ON SOME GENERALIZATION OF THE GOŁĄB–SCHINZEL EQUATION

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Abstract. Inspired by a problem posed by J. Matkowski in [10] we investigate the equation

\[ f(p(x,y)(xf(y) + y) + (1 - p(x,y))(yf(x) + x)) = f(x)f(y), \quad x, y \in \mathbb{R}, \]

where functions \( f: \mathbb{R} \to \mathbb{R} \), \( p: \mathbb{R}^2 \to \mathbb{R} \) are assumed to be continuous.

1. Introduction

The composite functional equation

\[ f(x + yf(x)) = f(x)f(y), \quad x, y \in X, \]

where \( X \) is a real linear space and \( f: X \to \mathbb{R} \) is an unknown function, is the well-known Gołąb–Schinzel equation. For details concerning this equation, its origin, generalizations and applications, we refer e.g. to J. Aczél [1], J. Aczél [2, pp. 132-135], J. Aczél, J. Dhombres [3, Chapter 19], J. Aczél, S. Gołąb [4], S. Gołąb, A. Schinzel [5], K. Baron [6], N. Brillouet, J. Dhombres [7], J. Brzdżek [8], P. Javor [9], S. Wołodźko [12].

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There are several papers devoted to some generalizations of equation (1), cf. a survey paper Brzdęk [8], Mureńko [11], J. Matkowski [10]. The last one inspired our paper. In [10] the following generalization of (1) is considered:

\[(2) \quad f\left(p(xf(y) + y) + (1-p)(yf(x) + x)\right) = f(x)f(y), \quad x, y \in X.\]

Roughly speaking, it turns out that the continuous solutions of (2) are the same as the continuous solutions of (1). To be more precise, the main result of J. Matkowski [10] reads as follows:

**Theorem 1 ([10]).** Let \(X\) be a real linear topological space and \(p \in \mathbb{R}\) be fixed. A continuous function \(f: X \to \mathbb{R}\) satisfies the equation

\[f\left(p(xf(y) + y) + (1-p)(yf(x) + x)\right) = f(x)f(y), \quad x, y \in X,\]

if, and only if, either

\[f(x) = 0, \quad x \in X,\]

or there is an \(x^* \in X^* \setminus \{0\}\) such that

\[f(x) = 1 + x^*(x), \quad x \in X,\]

or \(p \in [0, 1]\) and there exists \(x^* \in X^* \setminus \{0\}\) such that

\[f(x) = \sup(1 + x^*(x), 0), \quad x \in X.\]

Let a function \(f: \mathbb{R} \to \mathbb{R}\) be continuous and a function \(p: \mathbb{R}^2 \to \mathbb{R}\) be continuous with respect to each variable. Let \(F_{f,p}: \mathbb{R}^2 \to \mathbb{R}\) be defined by the formula

\[(3) \quad F_{f,p}(x, y) = p(x,y)(xf(y) + y) + (1-p(x,y))(yf(x) + x), \quad x, y \in \mathbb{R}.\]

In this note we consider the generalization of (2) of the form:

\[(4) \quad f(F_{f,p}(x, y)) = f(x)f(y), \quad x, y \in \mathbb{R}.\]

The following question naturally arises and was posed in [10]: what are the solutions of equation (4)? Our main result (Theorem 4) states that any real continuous function \(f\) fulfilling equation (4) is also of one of the forms described in the Theorem 1.
2. Technical lemmas

For arbitrary function \( f: \mathbb{R} \to \mathbb{R} \) and \( c \in \mathbb{R} \) let denote

\[
A^f_c = f^{-1}(\{c\})
\]

and define \( g_f: \mathbb{R} \setminus A^f_1 \to \mathbb{R} \) by

\[
g_f(x) = \frac{x}{1 - f(x)}.
\]

2.1. Part I: We establish a form of the function \( f \) on the set \( f^{-1}((-1, 1)) \) and a form of the set \( A^f_0 \).

**Lemma 1.** Let \( f: \mathbb{R} \to \mathbb{R}, \ p: \mathbb{R}^2 \to \mathbb{R} \) satisfy equation (4). Then

1. \[
\prod_{j=0}^{n-1} (1 + f(x)^{2^j}) = \frac{1 - f(x)^{2^n}}{1 - f(x)}, \quad x \not\in A^f_1, \ n \in \mathbb{N},
\]
2. \[
f\left(\prod_{j=0}^{n-1} (1 + f(x)^{2^j})x\right) = f(x)^{2^n}, \quad x \in \mathbb{R}, \ n \in \mathbb{N}.
\]

**Proof.** By induction and by using \( F_{f,p}(z, z) = z(1 + f(z)) \) with

\[
z = \prod_{j=0}^{n-1} (1 + f(x)^{2^j})x.
\]

\( \square \)

**Lemma 2.** Suppose that a continuous function \( f: \mathbb{R} \to \mathbb{R} \) and a function \( p: \mathbb{R}^2 \to \mathbb{R} \) satisfy equation (4). Then \( g_f(f^{-1}((-1, 1))) \subseteq A^f_0 \).

