

3D THERMAL WAVE MODEL OF BIOHEAT TRANSFER - SOLUTION BY MEANS OF FINITE DIFFERENCE METHOD

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Abstract. Up to the present, the models describing temperature distribution in the biological tissue as a rule based on the Pennes bioheat transfer equation. Taking into account the nonhomogeneous inner structure of tissue the heat conduction proceeding in this domain should be described by the hyperbolic equation. In the paper the algorithm of numerical solution of hyperbolic heat conduction equation is presented. The explicit variant of finite differences method is applied and the results of computations are shown.

1. Thermal wave model of bioheat transfer

Heat transfer in biological system is usually described by the Pennes equation basing on the classical Fourier law (e.g. [1]). Because the biological tissues are the materials with nonhomogeneous inner structure therefore the modified unsteady heat conduction equation (Cattaneo and Vernotte (CV) equation, hyperbolic heat conduction equation, non-Fourier heat conduction equation) should be taken into account (e.g. [2-4]). A general form of the thermal wave model of bioheat transfer in living tissues is following

$$c \left[\tau \frac{\partial^2 T(x, t)}{\partial t^2} + \frac{\partial T(x, t)}{\partial t} \right] = \lambda \nabla^2 T(x, t) + Q(x, t) + \tau \frac{\partial Q(x, t)}{\partial t} \quad (1)$$

where c , λ denote the volumetric specific heat and thermal conductivity of tissue, $Q(x, t)$ is the volumetric heat due to metabolism and blood perfusion, τ is the relaxation time. The function $Q(x, t)$ is equal to

$$Q(x, t) = G_B c_B [T_B - T(x, t)] + Q_m \quad (2)$$

where G_B is the blood perfusion rate, c_B is the volumetric specific heat of blood, T_B is the artery temperature and Q_m is the metabolic heat source.

It should be pointed out that for $\tau = 0$ the equation (1) reduces to the well-known Pennes bioheat equation.

The equation (1) is supplemented by the boundary condition

$$x \in \Gamma: T(x, t) = T_b(x) \quad (3)$$

and initial ones

$$t = 0: T(x, t) = T_0, \quad \left. \frac{\partial T(x, t)}{\partial t} \right|_{t=0} = 0 \quad (4)$$

where $T_b(x)$ is known boundary temperature and T_0 is known initial temperature of biological tissue.

The dependence between relaxation time τ and thermal wave velocity C is following

$$C^2 = \frac{a}{\tau} \quad (5)$$

where $a = \lambda/c$ is the thermal diffusivity.

Taking into account formula (2) the equation (1) can be written in the form

$$\tau \frac{\partial^2 T(x, t)}{\partial t^2} + \frac{\partial T(x, t)}{\partial t} = a \nabla^2 T(x, t) + \frac{k}{c} [T_b - T(x, t)] + \frac{Q_m}{c} - \frac{\tau k}{c} \frac{\partial T(x, t)}{\partial t} \quad (6)$$

or

$$\tau \frac{\partial^2 T(x, t)}{\partial t^2} + \left(1 + \frac{\tau k}{c}\right) \frac{\partial T(x, t)}{\partial t} = a \nabla^2 T(x, t) + \frac{k}{c} [T_b - T(x, t)] + \frac{Q_m}{c} \quad (7)$$

where $k = G_{BCB}$.

2. Approximation of time derivatives

In order to solve the problem (7), (3), (4) the time grid

$$0 = t^0 < t^1 < \dots < t^{f-2} < t^{f-1} < t^f < \dots < t^F < \infty \quad (8)$$

with constant step $\Delta t = t^f - t^{f-1}$ is introduced.

