ANALYSIS OF HM-NETWORKS WITH STOCHASTIC INCOMES FROM TRANSITIONS BETWEEN STATES

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Abstract. In article method of finding of expected incomes in systems of HM-network of arbitrary topology when incomes from transitions between network states are stochastic variables with given mean values is proposed. For expected incomes the system of linear non-homogeneous ordinary differential equations was obtained, to solve it we can find incomes in network systems.

Introduction

Markov queueing networks (QN) with incomes were examined in works \cite{1-3} at the first time. They are described with help of Markov chains with continuous time and incomes which were introduced by R. Howard \cite{4}. So at the recent time they are called HM(Howard-Matalytski)-networks \cite{5, 6}. Before investigation of closed networks with account number of states was carry out. Herewith the next cases were examined: a) incomes from transitions between network states depend on states and time or b) incomes are stochastic variables (SV) with known finite moments of the first and the second orders.

For expected incomes of network systems in case a) the system of difference-differential equations which are reduced to the system of linear non-homogeneous ordinary differential equations (ODE) can be obtained. For their solution different methods - method of multidimensional z-transformations and known methods: method of Laplace transformation, matrix method, numerical methods were proposed \cite{7-9}.

In \cite{5, 10} approximate relations for expected incomes and income variations in systems of exponential HM-networks in case b) were obtained. Technique of receiving of these relations is based on interval partition of the network functioning by big number \( m \) of small intervals of size \( \Delta t \), income estimation on every interval and summing of these incomes by means of passage to the limit \( m \to \infty, \Delta t \to 0 \). Herewith mean value of messages in network system in unsteady condition was found with help of developed recurrence by time moments method. Notice that different methods of analysis and optimization of Markov HM-networks and their
application were described in [11]. In the present work method of finding of expected incomes of network systems in case b) which is based on solution of the linear non-homogeneous ODE, which were received for expected incomes and mean values of messages, is proposed.

1. Finding of expected incomes in systems

Let us examine open exponential QN of arbitrary topology with one type messages which consists of \( n \) queueing systems (QS) \( S_1, S_2, \ldots, S_n \) with \( m_i \) service channels in system \( S_i, \ i = 1, n \). Denote via \( k(t) = (k_1(t), k_2(t), \ldots, k_n(t)) \) - vector of network states, where \( k_i(t) \) - message number in system \( S_i \) (in queue and service) at the moment \( t \). Poisson flow of messages of the rate \( \lambda \) enters the network. Service rate of messages at the moment \( t \mu(k(t)) \) in system \( S_i \) depends on message number in this system, \( i = 1, n \). Message when transiting from one QS to another brings to the last system some stochastic income and income of the first system reduces by this value correspondingly.

Let us consider dynamics of income changes of some network system \( S_i \). Denote it’s income at the moment \( t \) as \( V_i(t) \). Let at the initial moment income of system equals \( V_i(0) = v_{i0} \). Income of this QS at the moment \( t + \Delta t \) can be presented as

\[
V_i(t + \Delta t) = V_i(t) + \Delta V_i(t, \Delta t)
\]

where \( \Delta V_i(t, \Delta t) \) - income change of system \( S_i \) on time interval \([t, t+\Delta t]\). For finding of this value we write probabilities of events which can appear during time \( \Delta t \) and changes of incomes of system \( S_i \), which are connected with these events.

1. Message from the outside with probability \( \lambda p_{i0} \Delta t + o(\Delta t) \) will enter to system \( S_i \) and will bring to it income of size \( r_{i0} \), where \( r_{i0} \) - RV with mathematical expectation \( \{\mu\} M\{r_{i0}\} = a_{i0} \), \( p_{i0} \) - probability of message enter from outside to the system \( S_i, i = 1, n \).