**Proof.** Take arbitrary \( x_0 \in f^{-1}((-1, 1)) \). Then \( \lim_{n \to +\infty} f(x_0)^{2^n} = 0 \), so Lemma 1 and continuity of \( f \) imply that

\[
0 = \lim_{n \to +\infty} f(x_0)^{2^n} = \lim_{n \to +\infty} f\left(\prod_{j=0}^{n-1} (1 + f(x_0)^{2^j})x_0\right)
\]

\[
= f\left(\lim_{n \to +\infty} \prod_{j=0}^{n-1} (1 + f(x_0)^{2^j})x_0\right)
\]

\[
= f\left(\lim_{n \to +\infty} \frac{1 - f(x_0)^{2^n}}{1 - f(x_0)x_0}\right) = f\left(\frac{x_0}{1 - f(x_0)}\right).
\]

Hence \( g_f(x_0) \in A_0 \). \( \square \)
Lemma 3. Let \( f: \mathbb{R} \to \mathbb{R}, \ p: \mathbb{R}^2 \to \mathbb{R} \) satisfy equation (4). Then \( f(0) = 0 \) or \( f(0) = 1 \).

Proof. Put \( x = y = 0 \) in (4) in order to obtain \( f(0) = f(0)^2 \). \( \square \)

Lemma 4. Let \( f: \mathbb{R} \to \mathbb{R}, \ p: \mathbb{R}^2 \to \mathbb{R} \) satisfy equation (4). If there exists \( x_0 \in \mathbb{R} \) such that \( f(x_0) = -1 \), then \( f(0) = 1 \).

Proof. Put \( x = y = x_0 \) in (4) in order to get
\[
f(0) = f((1 + f(x_0))x_0) = f(x_0)^2 = (-1)^2 = 1.
\]
\( \square \)

Lemma 5. Suppose that a continuous function \( f: \mathbb{R} \to \mathbb{R} \) and a function \( p: \mathbb{R}^2 \to \mathbb{R} \) satisfy equation (4). If \( f \) is not identically equal zero, then \( f(0) = 1 \).

Proof. Assume, in search of a contradiction, that \( f \) is not identically equal to 0 and \( f(0) = 0 \) (cf. Lemma 3). Let \( S_0 = (A,B) \) with some \(-\infty \leq A < 0 < B \leq \infty \) be a component of \( f^{-1}((-1,1)) \) which contains 0. Then from Lemma 2 it follows that \( g_f(S_0) \subseteq A_f^0 \) and \( 0 = g_f(0) \in g_f(S_0) \). Moreover, \( g_f \) is continuous on \( f^{-1}((-1,1)) \). So, \( g_f(S_0) \) is an interval contained in \( A_f^0 \). Since \( g_f(0) = 0 \), we have \( g_f(S_0) = [C,D] \) with some \( C \leq 0 \leq D \). If \( C = 0 \), then for every \( x \in S_0 \) we have \( g_f(x) = \frac{x}{1-f(x)} \geq 0 \), which can occur (in the set \( f^{-1}((-1,1)) \)) only when \( x \geq 0 \) for every \( x \in S_0 \), which is impossible since \( S_0 \) is open and contains 0. Analogically, \( D = 0 \) can be excluded. Thus \( C < 0 < D \) and at least one of numbers \( C,D \) is real (because \( f \not\equiv 0 \)). If for example \( D \in \mathbb{R} \) then for every \( x \in S_0 \) we have \( \frac{x}{1-f(x)} \leq D \), which is equivalent to \( f(x) \leq 1 - \frac{x}{D} \). Regarding \( f(x) \in (-1,1) \) for every \( x \in (A,B) \), we conclude that \( B \in \mathbb{R} \) and \( f(B) = -1 \). Then from Lemma 4 we get \( f(0) = 1 \), which contradicts with our assumption. \( \square \)

Lemma 6. Suppose that a continuous function \( f: \mathbb{R} \to \mathbb{R} \) and a function \( p: \mathbb{R}^2 \to \mathbb{R} \) continuous with respect to each variable satisfy equation (4). Then set \( A_f^0 \) is a closed interval or is empty.

Proof. Assume that \( A_f^0 \neq \emptyset \). If \( f \) is identically equal to 0, then \( A_f^0 = \mathbb{R} \) and the thesis holds.

If \( f \neq 0 \), then \( f(0) = 1 \) (cf. Lemma 5). Let \( x_0, \ x_1 \in A_f^0, \ x_0 < x_1 \). For every \( y \in \mathbb{R} \) we have
\[
f(F_{f,p}(x_0,y)) = f(x_0)f(y) = 0 \quad \text{and} \quad f(F_{f,p}(y,x_1)) = f(y)f(x_1) = 0,
\]
so $F_{f,p}(x_0, \mathbb{R})$ and $F_{f,p}(\mathbb{R}, x_1)$ are intervals contained in $A^f_0$. Obviously,

$$F_{f,p}(0, x_1) = x_1 \quad \text{and} \quad F_{f,p}(x_0, 0) = x_0.$$  

Furthermore, $F_{f,p}(x_0, x_1) \in F_{f,p}(x_0, \mathbb{R}) \cap F_{f,p}(\mathbb{R}, x_1)$. Thus,

$$[x_0, x_1] \subseteq F_{f,p}(x_0, \mathbb{R}) \cup F_{f,p}(\mathbb{R}, x_1) \subseteq A^f_0.$$  

