IDENTIFICATION OF THERMAL CONDUCTIVITY BY MEANS OF THE GRADIENT METHOD AND THE BEM

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Abstract. In the paper the application of the gradient method coupled with the boundary element method for numerical solution of the inverse parametric problem is presented. On the basis of the knowledge of temperature field in the domain considered the temperature dependent thermal conductivity is identified. The non-steady state is considered and 1D problem is discussed. In the final part of the paper the results of computations are shown.

1. Direct problem

The following boundary initial problem is considered

$$\begin{array}{ll} 0 < x < L: & c\dfrac{\partial T(x,t)}{\partial t} = \dfrac{\partial}{\partial x}\left[\lambda(T)\dfrac{\partial T(x,t)}{\partial x}\right] \\ x = 0: & q(x,t) = -\lambda(T)\dfrac{\partial T(x,t)}{\partial x} = q_b \\ x = L: & q(x,t) = -\lambda(T)\dfrac{\partial T(x,t)}{\partial x} = 0 \\ t = 0: & T(x,t) = T_0 \end{array} \tag{1}$$

where c is the volumetric specific heat, $\lambda(T)$ is the thermal conductivity, T, x, t denote temperature, spatial co-ordinate and time, q_b is the boundary heat flux and T_0 is the initial temperature.

In order to solve the problem (1) by means of the boundary element method the Kirchhoff transformation is introduced [1, 2]

$$U(T) = \int_0^T \lambda(\mu)\,\mathrm{d}\mu \tag{2}$$

and then the governing equations (1) take a form

$$
\begin{array}{ll}
0<x<L: & c\dfrac{\partial U(x,t)}{\partial t}=\lambda(T)\dfrac{\partial^2 U(x,t)}{\partial x^2} \\
x=0: & q(x,t)=-\dfrac{\partial U(x,t)}{\partial x}=q_b \\
x=L: & q(x,t)=-\dfrac{\partial U(x,t)}{\partial x}=0 \\
t=0: & U(x,t)=U_0
\end{array}
\tag{3}
$$

where $U_0 = U(T_0)$. We assume that

$$
\lambda(T)=b_1+b_2T+b_3T^2 \tag{4}
$$

where b_1, b_2, b_3 are the coefficients.

If the direct problem is considered then all geometrical and thermophysical parameters appearing in the mathematical model are known.

2. Sensitivity coefficients

In this chapter the sensitivity analysis of function $U(x, t)$ (c.f. governing equations (3)) with respect to the coefficients b_e, $e = 1, 2, 3$ (c.f. equation (4)) is discussed. Here the direct approach is applied [3, 4]. Additionally, it is assumed that the volumetric specific heat c is a constant value.

Differentiation of equations (3) with respect to b_e, $e = 1, 2, 3$ gives

$$
\begin{array}{ll}
0<x<L: & c\dfrac{\partial}{\partial t}\left[\dfrac{\partial U(x,t)}{\partial b_e}\right]=\dfrac{\partial \lambda(T)}{\partial b_e}\dfrac{\partial^2 U(x,t)}{\partial x^2}+\lambda(T)\dfrac{\partial^2}{\partial x^2}\left[\dfrac{\partial U(x,t)}{\partial b_e}\right] \\
x=0: & \dfrac{\partial q(x,t)}{\partial b_e}=-\dfrac{\partial}{\partial x}\left[\dfrac{\partial U(x,t)}{\partial b_e}\right]=\dfrac{\partial q_b}{\partial b_e} \\
x=L: & \dfrac{\partial q(x,t)}{\partial b_e}=-\dfrac{\partial}{\partial x}\left[\dfrac{\partial U(x,t)}{\partial b_e}\right]=0 \\
t=0: & \dfrac{\partial U(x,t)}{\partial b_e}=\dfrac{\partial U_0}{\partial b_e}
\end{array}
\tag{5}
$$

