ABOUT THE EQUIVALENCE OF THE TANGENCY RELATION OF ARCS

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Abstract. In this paper the problem of the equivalence of the tangency relation $T_l(a, b, k, p)$ of the rectifiable arcs in the generalized metric spaces is considered. Some sufficient conditions for the equivalence of this relation of the rectifiable arcs have been given here.

Introduction

Let $E$ be an arbitrary non-empty set, and $E_0$ the family of all non-empty subsets of the set $E$. Let $l$ be a non-negative real function defined on the Cartesian product $E_0 \times E_0$, and let $l_0$ be the function of the form:

$$l_0(x, y) = l(\{x\}, \{y\}) \quad \text{for} \quad x, y \in E$$

(1)

If we put some conditions on the function $l$, then the function $l_0$ defined by the formula (1) will be the metric of the set $E$. For this reason the pair $(E, l)$ can be treated as a certain generalization of the metric space and we will call it (see [1]) the generalized metric space.

Using (1) we may define in the space $(E, l)$, similarly as in a metric space, the following notions: the sphere $S_l(p, r)$ and the ball $K_l(p, r)$ with centre at the point $p$ and radius $r$.

Let $a, b$ be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of $0$ such that

$$a(r) \xrightarrow{r \to 0^+} 0 \quad \text{and} \quad b(r) \xrightarrow{r \to 0^+} 0$$

(2)

By $S_l(p, r)_u$ (see [1, 2]) we will denote the so-called $u$-neighbourhood of the sphere $S_l(p, r)$ in the space $(E, l)$ defined by the formula:

$$S_l(p, r)_u = \begin{cases} \bigcup_{q \in S_l(p, r)} K_l(q, u) & \text{for} \quad u > 0 \\ S_l(p, r) & \text{for} \quad u = 0 \end{cases}$$

(3)
We say that the pair \((A, B)\) of sets \(A, B \in E_0\) is \((a, b)\)-clustered at the point \(p\) of the space \((E, l)\), if 0 is the cluster point of the set of all real numbers \(r > 0\) such that
\[
A \cap S_l(p, r)_{a(r)} \neq \emptyset \quad \text{and} \quad B \cap S_l(p, r)_{b(r)} \neq \emptyset.
\]
Let (see [3, 4])
\[
T_l(a, b, k, p) = \{(A, B) : A, B \in E_0, \text{the pair } (A, B) \text{ is } (a, b)\text{-clustered at the point } p \text{ of the space } (E, l) \text{ and}

\]
\[
\frac{1}{r^k} l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \to 0 \quad r \to 0^+ (4)
\]
If \((A, B) \in T_l(a, b, k, p)\), then we say that the set \(A \in E_0\) is \((a, b)\)-tangent of order \(k > 0\) to the set \(B \in E_0\) at the point \(p\) of the space \((E, l)\).
The set \(T_l(a, b, k, p)\) defined by (4) we will call the \((a, b)\)-tangency relation of order \(k\) of sets at the point \(p\) in the generalized metric space \((E, l)\).
We say that the tangency relation \(T_l(a, b, k, p)\) is the equivalence in the set \(E_0\), if is reflexive, symmetric and transitive relation in this set.

Let \(\rho\) be a metric of the set \(E\) and let \(A, B\) be arbitrary sets of the family \(E_0\). Let us denote
\[
\rho(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}, \quad d_\rho A = \sup\{\rho(x, y) : x, y \in A\} (5)
\]
By \(\mathfrak{F}_\rho\) we shall denote the class of all functions \(l\) fulfilling the conditions:
1. \( l : E_0 \times E_0 \to [0, \infty)\),
2. \( \rho(A, B) \leq l(A, B) \leq d_\rho(A \cup B) \quad \text{for } A, B \in E_0.\)
From (1) and from the condition 2. we get the equality:
\[
l(\{x\}, \{y\}) = l_0(x, y) = \rho(x, y) \quad \text{for } l \in \mathfrak{F}_\rho \text{ and } x, y \in E
\]
From the above equality it follows that every function \(l \in \mathfrak{F}_\rho\) generates on the set \(E\) the metric \(\rho\).
In this paper the problem of the equivalence of the tangency relation \(T_l(a, b, k, p)\) of the rectifiable arcs in the spaces \((E, l)\), for the functions \(l\) belonging to the class \(\mathfrak{F}_\rho\) is considered.

