ORTHOGONALLY PEXIDER FUNCTIONS MODULO
A DISCRETE SUBGROUP

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Abstract. Under appropriate conditions on abelian topological groups $G$ and $H$, an orthogonality $\perp \subset G^2$ and a $\sigma$-algebra $\mathfrak{M}$ of subsets of $G$ we prove that if at least one of the functions $f, g, h : G \to H$ satisfying

$$f(x + y) - g(x) - h(y) \in K \quad \text{for } x, y \in G \text{ such that } x \perp y,$$

where $K$ is a discrete subgroup of $H$, is continuous at a point or $\mathfrak{M}$-measurable, then there exist: a continuous additive function $A : G \to H$, a continuous biadditive and symmetric function $B : G \times G \to H$ and constants $a, b \in H$ such that

$$\begin{cases} f(x) - B(x, x) - A(x) - a \in K, \\ g(x) - B(x, x) - A(x) - b \in K, \\ h(x) - B(x, x) - A(x) - a + b \in K \end{cases}$$

for $x \in G$ and

$$B(x, y) = 0 \quad \text{for } x, y \in G \text{ such that } x \perp y.$$

Let $G$ and $H$ be groups and $\perp \subset G^2$ an orthogonality. We say that a function $f : G \to H$ is orthogonally additive, if

$$f(x + y) = f(x) + f(y) \quad \text{for } x, y \in G \text{ such that } x \perp y.$$
In the paper [3] J. Brzdęk considers the Rätz orthogonality (cf.[5]) and, under some assumptions, gives a description of orthogonally additive functions modulo a discrete subgroup, i.e. functions \( f : G \to H \) such that

\[ f(x + y) - f(x) - f(y) \in K \quad \text{for} \quad x, y \in G \quad \text{such that} \quad x \perp y, \]

where \( K \) is a discrete subgroup of \( H \). In the papers [7] and [4] authors prove similar theorems (for continuous or measurable functions), but for the orthogonality defined by K. Baron and P. Volkmann in [2], which includes the Rätz orthogonality.

Now we would like to obtain some similar results for the Pexider difference instead of the Cauchy difference, i.e. we assume that functions \( f, g, h : G \to H \) are orthogonally Pexider modulo a discrete subgroup, which means that they satisfy

\[ f(x + y) - g(x) - h(x) \in K \quad \text{for} \quad x, y \in G \quad \text{such that} \quad x \perp y, \]

where \( K \) is a discrete subgroup of \( H \). We start with the following result.

**Lemma.** Let \( G \) be a groupoid with a neutral element, \( H \) an abelian group, \( K \) a subgroup of \( H \). Let \( \Delta \subset G \times G \) be a set with

\[ (0, x), (x, 0) \in \Delta \quad \text{for all} \quad x \in G. \]

If functions \( f, g, h : G \to H \) satisfy

\[ f(x + y) - g(x) - h(y) \in K \quad \text{for} \quad (x, y) \in \Delta, \]

then the following are true:

(a) There are functions \( k_1, l_1 : G \to K, \varphi_1 : G \to H \) and constants \( a, b \in H \) such that

\[ \varphi_1(x + y) - \varphi_1(x) - \varphi_1(y) \in K \quad \text{for} \quad (x, y) \in \Delta \]

and

\[ \begin{cases} f(x) = \varphi_1(x) + a, \\ g(x) = \varphi_1(x) + k_1(x) + b, \\ h(x) = \varphi_1(x) - k_1(x) + l_1(x) + a - b \end{cases} \]

for all \( x \in G \).
There are functions \( k_2, l_2 : G \to K \), \( \varphi_2 : G \to H \) and constants \( a, b \in H \) such that

\[
\varphi_2(x + y) - \varphi_2(x) - \varphi_2(y) \in K \quad \text{for } (x, y) \in \Delta
\]

and

\[
\begin{align*}
  f(x) &= \varphi_2(x) + k_2(x) + a, \\
  g(x) &= \varphi_2(x) + b, \\
  h(x) &= \varphi_2(x) + l_2(x) + a - b
\end{align*}
\]

for all \( x \in G \).

There are functions \( k_3, l_3 : G \to K \), \( \varphi_3 : G \to H \) and constants \( a, b \in H \) such that

\[
\varphi_3(x + y) - \varphi_3(x) - \varphi_3(y) \in K \quad \text{for } (x, y) \in \Delta
\]

and

\[
\begin{align*}
  f(x) &= \varphi_3(x) + k_3(x) + a, \\
  g(x) &= \varphi_3(x) + l_3(x) + b, \\
  h(x) &= \varphi_3(x) + a - b
\end{align*}
\]

for all \( x \in G \).

Moreover, each of assertions (a), (b), (c) gives a complete description of solutions of (2), that is, every triple \((f, g, h)\), being of one of the forms described above, is a solution of (2).