Because (c.f. equation (3))

$$
\frac{\partial^2 U(x,t)}{\partial x^2}=\frac{c}{\lambda(T)}\frac{\partial U(x,t)}{\partial t} \tag{6}
$$

so

$$
\begin{aligned}
&0<x<L: && c\frac{\partial Z_e(x,t)}{\partial t}=\lambda(T)\frac{\partial^2 Z_e(x,t)}{\partial x^2}+\frac{\partial\lambda(T)}{\partial b_e}\frac{c}{\lambda(T)}\frac{\partial U(x,t)}{\partial t}\\
&x=0: && -\frac{\partial Z_e(x,t)}{\partial x}=0\\
&x=L: && -\frac{\partial Z_e(x,t)}{\partial x}=0\\
&t=0: && Z_e(x,t)=0
\end{aligned}
\tag{7}
$$

where

$$
Z_e(x,t)=\frac{\partial U(x,t)}{\partial b_e},\quad e=1,2,3 \tag{8}
$$

We calculate

$$
\frac{\partial\lambda(T)}{\partial b_1}=1+b_2\frac{\partial T}{\partial b_1}+2b_3T\frac{\partial T}{\partial b_1}=1+b_2\frac{\mathrm{d}T}{\mathrm{d}U}\frac{\partial U}{\partial b_1}+2b_3T\frac{\mathrm{d}T}{\mathrm{d}U}\frac{\partial U}{\partial b_1} \tag{9}
$$

$$
\frac{\partial\lambda(T)}{\partial b_2}=T+b_2\frac{\partial T}{\partial b_2}+2b_3T\frac{\partial T}{\partial b_2}=T+b_2\frac{\mathrm{d}T}{\mathrm{d}U}\frac{\partial U}{\partial b_2}+2b_3T\frac{\mathrm{d}T}{\mathrm{d}U}\frac{\partial U}{\partial b_2} \tag{10}
$$

$$
\frac{\partial\lambda(T)}{\partial b_3}=b_2\frac{\partial T}{\partial b_3}+T^2+2b_3T\frac{\partial T}{\partial b_3}=T^2+b_2\frac{\mathrm{d}T}{\mathrm{d}U}\frac{\partial U}{\partial b_3}+2b_3T\frac{\mathrm{d}T}{\mathrm{d}U}\frac{\partial U}{\partial b_3} \tag{11}
$$

this means

$$
\begin{aligned}
&\frac{\partial\lambda(T)}{\partial b_e}=T^{e-1}+\frac{b_2}{\lambda(T)}Z_e+2\frac{b_3}{\lambda(T)}T\,Z_e=\\
&\frac{1}{\lambda(T)}\left(b_2+2b_3T\right)Z_e+T^{e-1}=\frac{1}{\lambda(T)}\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}Z_e+T^{e-1}
\end{aligned}
\tag{12}
$$

Finally, the equations (7) can be written in the form

$$
\begin{aligned}
0<x<L:\quad & c\frac{\partial Z_e(x,t)}{\partial t}=\lambda(T)\frac{\partial^2 Z_e(x,t)}{\partial x^2}+\\
& \frac{c}{\lambda(T)}\left[\frac{1}{\lambda(T)}\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}Z_e(x,t)+\left[T(x,t)\right]^{e-1}\right]\frac{\partial U(x,t)}{\partial t}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&x = 0: \qquad W_e(x,t) = 0\\
&x = L: \qquad W_e(x,t) = 0\\
&t = 0: \qquad Z_e(x,t) = 0
\end{aligned}
$$

where $W_e(x,t) = -\partial Z_e(x,t) / \partial x$

3. Boundary element method

The basic problem (3) and additional problems (13) connected with the sensitivity functions have been solved by means of the first scheme of the boundary element method. This method is presented for the following equation

$$\frac{\partial F(x,t)}{\partial t} = a(T) \frac{\partial^2 F(x,t)}{\partial x^2} + \frac{1}{c} Q(x,t) \tag{14}$$

where $a(T) = \lambda(T)/c$ and for the basic problem (3): $F(x, t) = U(x, t)$, $Q(x, t) = 0$, while for the additional problems (13): $F(x, t) = Z_e(x, t)$ and

$$Q(x,t) = \frac{c}{\lambda(T)} \left[\frac{1}{\lambda(T)} \frac{\mathrm{d}\lambda(T)}{\mathrm{d}T} Z_e(x,t) + \left[T(x,t)\right]^{e-1} \right] \frac{\partial U(x,t)}{\partial t} \tag{15}$$