Using the Lagrange interpolation [4] for the points (t^{f-2}, T^{f-2}) , (t^{f-1}, T^{f-1}) , (t^f, T^f) , where $T^{f-2} = T(x, t^{f-2})$, $T^{f-1} = T(x, t^{f-1})$, $T^f = T(x, t^f)$ one obtains

$$\begin{aligned} t \in [t^{f-2}, t^f]: T(x, t) = & T^{f-2} \frac{(t - t^{f-1})(t - t^f)}{(t^{f-2} - t^{f-1})(t^{f-2} - t^f)} + \\ & T^{f-1} \frac{(t - t^{f-2})(t - t^f)}{(t^{f-1} - t^{f-2})(t^{f-1} - t^f)} + T^f \frac{(t - t^{f-2})(t - t^{f-1})}{(t^f - t^{f-2})(t^f - t^{f-1})} \end{aligned} \quad (9)$$

or

$$t \in [t^{f-2}, t^f]: T(x, t) = T^{f-2} \frac{(t - t^{f-1})(t - t^f)}{2(\Delta t)^2} - T^{f-1} \frac{(t - t^{f-2})(t - t^f)}{(\Delta t)^2} + T^f \frac{(t - t^{f-2})(t - t^{f-1})}{2(\Delta t)^2} \quad (10)$$

On the basis of (10) the time derivative is calculated

$$t \in [t^{f-2}, t^f]: \frac{\partial T(x, t)}{\partial t} = T^{f-2} \frac{2t - t^f - t^{f-1}}{2(\Delta t)^2} - T^{f-1} \frac{2t - t^f - t^{f-2}}{(\Delta t)^2} + T^f \frac{2t - t^{f-1} - t^{f-2}}{2(\Delta t)^2} \quad (11)$$

and then

$$\left. \frac{\partial T(x, t)}{\partial t} \right|_{t=t^f} = \frac{T^{f-2} - 4T^{f-1} + 3T^f}{2\Delta t} \quad (12)$$

while

$$\frac{\partial^2 T(x, t)}{\partial t^2} = \frac{T^{f-2} - 2T^{f-1} + T^f}{(\Delta t)^2} \quad (13)$$

For time derivative the following approximation can be also taken into account

$$\frac{\partial T}{\partial t} = \frac{T^f - T^{f-1}}{2\Delta t} \quad (14)$$

3. Finite differences method

For 3D problem and domain oriented in Cartesian co-ordinate system $x = \{x_1, x_2, x_3\}$ one has

$$\nabla^2 T(x_1, x_2, x_3, t) = \frac{\partial^2 T(x_1, x_2, x_3, t)}{\partial x_1^2} + \frac{\partial^2 T(x_1, x_2, x_3, t)}{\partial x_2^2} + \frac{\partial^2 T(x_1, x_2, x_3, t)}{\partial x_3^2} \quad (15)$$

The following approximation of (15) with respect to the geometrical co-ordinates for constant mesh step h can be taken into account [5]

$$\begin{aligned} \left(\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} \right) = & \frac{T_{i-1,j,k} - 2T_{ijk} + T_{i+1,j,k}}{h^2} + \\ & \frac{T_{i,j-1,k} - 2T_{ijk} + T_{i,j+1,k}}{h^2} + \frac{T_{i,j,k-1} - 2T_{ijk} + T_{i,j,k+1}}{h^2} \end{aligned} \quad (16)$$

where $T_{ijk} = T(x_{1i}, x_{2j}, x_{3k}, t)$, $T_{i-1,j,k} = T(x_{1i-1}, x_{2j}, x_{3k}, t)$ etc.