**Proof.** Setting \( y = 0 \) in (2), by (1) we get

\[
\mu(x) := f(x) - g(x) - h(0) \in K \quad \text{for } x \in G
\]

and setting \( x = 0 \) we have

\[
\nu(y) := f(y) - g(0) - h(y) \in K \quad \text{for } y \in G.
\]

In particular,

\[
f(0) - g(0) - h(0) \in K.
\]
Denote $a = f(0)$, $b = g(0)$ and define $\varphi_i, k_i, l_i : G \to H$ for $i = 1, 2, 3$ by

\[
\begin{align*}
\varphi_1 &= f - a, \\
k_1 &= g - \varphi_1 - b, \\
l_1 &= h + k_1 - \varphi_1 - a + b, \\
\varphi_2 &= g - b, \\
k_2 &= f - \varphi_2 - a, \\
l_2 &= h - \varphi_2 - a + b, \\
\varphi_3 &= h - a + b, \\
k_3 &= f - \varphi_3 - a, \\
l_3 &= g - \varphi_3 - b.
\end{align*}
\]

Using (4), (5), (2) and (6) for every $(x, y) \in \Delta$ we get

\[
\begin{align*}
\varphi_1(x + y) - \varphi_1(x) - \varphi_1(y) &= f(x + y) - a - f(x) + a - f(y) + a \\
&= f(x + y) - \mu(x) - g(x) - h(0) - \nu(y) - g(0) - h(y) + a \in K; \\
\varphi_2(x + y) - \varphi_2(x) - \varphi_2(y) &= g(x + y) - b - g(x) + b - g(y) + b \\
&= f(x + y) - \mu(x + y) - h(0) - g(x) + \mu(y) - f(y) + h(0) + b \\
&= f(x + y) - \mu(x + y) - g(x) + \mu(y) - \nu(y) - g(0) - h(y) + b \in K; \\
\varphi_3(x + y) - \varphi_3(x) - \varphi_3(y) &= h(x + y) - a + b - h(x) + a - b - h(y) + a - b \\
&= f(x + y) - g(0) - \nu(x + y) + \nu(x) - f(x) + g(0) - h(y) + a - b \\
&= f(x + y) - \nu(x + y) + \nu(x) - \mu(x) - g(x) - h(0) - h(y) + a - b, \\
&\in K.
\end{align*}
\]

We also have

\[
\begin{align*}
k_1(x) &= g(x) - f(x) + a - b = -\mu(x) - h(0) + a - b \in K, \\
k_2(x) &= f(x) - g(x) + b - a = \mu(x) + h(0) + b - a \in K, \\
k_3(x) &= f(x) - h(x) + a - b - a = \nu(x) + g(0) - b \in K, \\
l_1(x) &= h(x) + k_1(x) - f(x) + a - a + b = -\nu(x) - g(0) + k_1(x) + b \in K, \\
l_2(x) &= h(x) + k_2(x) - f(x) + a - a + b = -\nu(x) - g(0) + k_2(x) + b \in K, \\
l_3(x) &= g(x) + k_3(x) - f(x) + a - b = -\mu(x) - h(0) + k_3(x) + a - b \in K
\end{align*}
\]

for $x \in G$. \qed

The part (b) of this lemma in the case when $\Delta = G^2$ was also obtained by K. Baron and PL. Kannappan in [1], even under some weaker assumptions. Some variations of (2) for functions with values in groupoids were studied by J. Sikorska in [6].

We work with the orthogonality proposed by K. Baron and P. Volkmann in [2], assuming additionally that the last condition in the following definition holds:
Let $G$ be a group such that the mapping

$$x \mapsto 2x, \quad x \in G,$$

is a bijection onto the group $G$. A relation $\perp \subset G^2$ is called orthogonality if it satisfies the following three conditions:

(i) $0 \perp 0$; and from $x \perp y$ the relations $-x \perp -y$, $\frac{x}{2} \perp \frac{y}{2}$ follow.

(ii) If an orthogonally additive function from $G$ to an abelian group is odd, then it is additive; if it is even, then it is quadratic.

(iii) $x \perp 0$ and $0 \perp x$ for every $x \in G$.

For a subset $U$ of a given group and for $n \in \mathbb{N}$ the symbol $nU$ denotes the set $\{nx : x \in U\}$.

**Theorem.** Assume $G$ is an abelian topological group such that the mapping (7) is a homeomorphism and every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that

$$U \subset 2U \quad \text{and} \quad G = \bigcup \{2^nU : n \in \mathbb{N}\}. $$

Let $\perp \subset G^2$ be an orthogonality, $H$ an abelian topological group and $K$ a discrete subgroup of $H$. Assume that functions $f, g, h : G \to H$ satisfy

$$f(x + y) - g(x) - h(y) \in K \quad \text{for} \quad x, y \in G \quad \text{such that} \quad x \perp y. $$