The time grid $t^0 < t^1 < ... < t^{f-1} < t^f < ... < t^F < \infty$ with constant step $\Delta t = t^f - t^{f-1}$ is introduced. The boundary integral equation for transition $t^{f-1} \to t^f$ has the following form [1, 5, 6]

$$
\begin{aligned}
F\left(\xi, t^f\right) + \left[a_f \int_{t^{f-1}}^{t^f} F^*\left(\xi, x, t^f, t\right) J(x,t) \,\mathrm{d}t \right]_{x=0}^{x=L} \\
= \left[a_f \int_{t^{f-1}}^{t^f} J^*\left(\xi, x, t^f, t\right) F(x,t) \,\mathrm{d}t \right]_{x=0}^{x=L} + \\
\int_0^L F^*\left(\xi, x, t^f, t^{f-1}\right) F\left(x, t^{f-1}\right) \mathrm{d}x + a_f \int_{t^{f-1}}^{t^f} \int_0^L Q(x,t) F^*\left(\xi, x, t^f, t\right) \mathrm{d}x \,\mathrm{d}t
\end{aligned}
\tag{16}
$$

where a_f is the mean value of thermal diffussivity for interval time $\left[t^{f-1}, t^f\right]$, ξ is the observation point. In equation (16) $F^*\left(\xi, x, t^f, t\right)$ is the fundamental solution [1, 5, 6]

$$F^*\left(\xi, x, t^f, t\right) = \frac{1}{2\sqrt{\pi a_f\left(t^f - t\right)}} \exp\left[-\frac{\left(x-\xi\right)^2}{4a_f\left(t^f - t\right)}\right] \tag{17}$$

$J^*\left(\xi, x, t^f, t\right)$ is the heat flux resulting from the fundamental solution

$$J^*\left(\xi, x, t^f, t\right) = \frac{x-\xi}{4\sqrt{\pi}\left[a_f\left(t^f - t\right)\right]^{3/2}} \exp\left[-\frac{\left(x-\xi\right)^2}{4a_f\left(t^f - t\right)}\right] \tag{18}$$

and

$$J\left(x,t\right) = -\frac{\partial F\left(x,t\right)}{\partial x} \tag{19}$$

In numerical realization of the BEM the constant elements with respect to time are introduced

$$t \in \left[t^{f-1}, t^f\right]: \quad \begin{cases} F\left(x,t\right) = F\left(x,t^f\right) \\ J\left(x,t\right) = J\left(x,t^f\right) \end{cases} \tag{20}$$

and then the equation (16) takes a form

$$\begin{gathered} F\left(\xi, t^f\right) + \left[a_f J\left(x, t^f\right) \int_{t^{f-1}}^{t^f} F^*\left(\xi, x, t^f, t\right) \mathrm{d}t\right]_{x=0}^{x=L} = \\ \left[a_f F\left(x, t^f\right) \int_{t^{f-1}}^{t^f} J^*\left(\xi, x, t^f, t\right) \mathrm{d}t\right]_{x=0}^{x=L} + \int_0^L F^*\left(\xi, x, t^f, t^{f-1}\right) F\left(x, t^{f-1}\right) \mathrm{d}x \\ + a_f \int_0^L Q\left(x, t^{f-1}\right)\left[\int_{t^{f-1}}^{t^f} F^*\left(\xi, x, t^f, t\right) \mathrm{d}t\right] \mathrm{d}x \end{gathered} \tag{21}$$

or

$$\begin{gathered} F\left(\xi, t^f\right) + g\left(\xi, L\right) J\left(L, t^f\right) - g\left(\xi, 0\right) J\left(0, t^f\right) = \\ h\left(\xi, L\right) F\left(L, t^f\right) - h\left(\xi, 0\right) F\left(0, t^f\right) + P\left(\xi\right) + R\left(\xi\right) \end{gathered} \tag{22}$$

where

$$g\left(\xi, x\right) = a_f \int_{t^{f-1}}^{t^f} F^*\left(\xi, x, t^f, t\right) \mathrm{d}t \tag{23}$$

and

$$h(\xi, x) = a_f \int_{t^{f-1}}^{t^f} J^*\left(\xi, x, t^f, t\right) \mathrm{d}\, t \tag{24}$$

while

$$P(\xi) = \int_0^L F^*\left(\xi, x, t^f, t^{f-1}\right) F\left(x, t^{f-1}\right) \mathrm{d}\, x \tag{25}$$

and

$$R(\xi) = \int_0^L Q\left(x, t^{f-1}\right) g\left(\xi, x\right) \mathrm{d}\, x \tag{26}$$