2. Message from the system \( S_i \) with probability \( \mu_i(k_i(t)) u(k_i(t)) p_{i0} \Delta t + o(\Delta t) \) will pass to the outside and income of the system \( S_i \) will decrease by value \( R_{i0} \), where \( R_{i0} \) - RV with m.e. \( M\{R_{i0}\} = b_{i0} \), \( p_{i0} \) - probability of message leaving from system \( S_i \) to the outside, \( i = 1, n \), \( u(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0, \end{cases} \) - Heavyside function.

3. Message from the system \( S_i \) with probability \( \mu_j(k_j(t)) u(k_j(t)) p_{j0} \Delta t + o(\Delta t) \) will pass to the system \( S_i \) and income of the system \( S_i \) will increase by value \( r_{ji} \) and income of the system \( S_j \) will decrease by this value, where \( r_{ji} \) - RV with m.e.
\[ M \{ r_{ji} \} = a_{ji}, \quad p_{ji} \] - probability of message transition from the system \( S_j \) to the system \( S_i \), \( i, j = 1, \ldots, n, i \neq j \).

4. Message from the system \( S_i \) with probability \( \mu_i(k_i(t))u(k_i(t))p_{ji}\Delta t + o(\Delta t) \) will pass to the system \( S_j \) and income of the system \( S_i \) will decrease by value \( R_{ij} \) and income of the system \( S_j \) increase by this value, where \( R_{ij} \) - RV with m.e. 
\[ M \{ R_{ji} \} = b_{ji}, \quad i, j = 1, \ldots, n, i \neq j. \]

5. State changes of system \( S_i \) on the time interval \([t, t+\Delta t])\) with probability 
\[
1 - \left( \lambda p_{io} + \mu_i(k_i(t))u(k_i(t)) + \sum_{j=1}^{n} \mu_j(k_j(t))u(k_j(t))p_{ji} \right) \Delta t + o(\Delta t) \text{ will not appear,}
\]
\[ i = 1, \ldots, n. \]

Besides that during each small time interval \( \Delta t \) the system \( S_i \) increases its income by value \( r_i \Delta t \), where \( r_i \) - RV with m.e. \( M \{ r_i \} = c_i \), \( i = 1, \ldots, n \). Let also suppose that RV \( r_{ji} \), \( R_{ji} \), \( r_{io} \), \( R_{io} \) are independent with respect to RV \( r_i \), \( i, j = 1, \ldots, n \).

Evidently that \( r_{ji} = R_{ji} \) with probability 1, i.e. 
\[ a_{ji} = b_{ji}, \quad i, j = 1, \ldots, n \] (2)

Then from said follows
\[
\Delta V_i(t, \Delta t) = \begin{cases} 
  r_i + r_i \Delta t & \text{with probability } \lambda p_{io} \Delta t + o(\Delta t), \\
  -R_{io} + r_i \Delta t & \text{with probability } \mu_i(k_i(t))u(k_i(t))p_{io} \Delta t + o(\Delta t), \\
  r_{ji} + r_i \Delta t & \text{with probability } \mu_j(k_j(t))u(k_j(t))p_{ji} \Delta t + o(\Delta t), \\
  -R_{ji} + r_i \Delta t & \text{with probability } \mu_i(k_i(t))u(k_i(t))p_{ji} \Delta t + o(\Delta t), \\
  r_i \Delta t & \text{with probability } 1 - \left( \lambda p_{io} + \mu_i(k_i(t))u(k_i(t)) + \sum_{j=1}^{n} \mu_j(k_j(t))u(k_j(t))p_{ji} \right) \Delta t + o(\Delta t) 
\end{cases}
\] (3)

