x)} A \cap F = \emptyset\}\) is a \(\lambda\)-complete ideal over \(x\) for which there is a family \(R\) which is both \(I_F\)-dense and \(I_F\)-proper.

**Proof.** If \(r : \beta x \to F\) is a retraction, then the family \(R\) in question consists of sets of the form \(x \cap r^{-1}(V)\), where \(V\) is a non-empty closed-open subset of \(F\). Indeed, \(x \cap r^{-1}(V)\) is a dense subset of \(r^{-1}(V)\) and \(\emptyset \neq V \subseteq r^{-1}(V)\); thus \(x \cap r^{-1}(V) \in I_F^+\). If \(V_1, \ldots, V_n \in R\), say \(V_i = x \cap r^{-1}(U_i)\), then \(V_1 \cap \ldots \cap V_n = x \cap r^{-1}(U_1 \cap \ldots \cap U_n)\) is either empty (if \(U_1 \cap \ldots \cap U_n = \emptyset\)) or belongs to \(I_F^+\) (if \(U_1 \cap \ldots \cap U_n \neq \emptyset\)). The above two properties of \(R\) show that \(R\) is \(I_F\)-proper. To prove that \(R\) is also \(I_F\)-dense, let \(A \in I_F\). Then \(U = \text{cl}_{P(x)} A \cap F \neq \emptyset\) is closed-open in \(F\). We will show that \(B = x \cap r^{-1}(U) - A \in I_F\). Assume otherwise, then \(V = \text{cl}_{P(x)} B \cap F \neq \emptyset\). Since \(A \cap B = \emptyset\), \(U \cap V = \emptyset\). Since \(r\) is a retraction, \(r^{-1}(U)\) is closed-open in \(\beta x\) and therefore it contains \(V\). Since \(r^{-1}(U) \cap F = U\), \(\emptyset \neq V \cap F \subset U\). But then \(U \cap V \neq \emptyset\); a contradiction.

**Note 2.3.** It should be remarked that if the retraction \(r\), as above, is \(1-1\) on \(x\), then the topology \(\tau\) on \(x\) induced by constructed family \(R\) for \(I_F\) is Hausdorff.

**Lemma 2.4.** Suppose that \(\tau\) is a \(\lambda\)-SIB topology on \(x\) such that any subset of \(x\) of size \(< \kappa\) is nowhere dense in \(\tau\). Then there exists a \(P(\lambda)\)-set in \(\mathcal{U}(x)\) which is a retract of \(\beta x\).

**Proof.** We may assume that \((x, \tau)\) is an extremally disconnected space. For \(\alpha < \kappa\) let \(\xi_\alpha = \{x \subseteq x : x \in \text{cl}_{\text{int}_{\tau}} x\}\). Then \(\xi_\alpha\) is a uniform ultrafilter on \(x\). Let \(Y = \{\xi_\alpha : \alpha < \kappa\} \subseteq \mathcal{U}(x)\).

**Claim.** For any \(x \subseteq x\), \(Y \cap \text{cl}_{P(x)} x = \emptyset\) iff \(x\) is nowhere dense in \(\tau\).

Indeed, \(Y \cap \text{cl}_{P(x)} x = \emptyset\) iff \(x \notin x\) for each \(\alpha < \kappa\) iff \(x \notin \text{cl}_{\text{int}_{\tau}} x\) for each \(\alpha < \kappa\) iff \(\text{cl}_{\text{int}_{\tau}} x = \emptyset\) iff \(\text{int}_{\tau} x = \emptyset\) iff \(x\) is boundary in \(\tau\) iff \(x\) is nowhere dense in \(\tau\).

Define \(r : x \cup Y \to Y\) by setting \(r(y) = \xi_y\) if \(y \in x\) and \(r(y) = y\) if \(y \in Y\). The function \(r\) is continuous. To see this, it suffices to show that if \(U\) is a closed-open subset of \(Y\), then \(Y \cap \text{cl}_{P(x)} x = \emptyset\) for each \(\alpha < \kappa\) iff \(x \notin \text{cl}_{\text{int}_{\tau}} x\) for each \(\alpha < \kappa\) iff \(\text{cl}_{\text{int}_{\tau}} x = \emptyset\) iff \(\text{int}_{\tau} x = \emptyset\) iff \(x\) is boundary in \(\tau\) iff \(x\) is nowhere dense in \(\tau\).