Therefore $A^f_0$ is an interval. It is closed, since $A^f_0 = f^{-1}(\{0\})$ and the function $f$ is continuous. \hfill $\square$

**Lemma 7.** Suppose that a continuous function $f: \mathbb{R} \to \mathbb{R}$ and a function $p: \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to each variable satisfy equation (4). If $f$ is not identically equal to 0, then $A^f_0$ is the empty set and $f(\mathbb{R}) \subseteq [1, +\infty)$ or there exists $\alpha \in \mathbb{R}^*$ such that either

1. $\alpha < 0$, $A^f_0 = (\alpha, 0]$ and $f(x) = 1 - \frac{x}{\alpha}$ for $x \in (\alpha, 0)$, $f(x) \geq 1$ for $x \geq 0$ or

2. $\alpha < 0$, $A^f_0 = \{\alpha\}$ and $f(x) = 1 - \frac{x}{\alpha}$ for $x \in (2\alpha, 0)$, $f(x) \leq -1$ for $x \leq 2\alpha$, $f(x) \geq 1$ for $x \geq 0$ or

3. $\alpha > 0$, $A^f_0 = [\alpha, +\infty)$ and $f(x) = 1 - \frac{x}{\alpha}$ for $x \in (0, \alpha)$, $f(x) \geq 1$ for $x \leq 0$ or

4. $\alpha > 0$, $A^f_0 = \{\alpha\}$ and $f(x) = 1 - \frac{x}{\alpha}$ for $x \in (0, 2\alpha)$, $f(x) \leq -1$ for $x \geq 2\alpha$ and $f(x) \geq 1$ for $x \leq 0$.

**Proof.** Assume in search of a contradiction that $A^f_0 = [\alpha, \beta]$ with some $-\infty < \alpha < \beta < +\infty$ (cf. Lemma 6).

If $f(x) > 0$ for $x > \beta$, $f(x) < 0$ for $x < \alpha$ (the case $f(x) < 0$ for $x > \beta$, $f(x) > 0$ for $x < \alpha$ can be treated similarly), then for $x, y < \alpha$ we have $f(F_{f,p}(x, y)) = f(x)f(y) > 0$, so $F_{f,p}(x, y) > \beta$. Hence for every $x < \alpha$ we get

$$F_{f,p}(x, \alpha) = \lim_{y \to \alpha^-} F_{f,p}(x, y) \geq \beta$$

and

$$\alpha = F_{f,p}(\alpha, \alpha) = \lim_{x \to \alpha^-} F_{f,p}(x, \alpha) \geq \beta,$$

which is a contradiction with $\alpha < \beta$.

If $f(x) < 0$ for $x \in (-\infty, \alpha) \cup (\beta, +\infty)$, then for $x, y < \alpha$, we have $f(F_{f,p}(x, y)) = f(x)f(y) > 0$, which is impossible.

To finish the proof of the first part of the thesis it is enough to consider the case $f(x) > 0$ for $x \in (-\infty, \alpha) \cup (\beta, +\infty)$. Let $(\gamma, \delta)$ be such a component of
\[ f^{-1}((-1,1)) \text{ that } [\alpha, \beta] \subseteq (\gamma, \delta). \] From Lemma 2 it follows that \( g_f((\gamma, \delta)) \subseteq [\alpha, \beta] \). Hence for \( x \in (\gamma, \delta) \) we have

\[ \alpha f(x) \geq \alpha - x \quad \text{and} \quad \beta f(x) \leq \beta - x. \]

If \( \alpha < 0 \), then \( f(x) \leq 1 - \frac{x}{\alpha} \), so for \( x \in (\gamma, \alpha) \) we would have \( f(x) < 0 \), which contradicts with the assumption.

If \( \alpha \geq 0 \), then \( \beta > 0 \) and \( f(x) \leq 1 - \frac{x}{\beta} \). Thus, for \( x \in (\beta, \delta) \) we would have \( f(x) < 0 \), which is again a contradiction with the assumption. Therefore either \( \alpha = \beta \in \mathbb{R} \) or \( \alpha = -\infty \) or \( \beta = +\infty \).

If \( A_0^f = \emptyset \), then from Lemma 2 it follows that \( f^{-1}((-1,1)) = \emptyset \). Lemma 3 and the continuity of \( f \) imply \( f(\mathbb{R}) \subseteq [1, +\infty) \).

Now assume that \( A_0^f \neq \emptyset \) and fix \( x_0 \in f^{-1}((-1,1)) \setminus A_0^f \). Then according to Lemma 2 \( g_f(x_0) \in A_0^f \). If \( A_0^f = \{\alpha\} \), then \( g_f(x_0) = \alpha \), so \( f(x_0) = 1 - \frac{x_0}{\alpha} \). If \( A_0^f = (-\infty, \alpha] \), then \( g_f(x_0) \leq \alpha \), so \( f(x_0) \geq 1 - \frac{x_0}{\alpha} \) (\( \alpha < 0 \) because \( f(0) = 1 \)). Hence \( f(x_0) = 1 - \frac{x_0}{c} \) with some \( c \leq \alpha \). Assume in search of a contradiction that \( c < \alpha \). From Lemma 1 it follows that