It should be pointed out that the integrals (23), (24) can be calculated in analytical way [1, 6].
For $\xi \to 0^+$ and $\xi \to L^-$ one obtains the following system of equations

$$\begin{cases} F\left(0, t^f\right) + g\left(0, L\right) J\left(L, t^f\right) - g\left(0, 0\right) J\left(0, t^f\right) = \\ h\left(0^+, L\right) F\left(L, t^f\right) - h\left(0^+, 0\right) F\left(0, t^f\right) + P(0) + R(0) \\ F\left(L, t^f\right) + g\left(L, L\right) J\left(L, t^f\right) - g\left(L, 0\right) J\left(0, t^f\right) = \\ h\left(L^-, L\right) F\left(L, t^f\right) - h\left(L^-, 0\right) F\left(0, t^f\right) + P(L) + R(L) \end{cases} \tag{27}$$

which can be written in the matrix form

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} J\left(0, t^f\right) \\ J\left(L, t^f\right) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} F\left(0, t^f\right) \\ F\left(L, t^f\right) \end{bmatrix} + \begin{bmatrix} P(0) + R(0) \\ P(L) + R(L) \end{bmatrix} \tag{28}$$

This system of equations allows to find the “missing” boundary vales of *F* or *J*. The values of function *F* at the internal points ξ are calculated using the formula (c.f. equation (22))

$$\begin{gathered} F\left(\xi, t^f\right) = h(\xi, L) F\left(L, t^f\right) - h(\xi, 0) F\left(0, t^f\right) - \\ g\left(\xi, L\right) J\left(L, t^f\right) + g\left(\xi, 0\right) J\left(0, t^f\right) + P(\xi) + R(\xi) \end{gathered} \tag{29}$$

4. Gradient method of inverse problem solution

It is assumed that the coefficients b_1, b_2, b_3 in equation (4) are unknown. To solve the inverse problem formulated, the additional information is necessary. Let

$$T_{di}^{f} = T_d\left(x_i, t^f\right), \quad i = 1, 2, ..., M, \quad f = 1, 2, ..., F \tag{30}$$

are the known (measured) temperatures at the points x_i for times t^f.

The following least squares criterion has been taken into account [4]

$$S = \sum_{i=1}^{M}\sum_{f=1}^{F}\left(U_i^f - U_{di}^f\right)^2 \tag{31}$$

where $U_i^f = U\left(x_i, t^f\right)$ is the calculated value of function U at the point x_i for time t^f, $U_{d\,i}^f$ if is the value of function U corresponding to the measured value of temperature $T_{d\,i}^f$ at the point x_i for time t^f.

The necessary condition of optimum of function S leads to the following system of equations

$$\frac{\partial S}{\partial b_l} = 2\sum_{i=1}^{M}\sum_{f=1}^{F}\left(U_i^f - U_{di}^f\right)\left.\frac{\partial U_i^f}{\partial b_i}\right|_{b_l = b_l^k} = 0, \quad l = 1, 2, 3 \tag{32}$$

where b_l^k for $k = 0$ are the arbitrary assumed values of parameters b_l, while for $k > 0$ they result from the previous iteration.

Function $U_i^f = U\left(x_i, t^f\right)$ is expanded into the Taylor series

$$U_i^f = \left(U_i^f\right)^k + \sum_{e=1}^{3}\left.\frac{\partial U_i^f}{\partial \lambda_e}\right|_{b_e = b_e^k}\left(b_e^{k+1} - b_e^k\right) \tag{33}$$

this means

$$U_i^f = \left(U_i^f\right)^k + \sum_{e=1}^{3}\left(Z_{i\,e}^f\right)\left(b_e^{k+1} - b_e^k\right) \tag{34}$$

where

$$\left(Z_{i\,e}^f\right)^k = \left.\frac{\partial U_i^f}{\partial \lambda_e}\right|_{b_e = b_e^k} \tag{35}$$

are the sensitivity coefficients.