Under fixed realization of the process \( k(t) \) and taking to account (3) it can be written:
Average the last relation by \( k(t) \) with taking to account normalization condition \( \sum_k P(k(t) = k) = 1 \) for income change of system \( S_i \) we will obtain

\[
M \{ \Delta V_i(t, \Delta t) / k(t) \} = \sum_k P(k(t) = k) M \{ \Delta V_i(t, \Delta t) / k(t) \} = \\
= \sum_{k_1=k_2=0}^{\infty} \sum_{k_n=0}^{\infty} \ldots \sum_{k_3=0}^{\infty} P(k(t) = (k_1(t), k_2(t), \ldots, k_n(t))) \times \\
\times M \{ \Delta V_i(t, \Delta t) / k(t) = (k_1(t), k_2(t), \ldots, k_n(t)) \} = \\
= \left[ \lambda p_{00} a_{0i} + c_i - \left( p_{0i} b_{0i} + \sum_{j \neq i}^{\infty} p_{ji} b_{ji} \right) \sum_k P(k(t) = k) \mu_i(k_i(t)) u(k_i(t)) + \\
+ \sum_{j \neq i}^{\infty} p_{ji} a_{ji} \sum_k P(k(t) = k) \mu_j(k_j(t)) u(k_j(t)) \right] \Delta t + o(\Delta t)
\]

Let the system \( S_i \) contains \( m_i \) identical service channels, time of message service in every channel is distributed under exponential law with parameter \( \mu_i \), \( i = 1, n \). In this case

\[
\mu_i(k_i(t)) = \begin{cases} 
\mu_i k_i(t), & k_i(t) \leq m_i, \\
\mu_i m_i, & k_i(t) > m_i
\end{cases}, \quad \mu_i(k_i(t)) u(k_i(t)) = \mu_i \min(k_i(t), m_i), \quad i = 1, n
\]

Let us suppose that averaging of expression \( \mu_i(k_i(t)) u(k_i(t)) \) brings \( \mu_i \min(N_i(t), m_i) \), i.e.

\[
M \min(k_i(t), m_i) = \min(N_i(t), m_i)
\] (4)
where \( N_i(t) \) - average number of messages (waiting and serving) in the system \( S_i \) at the moment \( t, \ i = 1, n \). Taking in account this assumption we obtain the next approximate relation

\[
M \{ \Delta V_i(t, \Delta t) \} = \left[ \lambda p_{0i} a_{ii} + c_i - \mu_i \min(N_i(t), m_i) \left( p_{i0} b_{i0} + \sum_{j \neq i} p_{ij} b_{ij} \right) + \right.
\]

\[
+ \sum_{j \neq i} \mu_j \min(N_j(t), m_j) p_{ji} a_{ji} \right] \Delta t + o(\Delta t)
\]

(5)

Since Poisson flow of messages with rate \( \lambda \) enters the network, i.e. probability of entering of \( l \) messages in the system \( S_i \) during time \( \Delta t \) has the appearance

\[
P_l(\Delta t) = \frac{(\lambda \Delta t)^l}{l!} e^{-\lambda \Delta t}, \ l = 0, 1, 2, \ldots, \ 
\]

so average number of messages which entered to the system \( S_i \) from the outside during time \( \Delta t \) equals \( \lambda p_{0i} \Delta t \). Denote the average number of busy service channels in system \( S_i \) at the moment \( t \) as \( \rho_i(t) \), \( i = 1, n \). Then \( \mu_i \rho_i(t) \Delta t \) - average number of messages that left the system \( S_i \) during time \( \Delta t \) and \( \sum_{j \neq i} \mu_j \rho_j(t) p_{ji} \Delta t \) - average number of messages that entered in the system \( S_i \) from another QS during time \( \Delta t \). So

\[
N_i(t + \Delta t) - N_i(t) = \lambda p_{0i} \Delta t + \sum_{j \neq i} \mu_j \rho_j(t) p_{ji} \Delta t - \mu_i \rho_i(t) \Delta t, \ i = 1, n
\]

whence with \( \Delta t \to 0 \) system of ODE for \( N_i(t) \) follows:

\[
\frac{dN_i(t)}{dt} = \sum_{j \neq i} \mu_j \rho_j(t) p_{ji} - \mu_i \rho_i(t) + \lambda p_{0i}, \ i = 1, n
\]

(6)

