FREE VIBRATION OF ANNULAR PLATES OF STEPPED THICKNESS RESTING ON WINKLER ELASTIC FOUNDATION

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Abstract. This paper concerns axisymmetric free vibration of annular plates of stepped thickness resting on Winkler elastic foundation. Exact solution to vibration problem was obtained by dividing of considered plate into uniform annular plates and by using the Green's function method. Numerical examples are presented.

Introduction

The problems of the free vibration of circular and annular plates of stepped thickness have been the subject of many papers [1-5]. Solutions to free vibration problems of stepped-thickness plates in papers [1, 2] were obtained by using finite element method and optimized Rayleigh-Ritz method. In articles [3, 4] a closed form solution to considered vibration problems was obtained by using Green's function method. The vibration problems of circular plates resting on elastic foundation have been investigated also by many authors, (e.g. [5-7]). The authors of article [5] obtained the solution by using the Galerkin's method. In paper [7] a two-parameter model was used to represent the foundation. In article [6] circular plate on elastic foundation was modeled as a series of simply supported annular plates resting on supporting springs along their common edges.

The present paper deals with a free vibration problem of stepped annular plates resting on Winkler elastic foundation. Exact solution to considered problem is obtained by dividing of the stepped plate into uniform annular plates. Characteristic equation is obtained in analytical form by using the Green's function method. Formulation and solution to the problem take into account arbitrary finite number of uniform plates composing the stepped plate. Analytical solution to presented vibration problem is used to perform numerical analysis of an influence of parameters characterizing the system on its natural frequencies. Numerical examples presented here deal with stepped annular plates composed of two uniform plates.

1. Formulation and solution to the problem

Consider an annular plate resting on Winkler elastic foundation. Thickness of a plate is varying stepwise along $(n-1)$ concentric circles as schematic shown in Figure 1. These circles mark out n plate elements - uniform annular plates of thickness h_j and radii a_{j-1}, a_j, ($a_{j-1} < a_j$, j = 1,...,n).

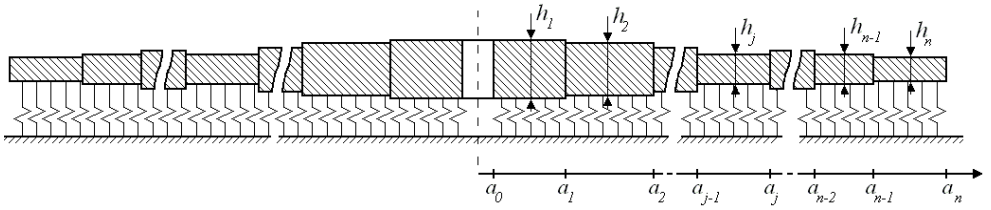


Fig. 1. A stepped annular plate resting on Winkler elastic foundation

Free vibration of j-th plate element is governed by differential equation:

$$\begin{aligned} &D_{\mathrm{j}}\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_{\mathrm{j}}}{\partial r}\right)\right]\right\}+\rho_{\mathrm{j}}h_{\mathrm{j}}\frac{\partial^2 w_{\mathrm{j}}}{\partial t^2}+k_{\mathrm{j}}w_{\mathrm{j}}= \\ &=-\mathrm{s}_{\mathrm{j}-1}\frac{1}{r}\delta\left(r-a_{\mathrm{j}-1}\right)+\mathrm{m}_{\mathrm{j}-1}\frac{1}{r}\delta'\left(r-a_{\mathrm{j}-1}\right)+\mathrm{s}_{\mathrm{j}}\frac{1}{r}\delta\left(r-a_{\mathrm{j}}\right)-\mathrm{m}_{\mathrm{j}}\frac{1}{r}\delta'\left(r-a_{\mathrm{j}}\right) \end{aligned} \tag{1}$$

where $w_j = w_j(r, t)$ is a transverse displacement of j-th plate, r, t - radial and time variable, $D_j = E_j h_j^3 / [12 (1 - v_j^2)]$ - bending rigidity of plate, E_j - Young modulus, v_j - Poisson ratio, ρ_j – mass per unit volume, k_j - stiffness coefficient of the foundations over a region of j-th plate, $s_j = s_j(t)$ - the shearing force, $m_j = m_j(t)$ - bending moment, $\delta(\,)$ is the Dirac delta function. It is also assumed that $s_0 = m_0 = s_n = m_n = 0$.