Using the explicit scheme of FDM one obtains the following approximation of equation (7)

$$\begin{aligned} \frac{\tau}{(\Delta t)^2} (T_{ijk}^{f-2} - 2T_{ijk}^{f-1} + T_{ijk}^f) + \left(\frac{c + \tau k}{2c\Delta t} \right) (T_{ijk}^{f-2} - 4T_{ijk}^{f-1} + 3T_{ijk}^f) = \\ \frac{a}{h^2} (T_{i-1,j,k}^{f-1} + T_{i+1,j,k}^{f-1} + T_{i,j-1,k}^{f-1} + T_{i,j+1,k}^{f-1} + T_{i,j,k-1}^{f-1} + T_{i,j,k+1}^{f-1} - 6T_{ijk}^{f-1}) + \\ \frac{k}{c} (T_B - T_{ijk}^{f-1}) + \frac{Q_m}{c} \end{aligned} \quad (17)$$

From equation (17) results that

$$\begin{aligned} \left[\frac{\tau}{(\Delta t)^2} + 3 \left(\frac{c + \tau k}{2c\Delta t} \right) \right] T_{ijk}^f = \left[\frac{2\tau}{(\Delta t)^2} + 4 \left(\frac{c + \tau k}{2c\Delta t} \right) - \frac{6a}{h^2} - \frac{k}{c} \right] T_{ijk}^{f-1} + \\ \frac{a}{h^2} (T_{i-1,j,k}^{f-1} + T_{i+1,j,k}^{f-1} + T_{i,j-1,k}^{f-1} + T_{i,j+1,k}^{f-1} + T_{i,j,k-1}^{f-1} + T_{i,j,k+1}^{f-1}) + \\ \frac{kT_B + Q_m}{c} - \frac{\tau}{(\Delta t)^2} T_{ijk}^{f-2} - \left(\frac{c + \tau k}{2c\Delta t} \right) T_{ijk}^{f-2} \end{aligned} \quad (18)$$

or

$$\begin{aligned} T_{ijk}^f = \frac{1}{A} \left[\frac{2\tau}{(\Delta t)^2} + 4 \left(\frac{c + \tau k}{2c\Delta t} \right) - \frac{6a}{h^2} - \frac{k}{c} \right] T_{ijk}^{f-1} + \\ \frac{a}{Ah^2} (T_{i-1,j,k}^{f-1} + T_{i+1,j,k}^{f-1} + T_{i,j-1,k}^{f-1} + T_{i,j+1,k}^{f-1} + T_{i,j,k-1}^{f-1} + T_{i,j,k+1}^{f-1}) + \\ \frac{kT_B + Q_m}{Ac} - \left[\frac{2c\tau + \Delta t(c + \tau k)}{2cA(\Delta t)^2} \right] T_{ijk}^{f-2} \end{aligned} \quad (19)$$

where

$$A = \frac{\tau}{(\Delta t)^2} + 3 \left(\frac{c + \tau k}{2c\Delta t} \right) \quad (20)$$

It should be pointed out that for explicit scheme of FDM the stability criterion should be formulated [5].

4. Results of computations

The biological tissue domain of dimensions $0.01 \text{ m} \times 0.01 \text{ m}$ ($L = 0.01 \text{ m}$) has been considered. Initial temperature of tissue equals $T_0 = 37^\circ\text{C}$. The following input data have been taken into account: $\lambda = 0.75 \text{ W/(mK)}$, $c = 3 \cdot 10^6 \text{ J/(m}^3\text{K)}$, $G_B = 0.0005 \text{ 1/s}$, $c_B = 3.9962 \text{ J/(m}^3\text{K)}$, $T_B = 37^\circ\text{C}$, $Q_m = 245 \text{ W/m}^3$.

On the upper boundary $x_3 = L/2$, $-L/2 \leq x_1 \leq L/2$, $-L/2 \leq x_2 \leq L/2$ the Dirichlet condition in the form

$$T\left(x_1, x_2, \frac{L}{2}\right) = \begin{cases} T_{\max} + (T_b - T_{\max}) \frac{x_1^2 + x_2^2}{9/64 L^2}, & x_1^2 + x_2^2 \leq \frac{9}{64} L^2 \\ T_b, & x_1^2 + x_2^2 > \frac{9}{64} L^2 \end{cases} \quad (21)$$

has been assumed - Figure 1. In equation (21) $T_{\max} = 60^\circ\text{C}$, $T_b = 37^\circ\text{C}$. On the remaining part of the boundary the constant temperature $T_b = 37^\circ\text{C}$ can be accepted.