Define \(r : x \cup Y \to Y\) by setting \(r(y) = \xi_y\) if \(y \in x\) and \(r(y) = y\) if \(y \in Y\). The function \(r\) is continuous. To see this, it suffices to show that if \(U\) is a closed-open subset of \(Y\), then \(Y \cap \text{cl}_{P(x)} (x \cap r^{-1}(U)) = U\). For this purpose take a set \(x \subseteq x\) such that \(U = Y \cap \text{cl}_{P(x)} x\). Then \(U = \{\xi_\alpha : \alpha \in \text{cl}_{\text{int}_{\tau}} x\}\) and hence \(x \cap r^{-1}(U) = \{\alpha : \xi_\alpha \in U\} = \{\alpha : \alpha \in \text{cl}_{\text{int}_{\tau}} x\}\). The set \(A = (x - \text{cl}_{\text{int}_{\tau}} x) \cup (\text{cl}_{\text{int}_{\tau}} x - x)\) is nowhere dense in \(\tau\) and, by Claim, \(Y \cap \text{cl}_{P(x)} (x \cap r^{-1}(U)) = Y \cap \text{cl}_{P(x)} x = U\); the continuity of \(r\) is proved.

There exists a continuous extension \(\beta r : \beta x \to \text{cl}_{P(x)} Y\) of \(r\). Since \(r|Y = \text{id}\), \(\beta r|\text{cl}_{P(x)} Y = \text{id}\). Thus \(\beta r\) is a retraction. Since the topology \(\tau\) is \(\lambda\)-SIB we get, by the claim, that \(\text{cl}_{P(x)} Y\) is a \(P(\lambda)\)-set in \(\mathcal{U}(x)\).

**Note 2.5.** It should be remarked that if the topology \(\tau\), above, is Hausdorff, then the constructed retraction is \(1-1\) on \(x\).

Finally, by Lemmas 2.2 and 2.4 and Notes 2.3 and 2.5 we have

**Theorem 2.6.** There exists a (Hausdorff) \(\lambda\)-SIB iff there exist a cardinal \(\kappa\), a \(P(\lambda)\)-set \(F \subseteq \mathcal{U}(x)\) and a retraction \(r : \beta x \to F\) (which is \(1-1\) on \(x\)).
It is a well-known fact (proved probably first by Gleason [3]) that any compact extremally disconnected space of the density $\kappa$ can be embedded in $\beta\kappa$ as a retract of $\beta\kappa$. Moreover, a retraction can be chosen to be $1-1$ on $\kappa$.

NOTE 2.7. In the case when an ideal $I$ has a family which is both $I$-dense and $I$-proper, the resulting $(\lambda)$-SIB topology $\tau$ is such that $I$ coincides with the family of all nowhere dense sets in $\tau$.

PROPOSITION 2.8. If $I$ is a $\kappa$-complete ideal over $\kappa$ which has an $I$-dense family of size $\leq \kappa$, then there is a family $R \subset I^+$ which is both $I$-dense and $I$-proper.

Proof. If $S = \{a_\alpha : \alpha < \kappa\}$ is $I$-dense, then $R = \{b_\alpha : \alpha < \kappa\}$, where $b_\alpha = a_\alpha - \bigcup \{a_\xi \cap \bigcap \{a_\zeta : \zeta \in s\} : s \in [\kappa]^{<\omega}\} \text{ and } a_\alpha \cap \bigcap \{a_\zeta : \zeta \in s\} \in I$ is both $I$-dense and $I$-proper.