Equation (1) is completed by boundary conditions and continuity conditions. The boundary conditions can be written symbolically in the form:

$$\mathbf{B}_0\left[w_0\right]\big|_{r=a_0}=0,\quad \mathbf{B}_n\left[w_n\right]\big|_{r=a_n}=0 \tag{2}$$

and continuity conditions are:

$$\begin{aligned} w_{\mathrm{j}}\left(a_{\mathrm{j}},t\right)&=w_{\mathrm{j}+1}\left(a_{\mathrm{j}},t\right) \\ \frac{\mathrm{d}}{\mathrm{d}r}w_{\mathrm{j}}\left(r,t\right)\Big|_{r=a_{\mathrm{j}}}&=\frac{\mathrm{d}}{\mathrm{d}r}w_{\mathrm{j}+1}\left(r,t\right)\Big|_{r=a_{\mathrm{j}}}, \qquad \mathrm{j}=1,\ldots,n-1 \end{aligned} \tag{3}$$

In case of free harmonic vibration of the system one assumes:

$$w_{\mathrm{j}}\left(r,t\right)=W_{\mathrm{j}}\left(r\right)e^{\mathbf{i}\omega t},\quad \mathrm{s}_{\mathrm{j}}=\mathrm{S}_{\mathrm{j}}e^{\mathbf{i}\omega t},\quad \mathrm{m}_{\mathrm{j}}=\mathrm{M}_{\mathrm{j}}e^{\mathbf{i}\omega t} \tag{4}$$

where ω is an eigenfrequency of the system. Introducing simultaneously dimensionless quantities:

$$\bar{r}_{\mathrm{j}} = r / a_{\mathrm{j}}, \quad \bar{r}_{\mathrm{j,i}} = r_{\mathrm{j,i}} / a_{\mathrm{j}}, \quad \overline{W}_{\mathrm{j}} = W_{\mathrm{j}} / a_{\mathrm{j}} \tag{5}$$

and taking into account (4), equation (1) and the continuity conditions take the form:

$$\begin{aligned} \mathbf{L}_{\mathrm{j}}\left[\overline{W}_{\mathrm{j}}\right] = -\overline{\mathrm{S}}_{\mathrm{j-1}} \frac{\Delta_{\mathrm{j}}}{\alpha_{\mathrm{j}}} \frac{1}{\bar{r}_{\mathrm{j}}} \delta\left(\bar{r}_{\mathrm{j}} - \alpha_{\mathrm{j}}\right) + \overline{\mathrm{M}}_{\mathrm{j-1}} \Delta_{\mathrm{j}} \frac{1}{\bar{r}_{\mathrm{j}}} \delta'\left(\bar{r}_{\mathrm{j}} - \alpha_{\mathrm{j}}\right) + \\ \overline{\mathrm{S}}_{\mathrm{j}} \frac{1}{\bar{r}_{\mathrm{j}}} \delta\left(\bar{r}_{\mathrm{j}} - 1\right) - \overline{\mathrm{M}}_{\mathrm{j}} \frac{1}{\bar{r}_{\mathrm{j}}} \delta'\left(\bar{r}_{\mathrm{j}} - 1\right) \end{aligned} \tag{6}$$