It is impossible to find variable \( \rho_i(t) \) exactly so as we did earlier we approximate it by expression

\[
\rho_i(t) = \begin{cases} 
N_i(t), & N_i(t) \leq m_i, \\
\min(N_i(t), m_i), & m_i, \ N_i(t) > m_i
\end{cases}
\]
Then system of equations (6) will take on form

\[
\frac{dN_i(t)}{dt} = \sum_{j=1}^{n} \mu_j p_{ji} \min(N_j(t),m_j) - \mu_i \min(N_i(t),m_i) + \dot{\lambda} p_{0i}, \quad i = 1, n
\]  

(7)

This system is system of linear ODE with discontinuous right parts. It’s necessary to solve it by means of segmentation of the phase space by set of the areas and finding solution in each of them. System (7) can be solved for example by using means of system of computer mathematics Maple 8.

Let us introduce denotation \( v_i(t) = M\{V_i(t)\}, \ i = 1, n \). From (1), (5) we obtain

\[
v_i(t + \Delta t) = v_i(t) + M\{\Delta V_i(t,\Delta t)\} = \\
v_i(t) + \dot{\lambda} p_{0i} a_{0i} + c_i - \mu_i \min(N_i(t),m_i) \left( p_{0i} b_{0i} + \sum_{j \neq i} p_{ji} b_{ji} \right) + \\
+ \sum_{j \neq i} \mu_j \min(N_j(t),m_j) p_{ji} a_{ji} \Delta t + o(\Delta t)
\]

Then pass to limit with \( \Delta t \to 0 \) we receive non-homogeneous linear ODE of the first order

\[
\frac{dv_i(t)}{dt} = -\mu_i \min(N_i(t),m_i) \left( p_{0i} b_{0i} + \sum_{j \neq i} p_{ji} b_{ji} \right) + \\
+ \sum_{j \neq i} \mu_j \min(N_j(t),m_j) p_{ji} a_{ji} + \dot{\lambda} p_{0i} a_{0i} + c_i, \ i = 1, n
\]  

(8)

Specify initial conditions \( v_i(0) = v_{i0}, \ i = 1, n \), it is possible to find expected incomes of network systems.

If the network is functioning so that queues are not observed in it upon the average, i.e. \( \min(N_i(t),m_i) = N_i(t), \ i = 1, n \), then systems (7), (8) will take the appearance:

\[
\frac{dN_i(t)}{dt} = \sum_{j \neq i} \mu_j p_{ji} N_j(t) - \mu_i N_i(t) + \dot{\lambda} p_{0i}, \ i = 1, n
\]  

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\[
\begin{align*}
\frac{dv_i(t)}{dt} &= -\mu_i \left( p_{i0} b_{i0} + \sum_{j=1}^{n} p_{ij} b_{ij} \right) N_i(t) + \sum_{j=1, j \neq i}^{n} \mu_j p_{ji} a_{ji} N_j(t) + \lambda p_{i0} a_{i0} + c_i, \\
v_i(0) &= v_{i0}, \quad i = 1, n
\end{align*}
\]  

System (9) can be rewrote in matrix form

\[
\frac{d\mathbf{N}(t)}{dt} = \mathbf{Q}\mathbf{N}(t) + \mathbf{f}
\]

where \( N^T(t) = (N_1(t), N_2(t), \ldots, N_n(t)) \), \( \mathbf{Q} \) - quadratic matrix which consists of elements \( q_{ij} = \mu_j p_{ji} \), if suppose \( p_{ii} = -1, \ i, j = 1, n \), \( \mathbf{f} \) - column vector with elements \( \lambda p_{i0} \), \( i = 1, n \). Solution of the system (11) is

\[
N(t) = N(0)e^{\mathbf{Q}t} + \int_0^t e^{\mathbf{Q}(t-\tau)} d\tau
\]

where \( N(0) \) - some given initial conditions, but finding of elements of matrix \( e^{\mathbf{Q}t} \) is difficult problem even for rather small values of \( n \).