Putting (34) into (32) one has ($l = 1, 2, 3$)

$$\sum_{i=1}^{M}\sum_{f=1}^{F}\left[\left(U_i^f\right)^k + \sum_{e=1}^{3}\left(Z_{i\,e}^f\right)^k\left(b_e^{k+1} - b_e^k\right) - U_{d\,i}^f\right]\left(Z_{i\,l}^f\right)^k = 0 \tag{36}$$

or

$$\sum_{i=1}^{M}\sum_{f=1}^{F}\sum_{e=1}^{3}\left(Z_{i\,e}^{f}\right)^{k}\left(Z_{i\,l}^{f}\right)^{k}\left(b_{e}^{k+1}-b_{e}^{k}\right)=\sum_{i=1}^{M}\sum_{f=1}^{F}\left[U_{d\,i}^{f}-\left(U_{i}^{f}\right)^{k}\right]\left(Z_{i\,l}^{f}\right)^{k} \tag{37}$$

where $l = 1, 2, 3$, while $k = 0, 1, ..., K$ is the number of iteration.
The system of equations (37) can be written in the matrix form

$$\left(\mathbf{Z}^{\mathrm{T}}\right)^{k}\mathbf{Z}^{k}\mathbf{b}^{k+1}=\left(\mathbf{Z}^{\mathrm{T}}\right)^{k}\mathbf{Z}^{k}\mathbf{b}^{k}+\left(\mathbf{Z}^{\mathrm{T}}\right)^{k}\left(\mathbf{U}_{d}-\mathbf{U}^{k}\right) \tag{38}$$

5. Results of computations

The plate of thickness $L = 0.05$ m has been considered. On the left surface $x = 0$ the Neumann condition $q_b = 9.2\cdot10^5$ W/m^2 is assumed, on the right surface $x = L$ the zero heat flux $q(L, t) = 0$ has been accepted. Initial condition $T_0 = 100$°C has been also given (equations (1)). Additionally, it was assumed that $c = 4\cdot10^6$ J/(m^3 K) and $b_1 = 52$, $b_2 = -0.02$, $b_3 = -0.00001$ (equation (4)).

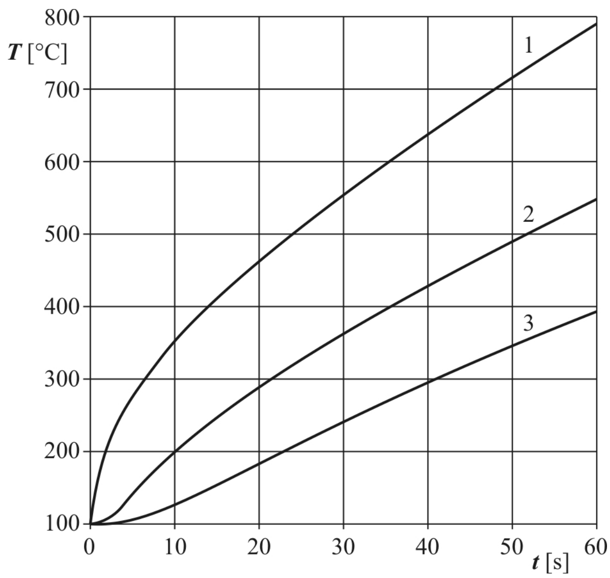

Fig. 1. Heating curves

The basic problem has been solved by means of the boundary element method. The domain has been divided into 100 internal cells and $\Delta t = 1$ s. In Figure 1 the heating curves at the points 1 – $x_1 = 0$, 2 – $x_2 = 0.01$ m and 3 – $x_3 = 0.02$ m are shown. Figures 2, 3, 4 illustrate the courses of sensitivity functions at the same points.