\[ f\left( x_0 \frac{1 - f(x_0)^{2n}}{1 - f(x_0)} \right) = f(x_0)^{2n} \]

for every \( n \in \mathbb{N} \). Thus \( f\left( x_0 \frac{1 - f(x_0)^{2n}}{1 - f(x_0)} \right) > 0 \) for every \( n \in \mathbb{N} \). On the other hand,

\[ \lim_{n \to +\infty} x_0 \frac{1 - f(x_0)^{2n}}{1 - f(x_0)} = \frac{x_0}{1 - f(x_0)} = c < \alpha, \]

so there exist \( N \in \mathbb{N} \) such that \( x_0 \frac{1 - f(x_0)^{2N}}{1 - f(x_0)} < \alpha \). Then

\[ f\left( x_0 \frac{1 - f(x_0)^{2N}}{1 - f(x_0)} \right) = f(x_0)^{2N} = 0, \]

which is not possible. To conclude, for every \( x_0 \in f^{-1}((-1,1)) \setminus A_0^f \) we have \( f(x_0) = 1 - \frac{x_0}{\alpha} \).

Furthermore, if \( A_0^f = \{\alpha\} \) and \( \alpha < 0 \), then for every \( x_0 \in f^{-1}((-1,1)) \setminus \{\alpha\} \) we have both \( f(x_0) = 1 - \frac{x_0}{\alpha} \) and \( f(x_0) \in (\alpha) \), which is possible if and only if \( x_0 \in (2\alpha, 0) \). Hence \( f^{-1}((-1,1)) = (2\alpha, 0) \). Moreover, \( f(2\alpha) = -1 \), \( f(0) = 1 \), so \( f((\alpha, 2\alpha]) \subseteq (-\infty, -1] \), \( f([0, +\infty) \subseteq [1, +\infty) \). If \( A_0^f = \{\alpha\} \) and \( \alpha > 0 \), then similarly as above we get \( f((\alpha, 0]) \subseteq [1, +\infty) \) and \( f([2\alpha, +\infty) \subseteq (-\infty, -1] \).
Finally, we consider the case of $A^f_0 = (-\infty, \alpha]$ with $\alpha < 0$ (the case of $A^f_0 = [\alpha, +\infty)$ with $\alpha > 0$ may be analyzed analogically). For every $x_0 \in f^{-1}((-1, 1)) \setminus (-\infty, \alpha]$ we have both $f(x_0) = 1 - \frac{x_0}{\alpha}$ and $f(x_0) \in (-1, 1)$, which is possible if and only if $x_0 \in (\alpha, 0)$. Hence $f^{-1}((-1, 1)) = (-\infty, 0)$ and $f([0, +\infty)) \subseteq [1, +\infty)$.

□

2.2. Part II: We prove that if $f \not\equiv 0$, $f \not\equiv 1$ is a solution of (4), then $A^f_1 = \{0\}$, so either $f$ takes values greater than 1 for positive arguments and smaller than 1 for negative arguments or the reverse.

**Lemma 8.** Let $f: \mathbb{R} \to \mathbb{R}$ and $p: \mathbb{R}^2 \to \mathbb{R}$ satisfy equation (4). The set $A^f_1$ is a semigroup.

**Proof.** Put in (4) $x, y \in A^f_1$ in order to obtain $f(x + y) = 1$.

**Lemma 9.** Suppose that a continuous function $f: \mathbb{R} \to \mathbb{R}$ and a function $p: \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to each variable satisfy equation (4). If for some $\varepsilon > 0$ we have $f((\varepsilon, \varepsilon)) \subseteq [1, +\infty)$ or $f((\varepsilon, \varepsilon)) \subseteq (0, 1]$, then $f \equiv 1$.

**Proof.** Assume that $f((\varepsilon, \varepsilon)) \subseteq [1, +\infty)$ for some $\varepsilon > 0$. Observe that $F_{f,p}(0, x) = x = F_{f,p}(x, 0)$ for every $x \in \mathbb{R}$. Continuity of $F_{f,p}(\cdot, \varepsilon)$ and $F_{f,p}(\cdot, -\varepsilon)$ at the point 0 implies that there exists $\delta > 0$, $\delta < \varepsilon$ such that for every $|x| < \delta$ we have

$$|F_{f,p}(x, \varepsilon) - \varepsilon| = |F_{f,p}(x, \varepsilon) - F_{f,p}(0, \varepsilon)| < \frac{\varepsilon}{2}$$

and

$$|F_{f,p}(x, -\varepsilon) + \varepsilon| = |F_{f,p}(x, -\varepsilon) - F_{f,p}(0, -\varepsilon)| < \frac{\varepsilon}{2}.$$

Hence

$$F_{f,p}(x, \varepsilon) \in \left(\frac{\varepsilon}{2}, \frac{3\varepsilon}{2}\right) \quad \text{and} \quad F_{f,p}(x, -\varepsilon) \in \left(-\frac{3\varepsilon}{2}, -\frac{\varepsilon}{2}\right), \quad |x| < \delta.$$

For every $|x| < \delta$ from Darboux property of function $F_{f,p}(x, \cdot)$ it follows that there exists $y(x) \in (-\varepsilon, \varepsilon)$ such that $F_{f,p}(x, y(x)) = 0$. Therefore from (4) we have

$$1 = f(0) = f(F_{f,p}(x, y(x))) = f(x)f(y(x)) \geq 1 \quad \text{for } |x| < \delta.$$
and equality holds if and only if \( f(x) = f(y(x)) = 1 \). Thus we have proved that \((-\delta, \delta) \subseteq A^f_1\). However, the set \( A^f_1 \) is a semigroup (cf. Lemma 8), so \( \mathbb{R} = A^f_1\).