1. The equivalence of the tangency relation of the rectifiable arcs
Let \(\rho\) be a metric of the set \(E\), and let \(A\) be any set of the family \(E_0\). By \(A'\) we shall denote the set of all cluster points of the set \(A\).
By \(\hat{A}_\rho\) we will denote the class of sets of the form (see [5, 6]):
\[
\tilde{A}_p = \{ A \in E_0: A \text{ is rectifiable arc with the origin at the point } p \in E \text{ and } \lim_{A \to p} \frac{\ell(\tilde{p}x)}{\rho(p,x)} = g < \infty \} \tag{7}
\]
where \(\ell(\tilde{p}x)\) denotes the length of the arc \(\tilde{p}x\) with the ends \(p\) and \(x\).

From the considerations of the paper [4] and from Lemma 2.1 of the paper [7] follows the following corollary:

**Corollary 1.** If the function \(a\) fulfills the condition
\[
\lim_{r \to 0^+} \frac{a(r)}{r} = 0 \tag{8}
\]
then for an arbitrary arc \(A \in \tilde{A}_p\)
\[
\lim_{r \to 0^+} \frac{1}{r} d_\rho(A \cap S_\rho(p,r)) = 0 \tag{9}
\]

We say that the tangency relation \(T_l(a, b, k, p)\) is reflexive in the set \(E\), if
\[
(A, A) \in T_l(a, b, k, p) \text{ for } A \in E_0 \tag{10}
\]

Using Corollary 1 we shall prove the following theorem:

**Theorem 1.** If \(l \in \mathfrak{S}_\rho\), functions \(a, b\) fulfill the condition
\[
\lim_{r \to 0^+} \frac{a(r)}{r} = 0 \text{ and } \lim_{r \to 0^+} \frac{b(r)}{r} = 0 \tag{11}
\]
then the tangency relation \(T_l(a, b, 1, p)\) is reflexive in the class \(\tilde{A}_p\) of the rectifiable arcs.

**Proof.** From the inequality
\[
d_\rho(A \cup B) \leq d_\rho A + d_\rho B + \rho(A, B) \text{ for } A, B \in E_0 \tag{12}
\]
and from the fact that
\[
\rho(A \cap S_\rho(p,r), A \cap S_\rho(p,r)) = 0 \text{ for } A \in E_0 \tag{13}
\]
we get
\[
0 \leq l(A \cap S_\rho(p,r), A \cap S_\rho(p,r)) \leq d_\rho((A \cap S_\rho(p,r)) \cup (A \cap S_\rho(p,r)))
\]
\[
\leq d_\rho(A \cap S_\rho(p, r)_{a(r)}) + d_\rho(A \cap S_\rho(p, r)_{b(r)}) \\
+ \rho(A \cap S_\rho(p, r)_{a(r)}, A \cap S_\rho(p, r)_{b(r)}) \\
= d_\rho(A \cap S_\rho(p, r)_{a(r)}) + d_\rho(A \cap S_\rho(p, r)_{b(r)})
\] (14)

From the assumption (8) and from Corollary 1 it follows that
\[
\frac{1}{r}d_\rho(A \cap S_\rho(p, r)_{a(r)}) \xrightarrow{r \to 0^+} 0
\] (15)

and
\[
\frac{1}{r}d_\rho(A \cap S_\rho(p, r)_{b(r)}) \xrightarrow{r \to 0^+} 0
\] (16)

From (15), (16) and from the inequality (14) we get
\[
\frac{1}{r}l(A \cap S_\rho(p, r)_{a(r)}, A \cap S_\rho(p, r)_{b(r)}) \xrightarrow{r \to 0^+} 0
\] (17)

Hence and from the fact that the pair of arcs \((A, A)\) is \((a, b)\)-clustered at the point \(p\) of the space \((E, l)\) it follows that \((A, A) \in T_l(a, b, 1, p)\), what means that the tangency relation \(T_l(a, b, 1, p)\) is reflexive in the class \(\tilde{A}_p\).