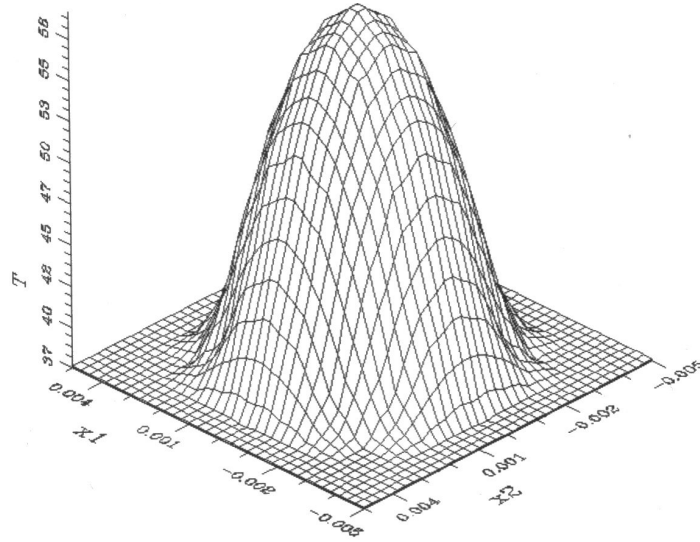


Fig. 1. Boundary condition on the upper surface of the domain considered

The domain has been divided into 1000 internal cells ($10 \times 10 \times 10$) - Figure 2, time step: $\Delta t = 0.1$ s. Figure 3 illustrates heating curves at the point (0, 0, 0) for $\tau = 0$ and $\tau = 20$ s. In Figure 4 the temperature distribution for $\tau = 20$ s (left hand side) and $\tau = 0$ s (right hand side) for times 10, 20 and 30 s in the section $x_1 = 0$ is shown.

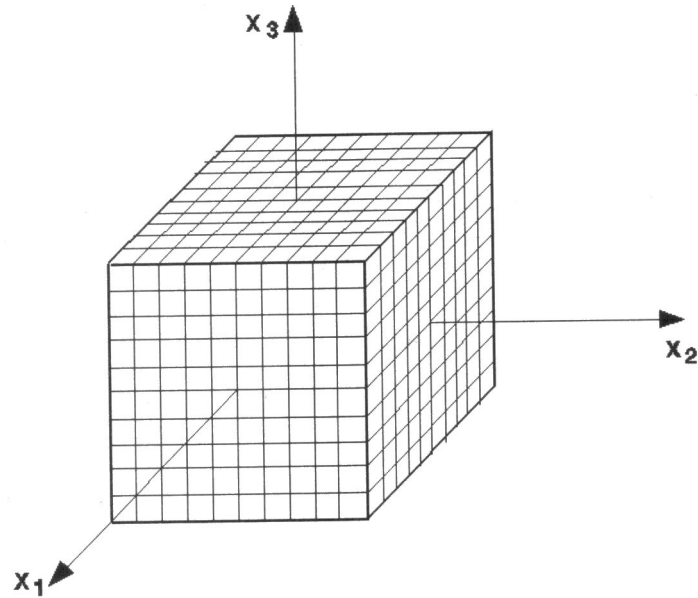


Fig. 2. Discretization

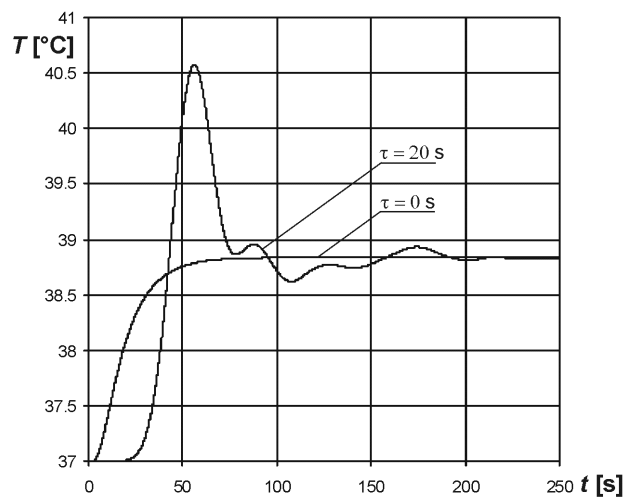


Fig. 3. Heating curves at the point (0, 0, 0)