**Corollary 1.** Suppose that a continuous function \( f: \mathbb{R} \to \mathbb{R} \) and a function \( p: \mathbb{R}^2 \to \mathbb{R} \) continuous with respect to each variable satisfy equation (4). If \( f^{-1}((-1, 1)) = \emptyset \), then \( f \equiv 1 \).

**Proof.** If \( f^{-1}((-1, 1)) = \emptyset \), then obviously \( A^f_0 = \emptyset \), so from Lemma 7 it follows that \( f(\mathbb{R}) \subseteq [1, +\infty) \). Therefore, Lemma 9 implies that \( f \equiv 1 \).

**Lemma 10.** Suppose that a continuous function \( f: \mathbb{R} \to \mathbb{R} \) and a function \( p: \mathbb{R}^2 \to \mathbb{R} \) satisfy equation (4). If \( 0 \) is a leftside accumulation point (rightside accumulation point) of \( A^f_1 \), then \( f([0, +\infty)) = \{1\} \) (\( f((0, +\infty)) = \{1\} \)).

**Proof.** Let \((x_n)_{n\in\mathbb{N}} \in (A^f_1)^\mathbb{N}\) be a decreasing sequence of points tending to \(0\). Fix \( g > 0 \). For every \( n \in \mathbb{N} \) there exists \( l(n) \in \mathbb{N} \) such that \((l(n) - 1)x_n < g \leq l(n)x_n\). Then \(|l(n)x_n - g| < x_n\), so

\[
\lim_{n\to+\infty} l(n)x_n = g.
\]

Moreover, \( A^f_1 \) is a semigroup, so \( l(n)x_n \in A^f_1 \). Thus, \( A^f_1 \) is dense in \([0, +\infty)\). On the other hand, \( A^f_1 = f^{-1}(\{1\}) \) is closed as a counterimage of a closed set by a continuous function. Hence \( f([0, +\infty)) = \{1\} \).

**Corollary 2.** Suppose that a continuous function \( f: \mathbb{R} \to \mathbb{R} \) and a function \( p: \mathbb{R}^2 \to \mathbb{R} \) continuous with respect to each variable satisfy equation (4). If condition (1) or (2) from Lemma 7 is satisfied, then there exists \( \varepsilon > 0 \) such that \( f((0, \varepsilon)) \subseteq (1, +\infty) \). If condition (3) or (4) from Lemma 7 is satisfied, then there exists \( \varepsilon > 0 \) such that \( f((-\varepsilon, 0)) \subseteq (1, +\infty) \).

**Proof.** Assume that condition (1) or (2) from Lemma 7 is fulfilled. From Lemma 7 follows that \( f((0, +\infty)) \subseteq (1, +\infty) \). If the thesis of the corollary did not hold, then \( 0 \) would be a righthand side accumulation point of the set \( A^f_1 \) and Lemma 1 would imply \( A^f_1 = [0, +\infty) \). Then we would have \( f(\mathbb{R}) \subseteq (-\infty, 1] \) and from Lemma 9 we would get \( f \equiv 1 \), which is a contradiction with the assumption of the lemma.

The proof is similar for condition (3) or (4).

**Lemma 11.** Suppose that a continuous function \( f: \mathbb{R} \to \mathbb{R} \) and a function \( p: \mathbb{R}^2 \to \mathbb{R} \) continuous with respect to each variable satisfy equation (4). If condition (1) or (2) from Lemma 7 is fulfilled, then \( f((0, +\infty)) \subseteq (1, +\infty) \). If condition (3) or (4) from Lemma 7 is fulfilled, then \( f((0, +\infty)) \subseteq (1, +\infty) \).
Proof. Without lost of generality we can assume that condition (1) or (2) from Lemma 7 is satisfied.

Assume for contradiction that \((0, +\infty) \cap A^f_1 \neq \emptyset\). From Corollary 2 it follows that \(\alpha = \inf((0, +\infty) \cap A^f_1) > 0\). Define \(h: \mathbb{R} \to \mathbb{R}\) by the formula \(h(x) = x(1 + f(x))\). Then \(h([0, \alpha])\) is a compact interval which contains \(h(0) = 0\) and \(h(\alpha) = 2\alpha\). If there is \(\beta \in (\alpha, 2\alpha)\) such that \(f(\beta) = 1\), then \(\beta = h(\gamma)\) with some \(\gamma \in (0, \alpha)\) and according to (4) we would have

\[
1 = f(\beta) = f(h(\gamma)) = f(\gamma)^2,
\]

which is equivalent to \(f(\gamma) = 1\) (cf. Lemma 7). However, this is a contradiction with the definition of \(\alpha\). Thus we proved that \(f((\alpha, 2\alpha)) \subseteq (1, +\infty)\).