PROPOSITION 2.9. Let $I$ be an ideal over $\kappa$ for which there is a family $R$ which is both $I$-dense and $I$-proper. Then $I$ is $\kappa^+$-saturated. If, in addition, $I$ is $\omega_1$-complete, then $I$ is weakly precipitous.

Proof. Both parts of the proposition follow easily from the following observation:

For any $I$-partition $P_1$ of $A \in I^+$ there exists an $I$-partition $P_2$ of $A$ which consists of disjoint sets and is a refinement of $P_1$.

Before we state our next result we introduce the following (standard) notation:

sat($I$) = $\inf \{\lambda : I$ is $\lambda$-saturated\}; if $X, Y, Z_t, t \in T$, are sets, then $\chi(Y) = \{f : f$ is a function and $\dom(f) = X$ and $\ran(f) \subset Y\}$, $\Pi(Z_t : t \in T) = \{f : f$ is a function and $\dom(f) = T$ and $f(t) \in Z_t\}$.

THEOREM 2.10. Let $I$ be a $\kappa$-complete ideal over $\kappa$ such that the Boolean algebra $\mathcal{P}(\kappa)/I$ is atomless. If either $\text{sat}(I) < \kappa$ or $\kappa$ is weakly compact and $\text{sat}(I) < \kappa$, then there are $I$-partitions $P_a, \alpha < \lambda$, of $\kappa$ such that:

(i) if $\alpha < \beta < \lambda$, then $P_\alpha \leq P_\beta$;

(ii) if $\varphi \in \Pi\{P_\alpha : \alpha < \lambda\}$, then $|\cap \{\varphi(\alpha) : \alpha < \lambda\}| < 1$.

Proof. In both cases $\kappa$ cannot be inaccessible. So there exists the smallest $\lambda$ that $2^\lambda \geq \kappa$. Let $f : \kappa \rightarrow 2^2$ be 1-1. For any $\alpha < \kappa$ and $\varphi \in 2^2, A_{\varphi, \alpha} = \{\psi \in 2^2 : \varphi \subset \psi\}$. We set $P_\alpha = \{f^{-1}(A_{\varphi, \alpha}) : \varphi \in 2^2$ and $f^{-1}(A_{\varphi, \alpha}) \in I^+\}$. Clearly, conditions (i) and (ii) are then satisfied. It remains to prove that each $P_\alpha, \alpha < \lambda$, is an $I$-partition of $\kappa$. To see this, note that for any $\alpha < \lambda$, $2^\alpha < \kappa$. Thus $|P_\alpha| < \kappa$ for every $\alpha < \lambda$. Since $\bigcup \{A_{\varphi, \alpha} : \varphi \in 2^2\} = \kappa^2$ and $I$ is $\kappa$-complete, $P_\alpha$ is an $I$-partition of $\kappa$, for every $\alpha < \kappa$.

3. Existence and consistency results. We shall present some methods for obtaining $\lambda$-complete ideals over regular cardinals having both dense and proper sets. Then we exhibit some consistent examples.

One of the methods is the one using maximal $\theta$-independent families, with $\theta \geq \omega_1$, introduced and considered by K. Kunen in [12].

Let $\kappa$ and $\theta$ be infinite cardinals with $\theta$ regular. A family $S \subset \mathcal{P}(\kappa)$ is called $\theta$-independent iff whenever $S_0, S_1 \subset S$ with $S_0 \cap S_1 = \emptyset$ and $|S_0 \cup S_1| < \theta$, we have $\bigcap \{A \in S_0\} \cap \bigcap \{\kappa - A : A \in S_1\} \neq \emptyset$. A family $S \subset \mathcal{P}(\kappa)$ is called maximal $\theta$-independent iff it is $\theta$-independent but no proper superset is.