$$\overline{W}_{\mathrm{j}}(1) = \frac{1}{\alpha_{\mathrm{j+1}}} \overline{W}_{\mathrm{j+1}}\left(\alpha_{\mathrm{j+1}}\right), \quad \frac{\mathrm{d}}{\mathrm{d}\bar{r}_{\mathrm{j}}} \overline{W}_{\mathrm{j}}\left(\bar{r}_{\mathrm{j}}\right)\Big|_{\bar{r}_{\mathrm{j}}=1} = \frac{\mathrm{d}}{\mathrm{d}\bar{r}_{\mathrm{j+1}}} \overline{W}_{\mathrm{j+1}}\left(\bar{r}_{\mathrm{j+1}}\right)\Big|_{\bar{r}_{\mathrm{j+1}}=\alpha_{\mathrm{j+1}}} \tag{7}$$

where: $\mathbf{L}_{\mathrm{j}} = \nabla^4 - \left(\Omega_{\mathrm{j}}^4 - \mathrm{K}_{\mathrm{j}}\right)$, $\nabla^2 = \mathrm{d}^2/\mathrm{d}r^2 + (1/r)\,\mathrm{d}/\mathrm{d}r$, $\Omega_{\mathrm{j}}^4 = \rho_{\mathrm{j}} h_{\mathrm{j}} \omega^2 a_{\mathrm{j}}^4 / D_{\mathrm{j}}$, $\mathrm{K}_{\mathrm{j}} = k_{\mathrm{j}} a_{\mathrm{j}}^4 / D_{\mathrm{j}}$, $\alpha_{\mathrm{j}} = a_{\mathrm{j-1}}/a_{\mathrm{j}}$, $\Delta_{\mathrm{j}} = D_{\mathrm{j-1}}/D_{\mathrm{j}}$, $\overline{\mathrm{S}}_{\mathrm{j}} = \mathrm{S}_{\mathrm{j}} a_{\mathrm{j}} / D_{\mathrm{j}}$, $\overline{\mathrm{M}}_{\mathrm{j}} = \mathrm{M}_{\mathrm{j}} / D_{\mathrm{j}}$, j = 1,...,$n$.

Applying to equations (6) the Green's function method and using properties of Dirac delta function allow us to obtain a set of equations:

$$\begin{aligned} W_{\mathrm{j}}\left(r_{\mathrm{j}}\right) = -\tilde{\mathrm{S}}_{\mathrm{j-1}} \frac{\Delta_{\mathrm{j}}}{\alpha_{\mathrm{j}}} G^{\mathrm{j}}\left(r_{\mathrm{j}}, \alpha_{\mathrm{j}}; \overline{\Omega}_{j}\right) - \tilde{\mathrm{M}}_{\mathrm{j-1}} \Delta_{\mathrm{j}} \left(\frac{\mathrm{d}}{\mathrm{d}\xi_{\mathrm{j}}} G^{\mathrm{j}}\left(r_{\mathrm{j}}, \xi_{\mathrm{j}}; \overline{\Omega}_{j}\right) \right)\Big|_{\xi_{\mathrm{j}}=\alpha_{\mathrm{j}}} + \\ + \tilde{\mathrm{S}}_{\mathrm{j}} G^{\mathrm{j}}\left(r_{\mathrm{j}}, 1; \overline{\Omega}_{j}\right) + \tilde{\mathrm{M}}_{\mathrm{j}} \left(\frac{\mathrm{d}}{\mathrm{d}\xi_{\mathrm{j}}} G^{\mathrm{j}}\left(r_{\mathrm{j}}, \xi_{\mathrm{j}}; \overline{\Omega}_{j}\right) \right)\Big|_{\xi_{\mathrm{j}}=1} \end{aligned} \tag{8}$$

where dashes over symbols r, W, S, M are omitted. G^{j} denotes the Green's function of differential operator $\mathbf{L}_{\mathrm{j}}$. Function G^{j} is defined over region $[\alpha_{\mathrm{j}}, 1]\times[\alpha_{\mathrm{j}}, 1]$, $(0 < \alpha_{\mathrm{j}} < 1)$. Green's functions of differential operators occurring in vibration problems of uniform annular plates were derived in papers [3, 4, 8, 9].