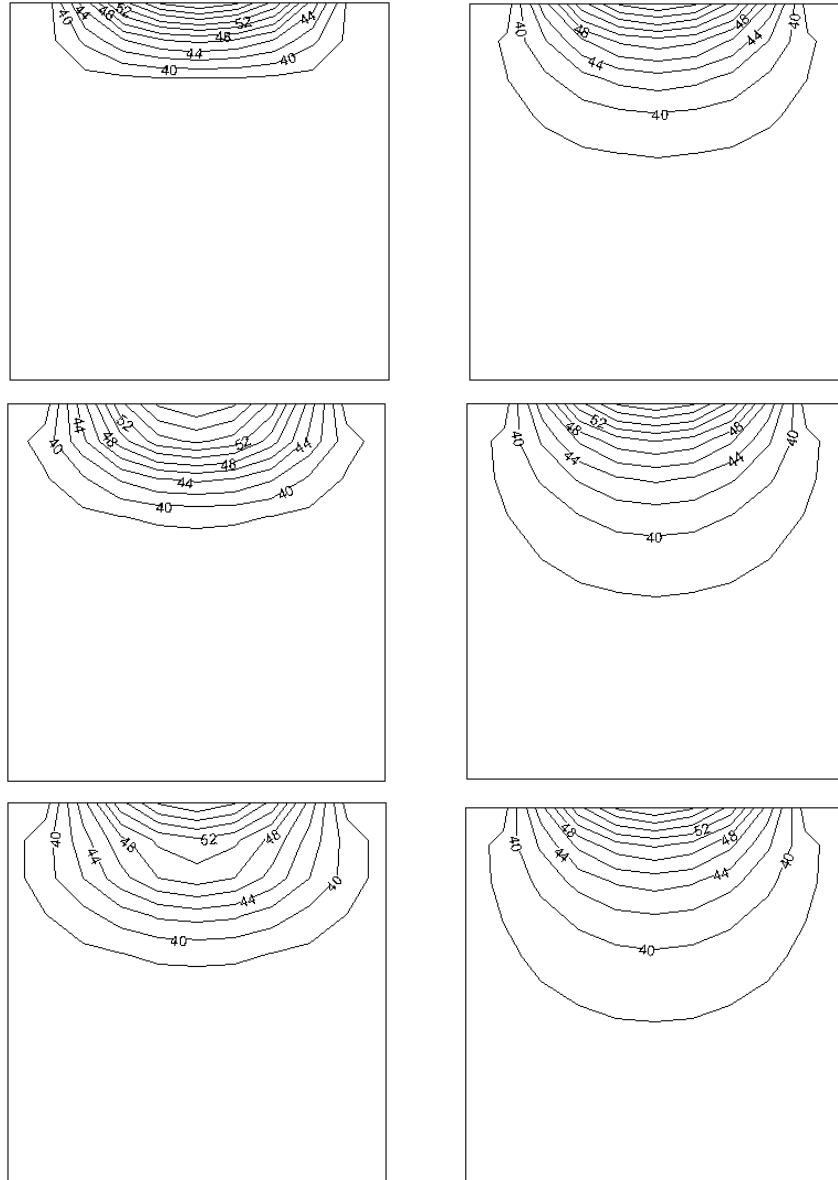


Fig. 4. Temperature distribution for $t = 10$ s, $t = 20$ s and $t = 30$ s ($\tau = 20$, $\tau = 0$)

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