It will be convenient to introduce the following notation. $\text{Fn}(X, Y, \theta) = \{p : p$ is a function and $\dom(p) \subset X$ and $\ran(p) \subset Y$ and $0 < |p| < \theta\}$. For $S \subset \mathcal{P}(\kappa)$ and
If \( p \in \text{Fn}(S, 2, \emptyset) \) we set \( \varphi(p) = \cap \{ A : A \in \text{dom}(p) \text{ and } p(A) = 1 \} \cap \{ x - A : A \in \text{dom}(p) \text{ and } p(A) = 0 \} \). Then \( S \subset \mathcal{P}(\kappa) \) is \( 0 \)-independent iff \( \varphi(p) \neq \emptyset \) for all \( p \in \text{Fn}(S, 2, \emptyset) \). The following is proved in [12].

**THEOREM 3.1.** Suppose \( S \subset \mathcal{P}(\kappa) \) is a maximal \( 0 \)-independent family with \( |S| \geq \theta \geq \omega_1 \). Then there is a \( \lambda \leq \kappa \) and \( S' \subset \mathcal{P}(\lambda) \) such that:

1. \( S' \) is maximal \( 0 \)-independent and \( |S'| \geq \theta \),
2. the family \( J = \{ X \subset \lambda : \neg (\exists p \in \text{Fn}(S', 2, \emptyset) (\varphi(p) \subset X)) \} \) is a \( \lambda \)-complete ideal over \( \lambda \).

From the definition of the ideal \( J \) it follows that the family \( R = \{ \varphi(p) : p \in \text{Fn}(S', 2, \emptyset) \} \) is \( J \)-dense. Moreover, if \( p, q \in \text{Fn}(S', 2, \emptyset) \) are such that \( \varphi(p) \cap \varphi(q) \neq \emptyset \), then there is an \( r \in \text{Fn}(S', 2, \emptyset) \) which extends \( p \) and \( q \) and \( \varphi(r) = \varphi(p) \cap \varphi(q) \). Thus \( R \) is \( J \)-proper. According to Theorem 2.1, there is a topology \( \tau \) on \( \lambda \) which is \( \lambda \)-SIB. Let us look at the topology \( \tau \) whose base is formed by the sets \( \varphi(p, p) \in \text{Fn}(S', 2, \emptyset) \). This topology is slightly weaker than the topology \( \tau \) but is still \( \lambda \)-SIB: \( B \) is boundary in \( \tau \) iff \( \neg (\exists p \in \text{Fn}(S', 2, \emptyset) (\varphi(p) \subset B)) \) iff \( B \in J \).

In the sense of the topology \( \tau \) the sets \( \varphi(p) \), \( p \in \text{Fn}(S', 2, \emptyset) \) are closed-open. We shall show that for some \( p_0 \in \text{Fn}(S', 2, \emptyset) \), the topology \( \tau \) restricted to \( \varphi(p_0) \) is \( T_1 \), thus \( 0 \)-dimensional and completely regular. For this purpose, for any \( \alpha < \lambda \) we put \( s_\alpha = \cap \{ \varphi(p) : p \in \text{Fn}(S', 2, \emptyset) \} \). Then \( \{ s_\alpha : \alpha < \lambda \} \) is a decomposition of \( \lambda \) into sets belonging to \( J \). Observe also that \( A = \{ \alpha : s_\alpha = \{ \alpha \} \} \in J^+ \). Indeed, let \( X \) be a set intersecting each \( s_\alpha \) in precisely one point. By maximality of \( S' \), there is a \( p_0 \in \text{Fn}(S', 2, \emptyset) \) such that \( \varphi(p_0) \subset X \) or \( \varphi(p_0) \subset \lambda - X \). Clearly, there must be \( \varphi(p_0) \subset A \). Therefore we have shown

**THEOREM 3.2.** If a maximal \( \theta \)-independent family \( S \) exists with \( |S| \geq \theta \geq \omega_1 \), then there is a cardinal \( \lambda > \omega \) and a topology \( \tau \) on \( \lambda \) which is completely regular \( 0 \)-dimensional and \( \lambda \)-SIB.

In [12] it is also shown that if ZFC plus the existence of measurable cardinal is consistent, so is ZFC plus the existence of a maximal \( \omega_1 \)-independent family \( S \subset \mathcal{P}(\mathcal{P}(\mathbb{N})) \). Hence

**THEOREM 3.3.** If ZFC is consistent with the existence of a measurable cardinal, so is ZFC with the existence of a completely regular \( 0 \)-dimensional SIB.