Characteristic equation to considered free vibration problem of stepped annular plate resting on elastic foundation is obtained by using equations (8) in continuity conditions (7). Using equations (8) in conditions (7) a system of equation is obtained. This system can be written in a matrix form:

$$\mathbf{A}\,\mathbf{X} = \mathbf{0} \tag{9}$$

where $\mathbf{X} = \left[\tilde{\mathrm{S}}_1 \quad \tilde{\mathrm{M}}_1 \quad \ldots \quad \tilde{\mathrm{S}}_{n-1} \quad \tilde{\mathrm{M}}_{n-1}\right]^T$ is the vector of unknown quantities, $\mathbf{A} = [\mathbf{A}_{\mathrm{ji}}]_{1 \le \mathrm{j,i} \le n\text{-}1}$ is a square-matrix where:

$$\mathbf{A}_{\mathrm{j,j-1}} = -\Delta_{\mathrm{j}} \begin{bmatrix} \dfrac{1}{\alpha_{\mathrm{j}}} G^{\mathrm{j}}(1,\alpha_{\mathrm{j}}) & G^{\mathrm{j}}_{,\xi_{\mathrm{j}}}(1,\alpha_{\mathrm{j}}) \\ \dfrac{1}{\alpha_{\mathrm{j}}} G^{\mathrm{j}}_{,r_{\mathrm{j}}}(1,\alpha_{\mathrm{j}}) & G^{\mathrm{j}}_{,\xi_{\mathrm{j}} r_{\mathrm{j}}}(1,\alpha_{\mathrm{j}}) \end{bmatrix}$$

$$\mathbf{A}_{\mathrm{j,j-1}} = \begin{bmatrix} G^{\mathrm{j}}(1,1) & G^{\mathrm{j}}_{,\xi_{\mathrm{j}}}(1,1) \\ G^{\mathrm{j}}_{,r_{\mathrm{j}}}(1,1) & G^{\mathrm{j}}_{,\xi_{\mathrm{j}} r_{\mathrm{j}}}(1,1) \end{bmatrix} + \Delta_{\mathrm{j+1}} \begin{bmatrix} \dfrac{1}{\alpha^2_{\mathrm{j+1}}} G^{\mathrm{j+1}}(\alpha_{\mathrm{j+1}},\alpha_{\mathrm{j+1}}) & \dfrac{1}{\alpha_{\mathrm{j+1}}} G^{\mathrm{j+1}}_{,\xi_{\mathrm{j+1}}}(\alpha_{\mathrm{j+1}},\alpha_{\mathrm{j+1}}) \\ \dfrac{1}{\alpha_{\mathrm{j+1}}} G^{\mathrm{j+1}}_{,r_{\mathrm{j+1}}}(\alpha_{\mathrm{j+1}},\alpha_{\mathrm{j+1}}) & G^{\mathrm{j+1}}_{,\xi_{\mathrm{j+1}} r_{\mathrm{j+1}}}(\alpha_{\mathrm{j+1}},\alpha_{\mathrm{j+1}}) \end{bmatrix} \tag{10}$$

$$\mathbf{A}_{\mathrm{j,j+1}} = -\begin{bmatrix} \dfrac{1}{\alpha_{\mathrm{j+1}}} G^{\mathrm{j+1}}(\alpha_{\mathrm{j+1}},1) & \dfrac{1}{\alpha_{\mathrm{j+1}}} G^{\mathrm{j+1}}_{,\xi_{\mathrm{j+1}}}(\alpha_{\mathrm{j+1}},1) \\ G^{\mathrm{j+1}}_{,r_{\mathrm{j+1}}}(\alpha_{\mathrm{j+1}},1) & G^{\mathrm{j+1}}_{,\xi_{\mathrm{j+1}} r_{\mathrm{j+1}}}(\alpha_{\mathrm{j+1}},1) \end{bmatrix}$$

$$\mathbf{A}_{\mathrm{ji}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{for i < (j – 1) and i > (j + 1)}$$

Equation (9) has non-trivial solution if and only if:

$$\det \mathbf{A} = 0 \tag{11}$$

Equation (11) is the characteristic equation of the considered vibration problem.