We call the tangency relation \(T_l(a, b, k, p)\) symmetric in the set \(E\), iff
\[
(A, B) \in T_l(a, b, k, p) \Rightarrow (B, A) \in T_l(a, b, k, p) \quad \text{for} \quad A, B \in E_0.
\] (18)

**Theorem 2.** If functions \(a, b\) fulfil the condition (11) and \(l \in \mathcal{F}_\rho\), then for arbitrary arcs of the class \(\tilde{A}_p\) the tangency relation \(T_l(a, b, 1, p)\) is symmetric.

**Proof.** We assume that \((A, B) \in T_l(a, b, 1, p)\) for \(A, B \in \tilde{A}_p\) and \(l \in \mathcal{F}_\rho\). From here and from the Theorem 2 of the paper [4] on the compatibility of the tangency relation of arcs it follows that \((A, B) \in T_l(b, a, 1, p)\). Therefore
\[
\frac{1}{r}l(A \cap S_\rho(p, r)_{b(r)}, B \cap S_\rho(p, r)_{a(r)}) \xrightarrow{r \to 0^+} 0
\] (19)

From the inequality (12) and from the assumption that \(l \in \mathcal{F}_\rho\), we get
\[
0 \leq l(B \cap S_\rho(p, r)_{a(r)}, A \cap S_\rho(p, r)_{b(r)}) \\
\leq d_\rho((B \cap S_\rho(p, r)_{a(r)}) \cup (A \cap S_\rho(p, r)_{b(r)})) \\
\leq d_\rho(A \cap S_\rho(p, r)_{b(r)}) + d_\rho(B \cap S_\rho(p, r)_{a(r)})
\]
About the equivalence of the tangency relation of arcs

\[ + \rho(A \cap S_\rho(p, r)_{b(r)}; B \cap S_\rho(p, r)_{a(r)}) \]
\[ \leq d_\rho(A \cap S_\rho(p, r)_{b(r)}) + d_\rho(B \cap S_\rho(p, r)_{a(r)}) \]
\[ + l(A \cap S_\rho(p, r)_{b(r)}; B \cap S_\rho(p, r)_{a(r)}). \]

Hence, from (19) and from Corollary 1 of this paper it follows that

\[ \frac{1}{r}l(B \cap S_\rho(p, r)_{a(r)}; A \cap S_\rho(p, r)_{b(r)}) \xrightarrow{r \to 0^+} 0 \]

(20)

Hence and from the fact that the pair of arcs \((B, A)\) is \((a, b)\)-clustered at the point \(p\) of the space \((E, l)\) it follows that \((B, A) \in T_i(a, b, 1, p)\). This means that the tangency relation \(T_i(a, b, 1, p)\) is symmetric in the class of arcs \(\tilde{A}_p\).

We say that the tangency relation \(T_i(a, b, k, p)\) is transitive in the set \(E\), if for \(A, B, C \in E_0\)

\[ [(A, B) \in T_i(a, b, k, p) \land (B, C) \in T_i(a, b, k, p)] \Rightarrow (A, C) \in T_i(a, b, k, p). \]

**Theorem 3.** If functions \(a, b\) fulfil the condition (11) and \(l \in \mathcal{F}_\rho\), then for arbitrary arcs of the class \(\tilde{A}_p\) the tangency relation \(T_i(a, b, 1, p)\) is transitive relation.

**Proof.** We assume that \((A, B) \in T_i(a, b, 1, p)\) and \((B, C) \in T_i(a, b, 1, p)\) for arbitrary arcs \(A, B, C \in \tilde{A}_p\) and the function \(l \in \mathcal{F}_\rho\).