In [15] it is shown that the consistency of ZFC + an inaccessible implies the consistency of ZF + "each subset of reals has the Baire property". By Note and Proposition 1.3 we have

**THEOREM 3.4.** \( \text{Con}(\text{ZFC} + \exists \text{inaccessible}) \Rightarrow \text{Con}(\text{ZF} + \exists \alpha \text{ SIB extension of the topology on reals}) \).

Let us make a remark on spaces being regular SIB's. If \( X \) is an irresolvable space such that any non-empty open set has size \( \geq \kappa \), then \( X \) has \( \pi \)-weight \( > \kappa \). Thus if \( X \) is also regular, the regular open algebra of \( X \) cannot have a dense set of size \( \leq \kappa \). On particular, the SIB topology obtained from Proposition 2.8 is not regular.

Now, we shall present a method which yields a \( T_2 \) \( \lambda \)-SIB.

**THEOREM 3.5.** Let \( I \) be a \( \kappa \)-complete ideal over \( \kappa \) such that the Boolean algebra
\(\mathcal{P}(x)/I\) is atomless and either sat(I) < x or sat(I) ≤ x and x is weakly compact. If I has a lifting, then there is a \(\kappa\)-SIB topology on some Z ∈ I* which is Hausdorff.

Proof. Let \(I\)-partitions \(P_\alpha, \alpha < \lambda,\) of \(x\) fulfil the conditions (i) and (ii) from Theorem 2.10. By Theorem 2.1 and Note 2.7, there is a \(\kappa\)-SIB topology \(\tau\) on \(x\) such that the ideal I coincides with the family of all nowhere densesets in \(\tau\). Let int\(P_\alpha = \{\text{int}X: X \in P_\alpha\}\). Thus \(x - \bigcup \text{int}P_\alpha \in I\) for all \(\alpha < \lambda\). Since \(\lambda < x\), the set \(Z = \bigcap \{\cup \text{int}P_\alpha: \alpha < \lambda\}\) is in \(I^*\). The subspace \(Z\) of the space \((x, \tau)\) is Hausdorff, since the condition (ii) holds.

COROLLARY 3.6. Con\((ZFC + \exists\) real-valued measurable cardinal) ⇒ Con\((ZFC + + \exists \omega < \kappa \leq 2^\omega \exists\) topology \(\tau\) on \(x\) ((\(x, \tau\) is Hausdorff \(\kappa\)-SIB)).

Proof. If it is consistent with ZFC that a real-valued measurable cardinal exists, then it is consistent with ZFC that there is a \(\kappa \leq 2^\omega\) and a measure \(\mu\) on \(\mathcal{P}(x)\) which is \(\kappa\)-complete, finite and atomless. Thus the ideal I of null sets with respect to \(\mu\) is \(\kappa\)-complete and the factor algebra \(\mathcal{P}(x)/I\) is atomless. Clearly, sat(I) = \(= \omega_1 < x\). Moreover, by Maharam’s result [13] (see also [19]), I has a lifting. Applying the preceding theorem to I, we get the required result.

COROLLARY 3.6. The following conditions are pairwise equiconsistent:

1. \(\exists\) measurable cardinal,
2. \(\exists\) real-valued measurable cardinal,
3. \(\exists\) SIB topology on a regular \(x\).

Proof. The equiconsistency of (1) and (2) is due to R. Solovay [16]. That the consistency of (2) implies the consistency of (3) is in Corollary 2.5. If we have a SIB topology on a regular cardinal \(x\), then, by Theorem 2.1 and Proposition 2.9, \(x\) admits a weakly precipitous ideal. But the existence of a regular cardinal \(x\) with such an ideal implies the consistency of a measurable cardinal (see Theorem 2.0).

We have constructed Hausdorff SIB topologies in two cases, both on cardinals which are not inaccessible. The Hausdorff property does not indicate greater strength in the next proposition.