2. Green's function

The Green's function $G(r,\xi)$, corresponding to an annular plate resting on Winkler elastic foundation is a solution of the equation:

$$\nabla^4 G(r,\xi) - (\Omega^4 - \mathrm{K}) G(r,\xi) = \frac{1}{r}\delta(r-\xi) \tag{12}$$

This function satisfies, with respect to variable r, boundary conditions along the plate's edges: $r = a$ and $r = b$. For example, the conditions for the annular plate with free edges are:

$$\frac{d^2G}{dr^2} + \nu \frac{1}{r}\frac{dG}{dr} = 0\,, \quad \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\right] = 0\,, \quad r = a,\ r = b \tag{13}$$

The solution of equation (12) can be written in the form of a sum [4]:

$$G(r,\xi) = G_0(r,\xi) + G_1(r,\xi)\, H(r-\xi) \tag{14}$$

where $G_0(r,\xi)$ and $G_1(r,\xi)$ are solutions of a homogeneous equation:

$$\nabla^4 G_i(r,\xi) - (\Omega^4 - K) G_i(r,\xi) = 0, \ (i = 0, 1) \tag{15}$$

Moreover, $G_1(r,\xi)$ satisfies following conditions [8]:

$$G_1(r,\xi)\big|_{r=\xi} = \frac{\partial}{\partial r} G_1(r,\xi)\bigg|_{r=\xi} = \frac{\partial^2}{\partial r^2} G_1(r,\xi)\bigg|_{r=\xi} = 0 \text{ and } \frac{\partial^3}{\partial r^3} G_1(r,\xi)\bigg|_{r=\xi} = \frac{1}{\xi} \tag{16}$$

Once the function G_1 is known, G_0 must be such function that G given by equation (14) satisfies the boundary conditions.

To solve the equation (15), two cases are considered:

Case 1: $\Omega^4 - K > 0$:

Denoting $\overline{\Omega}^4 = \Omega^4 - K$, equation (15) can be rewritten as [10]:

$$(\nabla^2 + \overline{\Omega}^2)(\nabla^2 - \overline{\Omega}^2) G_i(r,\xi) = 0 \tag{17}$$

and general solution $u(r)$ of equation (17) can be presented in the form:

$$u(r) = c_1 J_0(r\overline{\Omega}) + c_2 Y_0(r\overline{\Omega}) + c_3 I_0(r\overline{\Omega}) + c_4 K_0(r\overline{\Omega}) \tag{18}$$

where J_0, Y_0 are Bessel functions and I_0, K_0 are modified Bessel functions. Using (18) and the conditions (16), we find the function $G_1(r,\xi)$ as [5]:

$$G_1(r,\xi) = \frac{1}{2\overline{\Omega}^2}\Big[I_0(r\overline{\Omega}) K_0(\xi\overline{\Omega}) - I_0(\xi\overline{\Omega}) K_0(r\overline{\Omega}) + \frac{\pi}{2}\big(J_0(r\overline{\Omega}) Y_0(\xi\overline{\Omega}) - J_0(\xi\overline{\Omega}) Y_0(r\overline{\Omega})\big)\Big] \tag{19}$$

Hence, on the basis of equations (14) and (18), we have

$$G(r,\xi) = C_1 J_0(r\overline{\Omega}) + C_2 I_0(r\overline{\Omega}) + C_3 Y_0(r\overline{\Omega}) + C_4 K_0(r\overline{\Omega}) + G_1(r,\xi) H(r-\xi) \tag{20}$$

The constants C_1, C_2, C_3, C_4, are determined by using boundary conditions. In the considered case, the constants for classical boundary conditions are presented in reference [9].