Obviously \(h(\alpha) = 2\alpha\), \(h(2\alpha) = 4\alpha\), so \([2\alpha, 4\alpha] \subseteq h([\alpha, 2\alpha])\). Hence \(3\alpha = h(\gamma)\) with some \(\gamma \in (\alpha, 2\alpha)\) and \(f(3\alpha) = f(h(\gamma)) = f(\gamma)^2 > 1\). On the other hand \(3\alpha \in A^f_1\), because \(A^f_1\) is a semigroup (cf. Lemma 8).

\[\square\]

2.3. Part III: We establish the form of function \(f\) on the set \(f^{-1}(\mathbb{R} \setminus (-1, 1))\)

**Theorem 2.** Suppose that a continuous function \(f: \mathbb{R} \to \mathbb{R}\) and a function \(p: \mathbb{R}^2 \to \mathbb{R}\) continuous with respect to each variable satisfy equation (4). If condition (1) or (2) from Lemma 7 is fulfilled, then \(f(x) = 1 - \frac{x}{\alpha}\) for \(x > 0\). If condition (3) or (4) from Lemma 7 is fulfilled, then \(f(x) = 1 - \frac{x}{\alpha}\) for \(x < 0\).

Proof. Without lost of generality we can assume that condition (1) or (2) from Lemma 7 is satisfied.

Equation (4), Lemma 11 and Lemma 7 imply that for arbitrary \(x > 0\) there exists exactly one \(k(x) \in (\alpha, 0)\) such that \(f(x)f(k(x)) = 1\). Thus, \(f(x) = \frac{\alpha}{\alpha - k(x)}\) for every \(x > 0\).

Let \(x > 0\), \(\alpha < y < 0\). Then \(f(x) = \frac{\alpha}{\alpha - k(x)}\), \(f(y) = \frac{\alpha - y}{\alpha}\), so \(f(x)f(y) = \frac{\alpha - y}{\alpha - k(x)}\). Therefore, from Lemma 7 for \(x > 0\), \(y < 0\) we have

\[F_{f,p}(x, y) \in (\alpha, 0) \iff f(F_{f,p}(x, y)) = f(x)f(y) \in (0, 1) \iff y \in (\alpha, k(x))\]

and

\[F_{f,p}(x, y) > 0 \iff f(F_{f,p}(x, y)) = f(x)f(y) > 1 \iff y \in (k(x), 0)\]

Fix \(x > 0\), \(y \in (\alpha, k(x))\). Then

\[f(F_{f,p}(x, y)) = 1 - \frac{F_{f,p}(x, y)}{\alpha},\]
\[ F_{f,p}(x,y) = p(x,y) \left( x \left( 1 - \frac{y}{\alpha} \right) + y - y \frac{\alpha}{\alpha - k(x)} - x \right) + y \frac{\alpha}{\alpha - k(x)} + x \]

\[ = p(x,y) \left( y x k(x) - \alpha x - \alpha k(x) \right) \left( \frac{\alpha}{\alpha - k(x)} \right) + \frac{\alpha x + \alpha y - x k(x)}{\alpha - k(x)}. \]

Thus

\[ f(F_{f,p}(x,y)) = 1 - p(x,y) \frac{y (x k(x) - \alpha x - \alpha k(x))}{\alpha^2 (\alpha - k(x))} + \frac{x k(x) - \alpha x - \alpha y}{\alpha (\alpha - k(x))}, \]

so

\[ 1 - p(x,y) \frac{y (x k(x) - \alpha x - \alpha k(x))}{\alpha^2 (\alpha - k(x))} + \frac{x k(x) - \alpha x - \alpha y}{\alpha (\alpha - k(x))} = \frac{\alpha - y}{\alpha - k(x)} \]

and

\[ \alpha^2 (\alpha - k(x)) - p(x,y) y (x k(x) - \alpha x - \alpha k(x)) \]
\[ + \alpha (x k(x) - \alpha x - \alpha y) = \alpha^2 (\alpha - y), \]

which implies

\[ p(x,y) y (\alpha x + \alpha k(x) - x k(x)) = \alpha (\alpha x + \alpha k(x) - x k(x)). \]

Therefore, either

\[ k(x) = \frac{\alpha x}{x - \alpha}, \text{ which is equivalent to } f(x) = 1 - \frac{x}{\alpha}, \]

or

\[ p(x,y) = \frac{\alpha}{y}. \]

Assume that there exists a sequence \((x_n)_{n \in \mathbb{N}}\) decreasing to 0 such that \(f(x_n) \neq 1 - \frac{x_n}{\alpha}\). Fix \(y_0 \in (\alpha, 0)\). Since \(\lim_{x \to 0^+} k(x) = 0\), there is \(N \in \mathbb{N}\) such that for every \(n \geq N\) we have \(y_0 \in (\alpha, k(x_n))\), so \(p(x_n, y_0) = \frac{\alpha}{y_0}\). Then

\[ p(0, y_0) = \lim_{n \to +\infty} p(x_n, y_0) = \lim_{n \to +\infty} \frac{\alpha}{y_0} = \frac{\alpha}{y_0}. \]

Thus,

\[ p(0, 0) = \lim_{y_0 \to 0^-} p(0, y_0) = +\infty. \]