PROPOSITION 3.7. If there is a \(\kappa\)-SIB topology \(\tau\) on \(x\) and \(x\) is not inaccessible then there is a dense open in \(\tau\) set \(Z \subset x\) such that the subspace \(Z\) is Hausdorff.

Proof. Let \(\lambda < x\) be such that \(2^\lambda \geq x\). Take an arbitrary 1-1 function \(f: x \rightarrow \mathbb{Z}_2\). For \(\alpha < \lambda\) and \(i \in \{0, 1\}\) let \(A(\alpha, i) = \{p \in \mathbb{Z}_2: p(\alpha) = i\}\). Since \(\tau\) is strongly irresolvable, \(D_\alpha = \text{int}\left( f^{-1}(A(\alpha, 0)) \right) \cup \text{inf}(f^{-1}(A(\alpha, 1)))\) is dense. Since \(\lambda < x\) and \(\tau\) is \(\kappa\)-SIB, \(Z = \text{int}\left( \bigcap \{D_\alpha: \alpha < x\}\right)\) is dense. Since \(f\) is 1-1, the subspace \(Z\) is Hausdorff.

Observe that if we are given a \(\kappa\)-complete non-principal ultrafilter \(\xi\) on \(x\), where \(x > \omega\), then \(\xi \cup \{\emptyset\}\) is a \(\kappa\)-SIB topology on \(x\), since dense sets must be in the ultrafilter. This topology is only \(T_1\). One can ask whether the existence of a measurable cardinal implies the existence of a Hausdorff SIB topology on some cardinal \(x\). The answer is no under \(V = L[U]\) (see [10] or [6] for the definition).

THEOREM 3.8. If \(V = L[U]\), then there is no Hausdorff SIB on a regular cardinal.

Sketch of the proof. If \(U\) is the unique normal ultrafilter on \(x\) in \(L[U]\) then it can be shown by standard methods, that the only regular cardinal in \(L[U]\) which bears a weakly precipitous ideal is the measurable cardinal \(x\). Moreover, S. Wagon proved [20] that in \(L[U]\) any (weakly) precipitous ideal \(I\) over \(x\) is atomic,
i.e., there is an $I$-partition $P$ of $\kappa$ such that $I|A = \{B \in \kappa : B \cap A \in I\}$ is prime for all $A \in P$. But the existence of a Hausdorff SIB on a regular cardinal yields a weakly precipitous ideal on that cardinal which is nowhere atomic.

**PROPOSITION 3.9.** If $V=L$, then any space with $|X|$ a regular cardinal is the union of countably many boundary subsets.

**Proof.** Immediate, by Propositions 1.2 and 2.9 and Theorem 2.0.

4. **SIB’S ON $\omega_1$.** We shall now turn our attention to the SIB topologies on sets of size $\omega_1$. We do this again from the point of view of the theory of ideals. To explain our results we need some new definitions and facts.

A $\kappa$-complete ideal $I$ over $\kappa$ will be called a $P$-ideal (selective) provided that every $f \in {}^\kappa \kappa$ is either constant on a set from $I^+$ or there is an $X \in I^*$ such that $|f^{-1}(\alpha) \cap X| < \kappa$ $(|f^{-1}(\alpha) \cap X| \leq 1)$ for every $\alpha < \kappa$. A function $f \in {}^\kappa \kappa$ is called incompressible for the ideal $I$ provided that $f^{-1}(\alpha) \in I$ for every $\alpha < \kappa$ and for every $g \in {}^\kappa \kappa$, if $\{\alpha < \kappa : g(\alpha) < f(\alpha)\} \in I^+$, then $g$ is constant on a set from $I^+$.

A $\kappa$-complete ideal $I$ on $\kappa$ is normal provided that the identity on $\kappa$ is incompressible for $I$.

It is well-known that the ideal $\text{NS}_\kappa$ of non-stationary subsets of a regular cardinal $\kappa$ is normal and it is contained in any normal ideal over $\kappa$ [5].