From here it follows that

\[ \frac{1}{r}l(A \cap S_\rho(p, r)_{a(r)}; B \cap S_\rho(p, r)_{b(r)}) \xrightarrow{r \to 0^+} 0 \]

(21)

and

\[ \frac{1}{r}l(B \cap S_\rho(p, r)_{a(r)}; C \cap S_\rho(p, r)_{b(r)}) \xrightarrow{r \to 0^+} 0 \]

(22)

From (22) and from the Theorem 2 of the paper \([4]\) on the compatibility of the tangency relation of arcs it results

\[ \frac{1}{r}l(B \cap S_\rho(p, r)_{b(r)}; C \cap S_\rho(p, r)_{b(r)}) \xrightarrow{r \to 0^+} 0 \]

(23)

From the conditions (12), (13) and from the fact that \(l \in \mathcal{F}_\rho\), we get

\[ 0 \leq l(A \cap S_\rho(p, r)_{a(r)}; C \cap S_\rho(p, r)_{b(r)}) \]
\[ \leq d_\rho((A \cap S_\rho(p, r)_{a(r)}) \cup (C \cap S_\rho(p, r)_{b(r)})) \]
\[ \leq d_\rho(((A \cap S_\rho(p, r)_{a(r)}) \cup (B \cap S_\rho(p, r)_{b(r)})) \\
\cup((B \cap S_\rho(p, r)_{b(r)}) \cup (C \cap S_\rho(p, r)_{b(r)}))) \\
\leq d_\rho((A \cap S_\rho(p, r)_{a(r)}) \cup (B \cap S_\rho(p, r)_{b(r)})) \\
+ d_\rho((B \cap S_\rho(p, r)_{b(r)}) \cup (C \cap S_\rho(p, r)_{b(r)})) \\
\leq d_\rho(A \cap S_\rho(p, r)_{a(r)}) + d_\rho(B \cap S_\rho(p, r)_{b(r)}) \\
+ d_\rho(B \cap S_\rho(p, r)_{b(r)}) + d_\rho(C \cap S_\rho(p, r)_{b(r)}) \\
+ \rho(B \cap S_\rho(p, r)_{b(r)}, C \cap S_\rho(p, r)_{b(r)}) \\
\leq d_\rho(A \cap S_\rho(p, r)_{a(r)}) + 2d_\rho(B \cap S_\rho(p, r)_{b(r)}) + d_\rho(C \cap S_\rho(p, r)_{b(r)}) \\
+ l(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)}) + l(B \cap S_\rho(p, r)_{b(r)}, C \cap S_\rho(p, r)_{b(r)}) \\
\frac{1}{r} l((A \cap S_\rho(p, r)_{a(r)}, C \cap S_\rho(p, r)_{b(r)}) \xrightarrow{r \to 0^+} 0 \quad (24) \]

Because the pair \((A, C)\) of arcs of the class \(\tilde{A}_p\) is \((a, b)\)-clustered at the point \(p\) of the space \((E, l)\), then from here and from the condition (24) it follows that \((A, C) \in Tl(a, b, 1, p)\), what means that the tangency relation \(Tl(a, b, 1, p)\) is transitive relation for arbitrary arcs belonging to the class \(\tilde{A}_p\) and the function funkci \(l \in \mathfrak{F}_\rho\).

From the Theorems 1-3 of this paper we get the following corollary:

**Corollary 2.** If \(l \in \mathfrak{F}_\rho\) and the functions \(a, b\) fulfil the condition (11), then the tangency relation \(Tl(a, b, 1, p)\) is the equivalence in the class \(\tilde{A}_p\) of rectifiable arcs.

If
\[ \lim_{A \to p} \frac{l(p, x)}{\rho(p, x)} = 1 \quad (25) \]
then we say that the rectifiable arc \(A \in E_0\) with the origin at the point \(p \in E\) has the Archimedean property at the point \(p\) of the metric space \((E, \rho)\).

The class of all arcs having the Archimedean property at the point \(p \in E\) we denote by \(A_p\). Obvious is following inclusion: \(A_p \subset \tilde{A}_p\).
From here it follows that all results presented in this paper are true for the rectifiable arcs of the class $A_p$.

References