(i) If at least one of the functions $f, g, h$ is continuous at a point, then there exist: a continuous additive function $A : G \to H$, a continuous biadditive and symmetric function $B : G \times G \to H$ and constants $a, b \in H$ such that

$$\begin{cases} 
  f(x) - B(x, x) - A(x) - a \in K, \\
  g(x) - B(x, x) - A(x) - b \in K, \\
  h(x) - B(x, x) - A(x) - a + b \in K
\end{cases}$$

for $x \in G$ and

$$B(x, y) = 0 \quad \text{for} \quad x, y \in G \quad \text{such that} \quad x \perp y. $$

(ii) Let $\mathcal{M}$ be a $\sigma$-algebra of subsets of $G$ such that

$$x \pm 2A \in \mathcal{M} \quad \text{for all} \quad x \in G \quad \text{and} \quad A \in \mathcal{M}$$

and there is a proper $\sigma$-ideal $\mathcal{I}$ of subsets of $G$ with

$$0 \in \text{Int}(A - A) \quad \text{for} \quad A \in \mathcal{M} \setminus \mathcal{I}. $$
Assume moreover that $H$ is separable metric and the following condition (G) is fulfilled:

$(G)$ either $G$ is a first countable Baire group, or $G$ is metric separable, or $G$ is metric and $\mathcal{M}$ contains all Borel subsets of $G$.

If at least one of the functions $f, g, h$ is $\mathcal{M}$-measurable, then there exist: a continuous additive function $A: G \to H$, a continuous biadditive and symmetric function $B: G \times G \to H$ and constants $a, b \in H$ such that (10) and (11) hold.

Moreover, each of assertions (i), (ii) gives a complete description of solutions of (9).

**Proof.** (i): Case 1. Assume that $f$ is continuous at a point. Let $k_1, l_1 : G \to K, \varphi_1 : G \to H$ be as in Lemma (a). Then the function $\varphi_1$ is continuous at a point. According to Theorem 1 from [7] we get a continuous additive function $A: G \to H$ and a continuous biadditive and symmetric function $B: G \times G \to H$ such that

$$\varphi_1(x) - B(x, x) - A(x) \in K \quad \text{for } x \in G$$

and (11) hold. Then, according to (3),

$$f(x) - B(x, x) - A(x) - a = \varphi_1(x) + a - B(x, x) - A(x) - a \in K,$$

$$g(x) - B(x, x) - A(x) - b = \varphi_1(x) + k_1(x) + b - B(x, x) - A(x) - b \in K,$$

$$h(x) - B(x, x) - A(x) - a + b = \varphi_1(x) - k_1(x) + l_1(x) + a - b$$

$$- B(x, x) - A(x) - a + b \in K$$

for all $x \in G$.

Case 2. If the function $g$ is continuous at a point then instead of Lemma (a) we use Lemma (b).

Case 3. If the function $h$ is continuous at a point then we use Lemma (c).

(ii): If one of the functions $f, g, h$ is $\mathcal{M}$-measurable then we use Theorem 1 from [4] instead of Theorem 1 from [7].

For $\perp = G^2$ some special cases were obtained in [1] (cf. Corollaries 6 and 7 there).

If in the Theorem $G$ is Baire and we consider the Baire measurability, then we do not need to assume the first countability of $G$ in order to get the factorization with a separately continuous biadditive term only (cf. Corollary 2 in [4]).

**Corollary 1.** Assume $G$ is an abelian topological group such that the mapping (7) is a homeomorphism and every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that (8) holds. Let $\perp \subset G^2$ be an
orthogonality, $H$ an abelian separable metric group, $K$ a discrete subgroup of $H$ and functions $f,g,h: G \to H$ satisfy (9). If $G$ is Baire and at least one of the functions $f,g,h$ is Baire measurable, then there exist: a continuous additive function $A: G \to H$, a function $B: G \times G \to H$ biadditive, symmetric and continuous in each variable, and constants $a,b \in H$ such that (10) and (11) hold.

If we take $\perp = G^2$, then our Theorem gives us Corollary 2 below. It also leads to another conclusions in the case when we consider Baire or Christensen measurability.

**Corollary 2.** Assume $G$ is an abelian topological group such that the mapping (7) is a homeomorphism and every neighbourhood of zero in $G$ contains a neighbourhood $U$ of zero such that (8) holds. Let $H$ be an abelian separable metric group, $K$ a discrete subgroup of $H$, $\mathfrak{M}$ a $\sigma$-algebra of subsets of $G$ satisfying (12) and such that there is a proper $\sigma$-ideal $\mathfrak{I}$ of subsets of $G$ with property (13). If functions $f,g,h: G \to H$ satisfy

$$f(x + y) - g(x) - h(y) \in K \quad \text{for } x,y \in G$$

and at least one of them is $\mathfrak{M}$-measurable, then there exist a continuous additive function $A: G \to H$ and constants $a,b \in H$ such that

$$\begin{cases} f(x) - A(x) - a \in K, \\ g(x) - A(x) - b \in K, \\ h(x) - A(x) - a + b \in K \end{cases}$$

for $x \in G$.

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**References**


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