**THEOREM** (Taylor, [17]). If $I$ is a $\kappa$-complete $\kappa^+$-saturated ideal over a successor cardinal $\kappa$, then $I$ is a $P$-ideal.

**THEOREM** (Solovay, [16]). If $I$ is a $\kappa$-complete $\kappa^+$-saturated ideal over $\kappa$, then there is a function $f \in {}^\kappa \kappa$ that is incompressible for $I$ and such that $f_*^*(I) = \{X \subset \kappa : f^{-1}(X) \in I\}$ is a normal $\kappa^+$-saturated ideal over $\kappa$.

**THEOREM** (Węglorz [21]). If $I$ is a normal ideal over $\kappa$, then $I$ is a selective ideal.

**THEOREM 4.1.** Suppose there is a topology $\tau$ on $\omega_1$ which is SIB. Then there is a stationary set $S \subset \omega_1$ and a topology $\theta$ on $S$ which is SIB and such that $J = \{X \subset \omega_1 : X \cap S$ is nowhere dense in $\theta\}$ is a normal ideal over $\kappa$.

**Proof.** The ideal $I$ of nowhere dense in $\tau$ subsets of $\omega_1$ is an $\omega_1$-complete $\omega_2$-saturated ideal over $\omega_1$. By Solovay’s theorem, there is an $f \in {}^{\omega_1} \omega_1$ which is incompressible for $I$. By Taylor’s theorem, there is an $X \in I^*$ such that $|f^{-1}(\alpha) \cap X| \leq \omega$ for every $\alpha < \omega_1$. Hence $X$ is the union of countably many sets on which $f$ is 1-1. Thus one of them, say $A$, belongs to $I^+$. Clearly, $U = \text{int}_\tau A \neq \emptyset$, so $U \in I^+$. Since $f_*^*(I)$ is normal, $S = f(U)$ is stationary. The topology $\theta$ on $S$ is defined in such a way that $f|U$ becomes a homeomorphism, i.e., $V \subset S$ is in $\theta$ iff $f^{-1}(V)$ is in $\tau$. Since $U$, as an open subspace of a SIB, is SIB, $\theta$ is a SIB on $S$. Since the ideal $f_*^*(I)$ is normal, the ideal $J = f_*^*(I)|S$ is normal, as well.

Taking a function $h \in {}^{\omega_1} S$ which is 1-1 and onto and defining a topology $\Sigma$ on $\omega_1$ in such a way that $h$ becomes a homeomorphism from $(\omega_1, \Sigma)$ onto $(S, \theta)$, we get that $\Sigma$ is a SIB and the ideal of nowhere dense sets in $\Sigma$ is selective. As is easy to observe, such a topology must be rigid.

Now we shall consider the question of the existence of a SIB on $\omega_1$. Note that the existence of any SIB on $\omega_1$ induces a Hausdorff one, according to Proposition 3.7.
A special case of \( \omega_2 \)-saturated ideals over \( \omega_1 \) is those which have a dense set of size \( \omega_1 \) (see [22] for a relative consistency of the existence of such ideals). According to Proposition 2.8, if \( I \) is an \( \omega_1 \)-complete ideal over \( \omega_1 \) which has an \( I \)-dense set of size \( \omega_1 \), then there is a family which is both \( I \)-dense and \( I \)-proper. In such a case, by Theorem 2.1 there is a SIB topology on \( \omega_1 \). However we can obtain a SIB topology on \( \omega_1 \) from such a dense set in another fashion. First we need the following, due to A. Taylor [17, Theorems 7.4 and 7.7]

**Lemma 4.3.** If there is an \( \omega_1 \)-complete ideal over \( \omega_1 \) which has a dense set of size \( \omega_1 \), then there is one which is selective and which has a dense set \( \{ D_x : \alpha < \omega_1 \} \) such that either \( D_\alpha \cap D_\beta = \emptyset \) or \( D_\alpha \subset D_\beta \) or \( D_\beta \subset D_\alpha \) for all \( \alpha, \beta < \omega_1 \).