This contradiction proves that such a sequence \((x_n)_{n \in \mathbb{N}}\) does not exists. So, there is an \(\varepsilon > 0\) such that \(f(x) = 1 - \frac{x}{\alpha}\) for every \(x \in [0, \varepsilon]\).
If \( f(x) = 1 - \frac{x}{\alpha} \), \( f(y) = 1 - \frac{y}{\alpha} \), then 
\[
Ff,p(x,y) = x + y - \frac{xy}{\alpha} = 1 - \frac{x + y - \frac{xy}{\alpha}}{\alpha} = 1 - \frac{Ff,p(x,y)}{\alpha}.
\]

Therefore, \( f(z) = 1 - \frac{z}{\alpha} \) for every \( z \in Ff,p([0,\varepsilon]^2) \). In particular, for \( x \in [0,\varepsilon] \) we have 
\[
Ff,p([0,\varepsilon]^2) \ni Ff,p(x,x) = x(1 + f(x)) > 2x, \quad \text{so} \quad [0,2\varepsilon] \subseteq Ff,p([0,\varepsilon]^2)
\]
and \( f(z) = 1 - \frac{z}{\alpha} \) for every \( z \in [0,2\varepsilon] \). Repeating this reasoning, we get that 
\( f(z) = 1 - \frac{z}{\alpha} \) for every \( z > 0 \).

**Theorem 3.** Suppose that a continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \) and a function \( p : \mathbb{R}^2 \rightarrow \mathbb{R} \) continuous with respect to each variable satisfy equation (4). If condition (2) from Lemma 7 is fulfilled, then \( f(x) = 1 - \frac{x}{\alpha} \) for \( x < 0 \). If condition (4) from Lemma 7 is fulfilled, then \( f(x) = 1 - \frac{x}{\alpha} \) for \( x > 0 \).

**Proof.** Without lost of generality we can assume that condition (2) from Lemma 7 is satisfied.

Suppose that there exist \( x < 2\alpha \) such that \( f(x) < -1 \). Then \( x(f(x)+1) > 0 \), so from Theorem 2 and (4) we have 
\[
1 - \frac{x(1 + f(x))}{\alpha} = f(x(1 + f(x))) = f(x)^2,
\]
so \( \alpha f(x)^2 + xf(x) + x - \alpha = 0 \) and solving this quadratic equation we get 
\( f(x) = 1 - \frac{x}{\alpha} \) or \( f(x) = -1 \). We have chosen \( x \) such that \( f(x) < -1 \), so finally 
\( f(x) = 1 - \frac{x}{\alpha} \).

Let \( A = \{ x \in (-\infty,2\alpha) : f(x) = -1 \} \) and 
\[
B = \left\{ x \in (-\infty,2\alpha) : f(x) = 1 - \frac{x}{\alpha} \right\}.
\]

The sets \( A, B \) are disjoint, their union is \((-\infty,2\alpha)\) and they are closed in \((-\infty,2\alpha)\), since the function \( f \) is continuous. Connectedness of \((-\infty,2\alpha)\) implies that \( A = \emptyset \) or \( B = \emptyset \), so 
\[
f(x) = -1 \quad \text{for every } x < 2\alpha \quad \text{or} \quad f(x) = 1 - \frac{x}{\alpha} \quad \text{for every } x < 2\alpha.
\]

Now we show that the first case leads to a contradiction. Indeed, in this case 
we would have \( f(x) = \max\{-1,1 - \frac{x}{\alpha} \} \) and we could choose \( x_0 > 0, y_0 \leq 2\alpha \) and get 
\( f(Ff,p(x_0,y_0)) = f(x_0)f(y_0) = -(1 - \frac{x_0}{\alpha}) < -1 \). However, in the considered situation \( f(\mathbb{R}) \cap (-\infty,-1) = \emptyset \), which implies the desired contradiction. \( \square \)
3. Main result

Our main result reads as follows:

**Theorem 4.** Let a continuous function $f: \mathbb{R} \to \mathbb{R}$ and a continuous with respect to each variable function $p: \mathbb{R}^2 \to \mathbb{R}$ satisfy equation (4). Then one of the following conditions is satisfied:

1. $f \equiv 0$, $p$ arbitrary continuous function or
2. $f \equiv 1$, $p$ arbitrary continuous function or
3. $f(x) = 1 - \frac{x}{\alpha}$ with $\alpha \neq 0$, $p$ arbitrary continuous function or
4. $f(x) = \max\{0, 1 - \frac{x}{\alpha}\}$ with some $\alpha < 0$ and $p$ being a continuous function satisfying conditions:
   - if $x, y \geq \alpha$ or $x = y \leq \alpha$ or $x = 0$ or $y = 0$, then $p(x, y)$ is arbitrary,
   - if $x < y \leq \alpha$, then $p(x, y) \leq \frac{\alpha - x}{y - x}$,
   - if $y < x \leq \alpha$, then $p(x, y) \geq \frac{\alpha - x}{y - x}$,
   - if $x \in (\alpha, 0)$, $y < \alpha$, then $p(x, y) \geq 1 - \frac{\alpha}{x}$,
   - if $x > 0$, $y < \alpha$, then $p(x, y) \leq 1 - \frac{\alpha}{x}$,
   - if $x < \alpha$, $y \in (\alpha, 0)$, then $p(x, y) \leq \frac{\alpha}{y}$,
   - if $x < \alpha$, $y > 0$, then $p(x, y) \geq \frac{\alpha}{y}$, or
5. $f(x) = \max\{0, 1 - \frac{x}{\alpha}\}$ with some $\alpha > 0$ and $p$ being a continuous function satisfying conditions:
   - if $x, y \leq \alpha$ or $x = y \geq \alpha$ or $x = 0$ or $y = 0$, then $p(x, y)$ is arbitrary,
   - if $x > y \geq \alpha$, then $p(x, y) \leq \frac{\alpha - x}{y - x}$,
   - if $y > x \geq \alpha$, then $p(x, y) \geq \frac{\alpha - x}{y - x}$,
   - if $x < 0$, $y > \alpha$, then $p(x, y) \leq 1 - \frac{\alpha}{x}$,
   - if $x \in (0, \alpha)$, $y > \alpha$, then $p(x, y) \geq 1 - \frac{\alpha}{x}$,
   - if $x > \alpha$, $y \in (0, \alpha)$, then $p(x, y) \leq \frac{\alpha}{y}$,
   - if $x > \alpha$, $y < 0$, then $p(x, y) \geq \frac{\alpha}{y}$.

Conversely, if functions $f: \mathbb{R} \to \mathbb{R}$, $p: \mathbb{R}^2 \to \mathbb{R}$ satisfy one of the conditions (1)–(5), then $f$, $p$ is a solution of equation (4).

**Proof.** From Lemma 7, Theorem 2, Theorem 3 follows that if $f$ is not identically equal neither to 0 nor to 1, then $f(x) = 1 - \frac{x}{\alpha}$ or $f(x) = \max\{0, 1 - \frac{x}{\alpha}\}$. Obviously, if $f(x) = 1 - \frac{x}{\alpha}$, then the function $p$ is arbitrary. Therefore, to complete the proof it is enough to show that in cases (4) and (5) the function $p$ must satisfy conditions mentioned in, respectively, (4) or (5).

Now we consider the case $f(x) = \max\{0, 1 - \frac{x}{\alpha}\}$ and $\alpha < 0$. For $x, y \geq \alpha$ equation (4) is satisfied independently of $p(x, y)$. For $x, y \leq \alpha$ we have $f(F(x, y)) = 0$, so $F(x, y) \leq \alpha$ and $F(x, y) = p(x, y)(y - x) + x$. Thus, if $x < y \leq \alpha$, then $p(x, y) \leq \frac{\alpha - x}{y - x}$; if $x = y \leq \alpha$, then $p(x, y)$ is arbitrary; if
y \leq x \leq \alpha$, then \( p(x, y) \geq \frac{\alpha - x}{y - x} \). For \( x > \alpha, y < \alpha \) we have \( f(x)f(y) = 0 \), so \( F(x, y) \leq \alpha \). The definition of \( F \) gives

\[
F(x, y) = p(x, y) \left( y - y \left( 1 - \frac{x}{\alpha} \right) - x \right) + y \left( 1 - \frac{x}{\alpha} \right) + x
\]

so \(-xp(x, y) \frac{a-y}{a} \leq \frac{1}{a} (\alpha - x)(\alpha - y)\). Thus, \( p(0, y) \) are arbitrary; if \( x \in (\alpha, 0), y < \alpha \), then \( p(x, y) \geq 1 - \frac{\alpha}{x} \); if \( x > 0, y < \alpha \), then \( p(x, y) \leq 1 - \frac{\alpha}{x} \).

Similarly, if \( x < \alpha, y > \alpha \), then \( F(x, y) \leq \alpha \) and

\[
F(x, y) = p(x, y) \left( x \left( 1 - \frac{y}{\alpha} \right) + y - x \right) + x = yp(x, y) \left( 1 - \frac{x}{\alpha} \right) + x \leq \alpha,
\]

so \( yp(x, y) \frac{a-x}{a} \leq \alpha - x \). Thus, \( p(x, 0) \) are arbitrary; if \( x < \alpha, y \in (\alpha, 0) \), then \( p(x, y) \leq \frac{\alpha}{y} \); if \( x < \alpha, y > 0 \), then \( p(x, y) \geq \frac{\alpha}{y} \).

The case (5) is treated analogically to the case (4).

It is easy to check that function fulfilling one of the conditions (1)–(5) is a solution of (4). □

In the end observe, that there exist a lot of continuous functions \( p \) which satisfy conditions from (4) or (5) of Theorem 4, e.g. for \( \alpha > 0 \) one may take

\[
p_0(x, y) = \begin{cases} 
\frac{\alpha}{y}, & \text{for } |y| \geq \frac{\alpha}{2} \\
\frac{4y}{\alpha}, & \text{for } |y| < \frac{\alpha}{2}.
\end{cases}
\]

Let \( p_1: \mathbb{R}^2 \to \mathbb{R} \) be an arbitrary function continuous with respect to each variable and such that \( p_1(x, y) \neq 0 \) only for \( x < 0 \) and \( y < 0 \). Then the function \( p_0 + p_1 \) satisfies conditions (5) of Theorem 4, too.

References


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