GENERALIZATION OF TITCHMARSH’S THEOREM FOR
THE BESSEL TRANSFORM IN THE SPACE $L_{p,\alpha}(\mathbb{R}^+)$

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Abstract. In this paper, we prove a generalization of Titchmarsh’s theorem for the Bessel transform in the space $L_{p,\alpha}(\mathbb{R}^+)$ for functions satisfying the $(\psi,p)$-Bessel Lipschitz condition.

1. Introduction and preliminaries

In [2], we proved a generalization of Titchmarsh’s theorem for the Bessel transform in the space $L_{2,\alpha}(\mathbb{R}^+)$. In this paper we prove this generalization in the space $L_{p,\alpha}(\mathbb{R}^+)$, where $1 < p \leq 2$ and $\alpha > -\frac{1}{2}$. For this purpose, we use a Bessel generalized translation.

$L_{p,\alpha}(\mathbb{R}^+)$, $1 < p \leq 2$, is the Banach space of measurable functions $f(t)$ on $\mathbb{R}^+$ with the finite norm

$$\|f\|_{p,\alpha} = \left( \int_{0}^{\infty} |f(x)|^p x^{2\alpha+1} dx \right)^{1/p},$$

where $\alpha$ is a real number, $\alpha > -\frac{1}{2}$.

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Let
\[ B = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx} \]
be the Bessel differential operator.

For \( \alpha \geq -\frac{1}{2} \), we introduce the Bessel normalized function of the first kind \( j_\alpha \) defined by
\[ j_\alpha(z) = \left( \frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left( \frac{z}{2} \right)^{2n} \right) , \]
where \( \Gamma \) is the gamma-function (see [4]). Moreover, from (1) we see that
\[ \lim_{z \to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0 \]
by consequence, there exist \( c > 0 \) and \( \eta > 0 \) satisfying
\[ |z| \leq \eta \implies |j_\alpha(z) - 1| \geq c|z|^2. \]
The function \( y = j_\alpha(z) \) satisfies the differential equation
\[ By + y = 0 \]
with the initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \). \( j_\alpha(z) \) is function infinitely differentiable, even, and, moreover entire analytic.

In \( L_{p,\alpha}(\mathbb{R}+) \), consider the Bessel generalized translation \( T_h \) [4] defined by
\[ T_h f(t) = c_\alpha \int_0^\pi f(\sqrt{t^2 + h^2 - 2th\cos \varphi}) \sin^{2\alpha} \varphi d\varphi, \]
where
\[ c_\alpha = \left( \int_0^\pi \sin^{2\alpha} \varphi d\varphi \right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}. \]
The Bessel transform we call the integral transform from [3, 4, 5]
\[ \hat{f}(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt, \ \lambda \in \mathbb{R}^+. \]
The inverse Bessel transform is given by the formula
\[ f(t) = \left(2^\alpha \Gamma(\alpha + 1)\right)^{-2} \int_0^\infty \hat{f}(\lambda) j_\alpha(\lambda t) \lambda^{2\alpha+1} d\lambda. \]

The following relation connect the Bessel generalized translation and the Bessel transform, in [1] we have
\[ (\hat{T}_h f)(\lambda) = j_\alpha(\lambda h) \hat{f}(\lambda). \]

We have the Hausdorff–Young inequality
\[ \| \hat{f} \|_{q,\alpha} \leq C \| f \|_{p,\alpha}, \]
where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( C \) is a positive constant.

2. Main Result

In this section we give the main result of this paper. We need first to define \((\psi,p)\)-Bessel Lipschitz class.

**Definition 2.1.** A function \( f \in L_{p,\alpha}(\mathbb{R}_+) \) is said to be in the \((\psi,p)\)-Bessel Lipschitz class, denoted by \( \text{Lip}(\psi,\alpha,p) \), if
\[ \| T_h f(t) - f(t) \|_{p,\alpha} = O(\psi(h)) \quad \text{as } h \to 0, \]
where
1. \( \psi(t) \) is a continuous increasing function on \([0,\infty)\),
2. \( \psi(0) = 0 \),
3. \( \psi(ts) = \psi(t)\psi(s) \) for all \( t,s \in [0,\infty) \).

**Theorem 2.2.** Let \( f(t) \) belong to \( \text{Lip}(\psi,\alpha,p) \). Then
\[ \int_r^\infty |\hat{f}(\lambda)|^q \lambda^{2\alpha+1} d\lambda = O(\psi(r^{-q})) \quad \text{as } r \to +\infty. \]
Proof. Let \( f \in Lip(\psi,\alpha,p) \). Then we have
\[
\|T_h f(t) - f(t)\|_{p,\alpha} = O(\psi(h)) \quad \text{as } h \to 0.
\]
From formulas (3) and (4), we obtain
\[
\int_0^\infty |1 - j_\alpha(\lambda h)|^q \hat{f}(\lambda)^{q \lambda^{2\alpha + 1}} d\lambda \leq C^q \|T_h f(t) - f(t)\|_{p,\alpha}^q.
\]
From (2), we have
\[
\int_{\frac{n-1}{n}}^{\frac{n}{n}} |1 - j_\alpha(\lambda h)|^q \hat{f}(\lambda)^{q \lambda^{2\alpha + 1}} d\lambda \geq \frac{c^q \eta^2 q}{d^{2q}} \int_{\frac{n}{n}}^{\frac{n}{n}} \hat{f}(\lambda)^{q \lambda^{2\alpha + 1}} d\lambda,
\]
d > 1, 0 < h ≤ 1. It follows from the above consideration that there exists a positive constant \( K_d \) such that
\[
\int_{\frac{n}{n}}^{\infty} \hat{f}(\lambda)^{q \lambda^{2\alpha + 1}} d\lambda \leq K_d \psi(q)(h) = K_d \psi(h^q).
\]
Then
\[
\int_r^{d r} |\hat{f}(\lambda)|^{q \lambda^{2\alpha + 1}} d\lambda \leq C_d \psi(r^{-q}) \quad \text{for all } d > 1
\]
of course
\[
\int_r^{d^{n-1} r} |\hat{f}(\lambda)|^{q \lambda^{2\alpha + 1}} d\lambda = \left( \int_r^{d r} + \int_{d r}^{d^2 r} + \ldots + \int_{d^{n-1} r}^{d^n r} \right) |\hat{f}(\lambda)|^{q \lambda^{2\alpha + 1}} d\lambda.
\]
Therefore
\[
\int_r^{\infty} |\hat{f}(\lambda)|^{q \lambda^{2\alpha + 1}} d\lambda \leq C_d (1 + \psi(d^{-q}) + \psi^2(d^{-q}) + \ldots) \psi(r^{-q}).
\]
For fixed \( d_0 \) such that \( \psi(d_0^{-q}) < 1 \) we have
\[
\int_r^{\infty} |\hat{f}(\lambda)|^{q \lambda^{2\alpha + 1}} d\lambda \leq C_1 \psi(r^{-q}),
\]
where \( C_1 = C_{d_0}(1 - \psi(d_0^{-q}))^{-1} \).
Finally, we get
\[
\int_{r}^{\infty} |\widehat{f}(\lambda)|q\lambda^{2\alpha+1}d\lambda = O(\psi(r^{-q})) \quad \text{as } r \to \infty.
\]
Thus, the proof is finished. \qed

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References


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