Case 2: $\Omega^4 - K < 0$:

Introducing $\overline{\Omega}^4 = K - \Omega^4$ in equation (15), we can rewritten the equation in the form:

$$(\nabla^2 + \mathbf{i}\overline{\Omega}^2)(\nabla^2 - \mathbf{i}\overline{\Omega}^2) G_i(r,\xi) = 0 \tag{21}$$

and the general solution $u(r)$ of the equation is:

$$u(r) = \bar{c}_1 I_0\left(r\sqrt{\mathbf{i}}\,\overline{\Omega}\right) + \bar{c}_2 K_0\left(r\sqrt{\mathbf{i}}\,\overline{\Omega}\right) + \bar{c}_3 I_0\left(r\sqrt{-\mathbf{i}}\,\overline{\Omega}\right) + \bar{c}_4 K_0\left(r\sqrt{-\mathbf{i}}\,\overline{\Omega}\right) \quad (22)$$

The Bessel functions I_0 and K_0 of the complex arguments can be changed by Kelvin functions *ber*, *bei*, *ker*, *kei*, of a real argument by using the following relationships [10]:

$$I_0\left(x\sqrt{\pm i}\right) = ber(x) \pm i\,bei(x), \quad K_0\left(x\sqrt{\pm i}\right) = ker(x) \pm i\,kei(x) \quad (23)$$

Taking into account these relationships in equation (23) we obtain a real valued function for the real argument:

$$u(r) = c_1 bei\left(r\,\overline{\Omega}\right) + c_2 ber\left(r\,\overline{\Omega}\right) + c_3 kei\left(r\,\overline{\Omega}\right) + c_4\, ker\left(r\,\overline{\Omega}\right) \quad (24)$$

Using the conditions (16), we find the function $G_1(r,\xi)$ in the form:

$$G_1(r,\xi) = \frac{1}{\overline{\Omega}^2}\left[bei\left(r\,\overline{\Omega}\right)ker\left(\xi\,\overline{\Omega}\right) - kei\left(r\,\overline{\Omega}\right)ber\left(\xi\,\overline{\Omega}\right)\right.$$

$$\left. + ber\left(r\,\overline{\Omega}\right)kei\left(\xi\,\overline{\Omega}\right) - ker\left(r\,\overline{\Omega}\right)bei\left(\xi\,\overline{\Omega}\right)\right] \quad (25)$$

The general solution of equation (12), we obtain from equation (14) and in the considered case can be written as:

$$G(r,\xi) = C_1 bei\left(r\,\overline{\Omega}\right) + C_2 ber\left(r\,\overline{\Omega}\right) + C_3 kei\left(r\,\overline{\Omega}\right) + C_4\,\mathrm{ker}\left(r\,\overline{\Omega}\right) + G_1(r,\xi)H(r-\xi) \quad (26)$$

Similarly as in the case 1, the constants C_1, C_2, C_3, C_4, are determined by using boundary conditions.

3. Numerical examples

Presented here numerical examples deal with stepped annular plates having both edges free ($r = a$, $r = b$, $b < a$). The plate is subdivided into two uniform annular plates: an annular plate of thickness h_1 and radii $r = b$, $r = c$ and an annular plate of thickness h_2 and radii $r = c$, $r = a$ ($b < c < a$). In this case, characteristic equation to vibration problem (11) has the form:

$$\begin{vmatrix} G^1\left(1,1;\overline{\Omega}_1\right) + \dfrac{\Delta_2}{\alpha_2^2} G^2\left(\alpha_2,\alpha_2;\overline{\Omega}_2\right) & G^1_{,\xi_1}\left(1,1;\overline{\Omega}_1\right) + \dfrac{\Delta_2}{\alpha_2} G^2_{,\xi_2}\left(\alpha_2,\alpha_2;\overline{\Omega}_2\right) \\ G^1_{,r_1}\left(1,1;\overline{\Omega}_1\right) + \dfrac{\Delta_2}{\alpha_2} G^2_{,r_2}\left(\alpha_2,\alpha_2;\overline{\Omega}_2\right) & G^1_{,\xi_1 r_1}\left(1,1;\overline{\Omega}_1\right) + \Delta_2 G^2_{,\xi_2 r_2}\left(\alpha_2,\alpha_2;\overline{\Omega}_2\right) \end{vmatrix} = 0 \quad (27)$$