Fig. 1. Structure of the network with central QS

Let us consider closed network with central QS that consists of \( n \) systems (Fig. 1). Let suppose that queues are not observed in peripheral systems of the network under the average, i.e. \( \min(N_i(t), m_i) = N_i(t), \ \ i = 1, n - 1 \), and central QS functions in the condition of heavy traffic, i.e. \( \min(N_n(t), m_n) = m_n \). System (7) in this case will re-write as

\[
\begin{align*}
\frac{dN_i(t)}{dt} &= -\mu_i N_i(t) + \mu_o m_n p_{ni}, \quad i = 1, n - 1, \\
\frac{dN_n(t)}{dt} &= \sum_{i=1}^{n-1} \mu_i N_i(t) - \mu_n m_n
\end{align*}
\]  

(12)
General solution of the system (12) with initial conditions \( N_i(0), i = 1, n \), equals

\[
N_i(t) = g_i e^{-\mu_i t} + \frac{m_n \mu_n P_{m_i}}{\mu_i}, \quad i = 1, n-1,
\]

\[
N_n(t) = K - \sum_{i=1}^{n-1} \left( g_i e^{-\mu_i t} + \frac{m_n \mu_n P_{m_i}}{\mu_i} \right)
\]

where \( g_i = N_i(0) - \frac{m_n \mu_n P_{m_i}}{\mu_i} \), \( K = \sum_{i=1}^{n} N_i(t) \) - number of messages in the network.

For such \( N_i(t), i = 1, n \), system (10) for expected incomes of the network systems will take the appearance

\[
\frac{dv_i(t)}{dt} = -\mu_i b_{in} \left( g_i e^{-\mu_i t} + \frac{m_n \mu_n P_{m_i}}{\mu_i} \right) + \\
+ \mu_n p_{mi} a_{mi} \left[ K - \sum_{j=1}^{n-1} \left( g_j e^{-\mu_j t} + \frac{m_n \mu_n P_{m_j}}{\mu_j} \right) \right] + c_i, \quad i = 1, n-1
\]

\[
\frac{dv_n(t)}{dt} = \mu_n \sum_{j=1}^{n} p_{mj} b_{mj} \left[ \sum_{i=1}^{n-1} \left( g_i e^{-\mu_i t} + \frac{m_n \mu_n P_{m_i}}{\mu_i} \right) - K \right] + \\
+ \sum_{j=1}^{n-1} \mu_j a_{pj} \left( g_j e^{-\mu_j t} + \frac{m_n \mu_n P_{m_j}}{\mu_j} \right) + c_n
\]

Integrate given ODE with initial conditions \( v_i(0) = v_{i0}, i = 1, n \), we will obtain

\[
v_i(t) = \mu_n p_{mi} a_{mi} \sum_{j=1}^{n-1} g_j e^{-\mu_j t} + b_{in} g_i e^{-\mu_i t} + \\
+ \left[ \mu_n p_{mi} \left( K a_{mi} - \mu_n m_n a_{mi} \sum_{j=1}^{n-1} p_{mj} - m_n b_{in} \right) + c_i \right] t + \\
- \mu_n p_{mi} a_{mi} \sum_{j=1}^{n-1} g_j e^{-\mu_j t} - b_{in} g_i + v_{i0}, \quad i = 1, n-1
\]
\[ v_n(t) = -\mu_n \sum_{j=1}^{n-1} p_{nj} b_{nj} \sum_{i=1}^{n-1} \frac{g_i}{\mu_i} e^{-\mu_i t} - \sum_{j=1}^{n-1} a_{jn} g_j e^{-\mu_j t} + \]
\[ + \left[ \mu_n \left( m_n \mu_n \sum_{j=1}^{n-1} p_{nj} b_{nj} \sum_{i=1}^{n-1} \frac{P_{nj}}{\mu_i} - K \sum_{j=1}^{n-1} p_{nj} b_{nj} + m_n \sum_{j=1}^{n-1} P_{nj} a_{jn} \right) + c_n \right] t + \tag{14} \]
\[ + \mu_n \sum_{j=1}^{n-1} p_{nj} b_{nj} \sum_{j=1}^{n-1} \frac{g_j}{\mu_j} + \sum_{j=1}^{n-1} a_{jn} g_j + v_{n0} \]