**Theorem 4.4.** Suppose that there is an \( \omega_1 \)-complete ideal \( I \) over \( \omega_1 \) which has a dense set of size \( \omega_1 \). Then there is a topology \( \tau \) on \( \omega_1 \) which is Hausdorff and SIB.

**Proof.** Without loss of generality we may assume that \( I \) is selective and that \( \{ D_x : \alpha < \omega_1 \} \) is \( I \)-dense and for all \( \alpha, \beta < \omega_1 \), either \( D_\alpha \cap D_\beta = \emptyset \) or \( D_\alpha \subset D_\beta \) or \( D_\beta \subset D_\alpha \). Let us define a relation \( r \) on \( \omega_1 \) setting \( \alpha r \beta \) iff \( \alpha \in D_\gamma \iff \beta \in D_\gamma \) for all \( \gamma < \omega_1 \). Then \( r \) is an equivalence relation over \( \omega_1 \) and each equivalence class of \( r \) is in \( I \). By selectivity of \( I \), there is an \( S \in I^* \) which has precisely one point in common with each equivalence class of the relation \( r \). Since \( \{ D_x : \alpha < \omega_1 \} \) may be chosen so that \( D_\alpha \in I^* - I^* \) for all \( \alpha < \omega_1 \), the topology \( \tau \) on \( S \) generated by sets \( D_\alpha \) and their complements has no isolated points. Moreover, by the choice of \( S \) it is Hausdorff. Now, the topology \( \tau \) in question is generated by sets of the form \( U - A \), where \( U \) is open in \( \tau \) and \( A \in I \). Clearly, \( \tau \) being an extension of a \( T_2 \) topology is again \( T_2 \). If \( A \in I \), then \( A \cap S \) is nowhere dense in \( \tau \). To prove that \( \tau \) is a SIB it suffices to show, by virtue of \( \omega_1 \)-completeness of \( I \), that if \( B \) is boundary in \( \tau \), then \( B \in I \). To prove this, take \( B \in I^+ \). Then there is an \( \alpha < \omega_1 \) such that \( A = D_\alpha \setminus B \in I \). Hence \( D_\alpha - A \subset B \) and \( D_\alpha - A \) is non-empty and open in \( \tau \).

**Problem.** A family \( \{ U_x : \alpha < \lambda \}, \lambda \leq \kappa \), of ultrafilters on \( \kappa \) is a complete family of ultrafilters iff \( \bigcap \{ U_x : \alpha < \lambda \} \) is a \( \kappa \)-complete filter. A problem of Taylor is: what implications are reversible among: \( \exists \omega_1 \)-complete ideal on \( \omega_1 \) which has a dense set of size \( \omega_1 \) \( \Rightarrow \exists \) complete family of ultrafilters on \( \omega_1 \) \( \Rightarrow \exists \omega_2 \)-saturated \( \omega_1 \)-complete ideal on \( \omega_1 \)?

The statement: \( \exists \) SIB on \( \omega_1 \), falls between the first and second. Indeed, one of the implications is done in Theorem 4.4. Let us prove the second. Let \( \tau \) be a topology on \( \omega_1 \) which is SIB. As in Proposition 1.2, the dense sets form an \( \omega_1 \)-complete filter \( F \). Let \( N_\alpha \) be the neighborhood filter at \( \alpha \in \omega_1 \). For each \( \alpha \), \( N_\alpha \cup F \) generates a filter. Extend it to an ultrafilter \( U_\alpha \). Claim \( F = \bigcap \{ U_\alpha : \alpha < \omega_1 \} \). Only one direction requires proof. If \( D \) is in each \( U_\alpha \), then \( D \) meets every neighbourhood of every \( \alpha \), so \( D \) is dense, i.e., \( D \in F \). Thus \( \omega_1 \) admits a complete family of ultrafilters.

Again we do not know which new arrows reverse.
REFERENCES

[22] H. WOODIN, An $\aleph_1$-dense $\aleph_1$-complete ideal on $\aleph_1$, (preprint).