where $\alpha_2 = c/a$, $\Delta_2 = D_1/D_2$, $G^j(r_j,\xi_j)$ is a Green's function defined over region $[\alpha_j, 1]\times[\alpha_j, 1]$ (j = 1, 2). It is also assumed: $\rho_1 = \rho_2$, $\nu_1 = \nu_2$ and $E_1 = E_2$. Then $\Delta_2 = \alpha^3$, where $\alpha = h_1/h_2$.

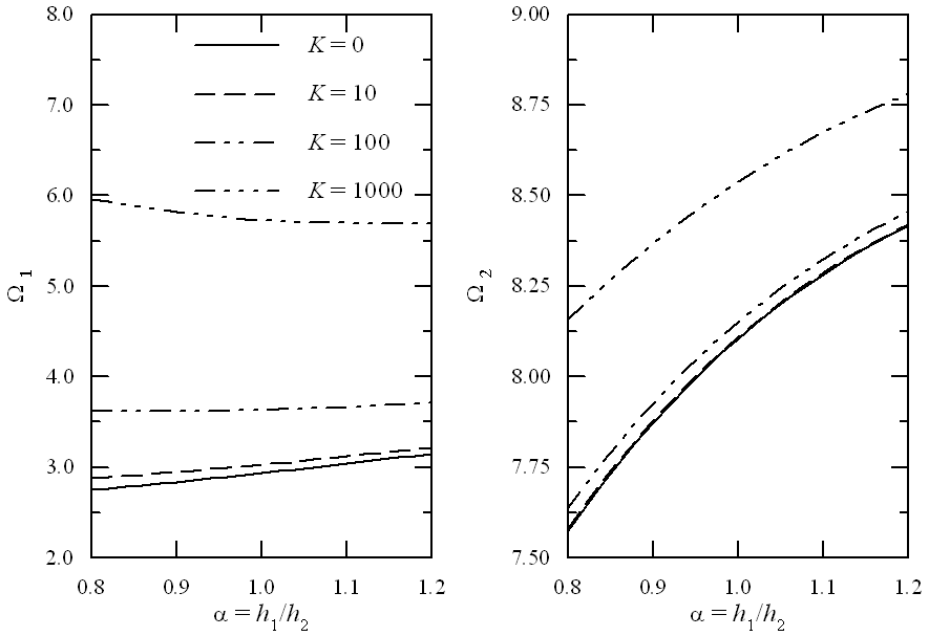


Fig. 2. Values of non-dimensional frequency parameter $\Omega_{2,i} = \sqrt[4]{\rho h_2 \omega^2 a^4 / D_2}$ (i = 1, 2) as a function of parameter $\alpha = h_1/h_2$; b/a = 0.4, c/a = 0.7, ν = 0.3

Using equation (27) numerical analysis of influence of parameters characterizing the system on its natural frequency was performed. Curves in Figure 2 present values of frequency parameter $\Omega_{2,i} = \sqrt[4]{\rho h_2 \omega^2 a^4 / D_2}$ (i = 1, 2) for free-free annular plate as a function of ratio $\alpha = h_1/h_2$ and for various values of foundation's parameter $K_2 = K$. Results were obtained for b/a = 0.4, c/a = 0.7, ν = 0.3. We can observe both the parameters α and K have significant effect on natural frequencies of stepped plate.

Conclusions

In this paper an exact solution to the problem of free vibration of a stepped annular plate resting on elastic foundation is presented. The formulation of the problem consists of the differential equations of motion of the plates, continuity conditions and boundary conditions. The solution to vibration problem is obtained by dividing of the stepped plate into uniform annular plates. Exact solu-

tion is obtained by using the Green's function method. The formulation and solution to the problem take into account an arbitrary number of uniform plates. Presented numerical example shows an influence of selected parameters characterizing the system on its natural frequencies.

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