2. Numerical example

Let us examine the network which was described in the previous section with \( n = 30, K = 61 \), where \( K \) - the number of messages in the network. Service rates of messages in channels of network systems equal: \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 6 \), \( \mu_7 = 7 \), \( \mu_8 = 8 \), \( \mu_9 = 9 \), \( \mu_{10} = 10 \), \( \mu_{11} = 11 \), \( \mu_{12} = 12 \), \( \mu_{13} = 13 \), \( \mu_{14} = 14 \), \( \mu_{15} = 15 \), \( \mu_{16} = 16 \), \( \mu_{17} = 17 \), \( \mu_{18} = 18 \), \( \mu_{19} = 19 \), \( \mu_{20} = 20 \), \( \mu_{21} = 21 \), \( \mu_{22} = 22 \), \( \mu_{23} = 23 \), \( \mu_{24} = 24 \), \( \mu_{25} = 25 \), \( \mu_{26} = 26 \), \( \mu_{27} = 27 \), \( \mu_{28} = 28 \), \( \mu_{29} = 29 \), \( \mu_{30} = 30 \), \( \mu_{31} = 31 \), \( \mu_{32} = 32 \), \( \mu_{33} = 33 \), \( \mu_{34} = 34 \), \( \mu_{35} = 35 \), \( \mu_{36} = 36 \), \( \mu_{37} = 37 \), \( \mu_{38} = 38 \), \( \mu_{39} = 39 \), \( \mu_{40} = 40 \), \( \mu_{41} = 41 \), \( \mu_{42} = 42 \), \( \mu_{43} = 43 \), \( \mu_{44} = 44 \), \( \mu_{45} = 45 \), \( \mu_{46} = 46 \), \( \mu_{47} = 47 \), \( \mu_{48} = 48 \), \( \mu_{49} = 49 \), \( \mu_{50} = 50 \), \( \mu_{51} = 51 \), \( \mu_{52} = 52 \), \( \mu_{53} = 53 \), \( \mu_{54} = 54 \), \( \mu_{55} = 55 \), \( \mu_{56} = 56 \), \( \mu_{57} = 57 \), \( \mu_{58} = 58 \), \( \mu_{59} = 59 \), \( \mu_{60} = 60 \), \( \mu_{61} = 61 \)

Charts of average number of messages in the QS are shown in Figures 2-5.

![Fig. 2. Average number of messages in network systems](image-url)
Fig. 3. Average number of messages in network systems $S_i$, $i = 11, 20$

Fig. 4. Average number of messages in network systems $S_i$, $i = 21, 29$
Specify values of m.e. of incomes from transitions between network states:

\[ c_i = 20 \cos \left( \frac{i \pi}{n} - \frac{\pi}{2} \right), \quad i = 1, 30 \]
\[ a_{30i} = 50 \sin \frac{i \pi}{n}, \quad a_{30} = 100 \exp \left( \frac{i}{n} \right), \quad i = 1, 29 \]

Then using relation (13), (14) with initial condition \( \nu_i(0) = 100, \ i = 1, 29, \nu_{30}(0) = 150 \), the expressions for expected incomes of network systems were obtained. For example expression for expected income of central system is

\[ \nu_{30}(t) = 795.4e^{-5t} + 2970.5e^{-3t} + 720.1e^{-5t} + 2354.9e^{-2t} + 149.7e^{-t} - 823.7e^{-7t} - 93.1e^{-7t} - 28654.2t + 15870.1 \]

Charts of expected incomes of the network systems are shown in Figures 6-9.
Fig. 6. Expected incomes of the systems $S_i$, $i = 1,10$

Fig. 7. Expected incomes of the systems $S_i$, $i = 11, 20$
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Fig. 8. Expected incomes of the systems $S_i$, $i = 20, 29$

Fig. 9. Expected incomes of the central QS
References