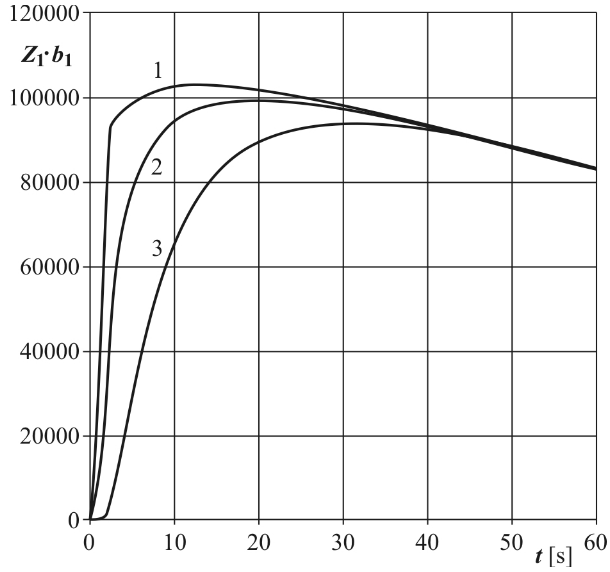

Fig. 2. Sensitivity function Z_1

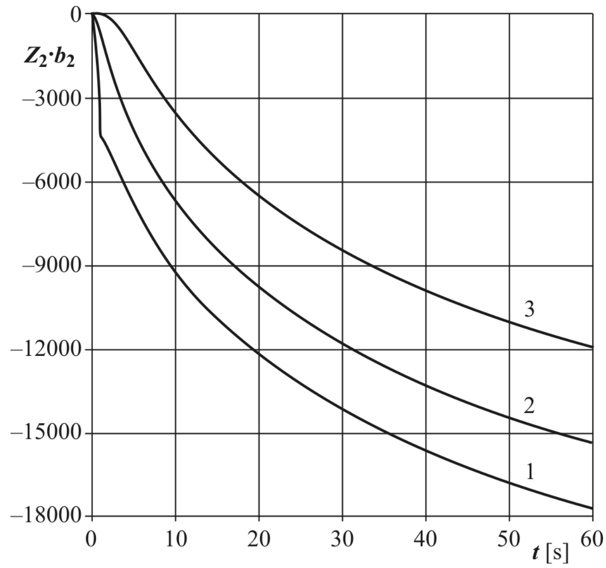

Fig. 3. Sensitivity function Z_2

Next, the inverse problem has been considered. The parameters $b1$, $b2$, $b3$ have been identified under the assumption that $b_1^0 = 51$, $b_2^0 = -0,017$, $b_3^0 = -0,000018$.

In the first variant of computations only one heating curve marked by 1 in Figure 1 is taken into account (curves marked by 1 in Figures 5, 6, 7), while in the second variant of computations three heating curves (Figure 1) are taken into account (curves marked by 3 in Figures 5, 6, 7).

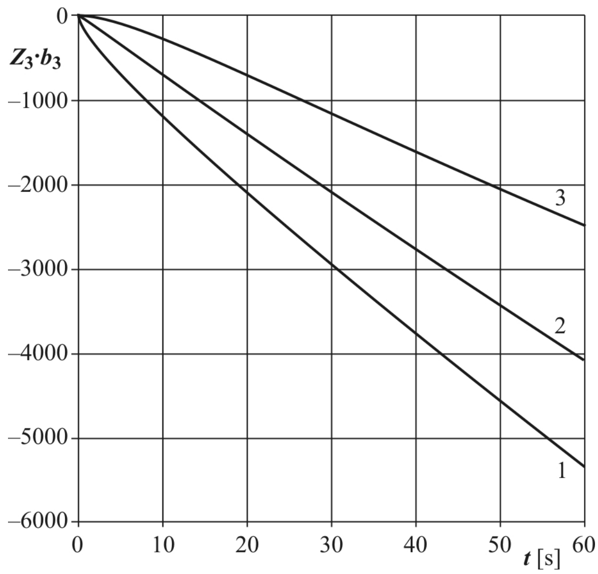

Fig. 4. Sensitivity function Z_3

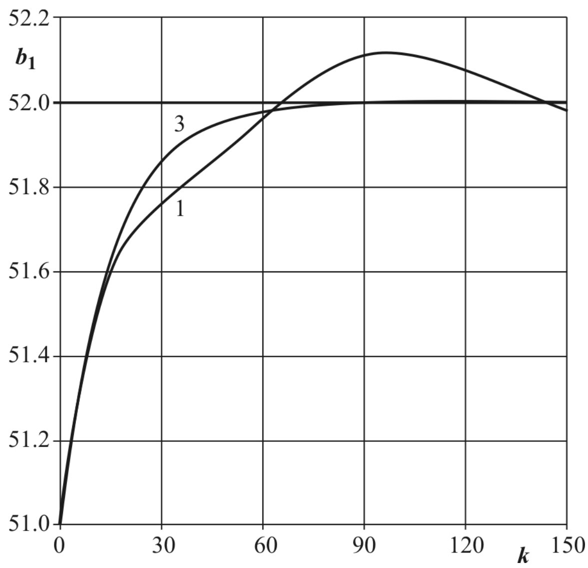

Fig. 5. Identification of b_1

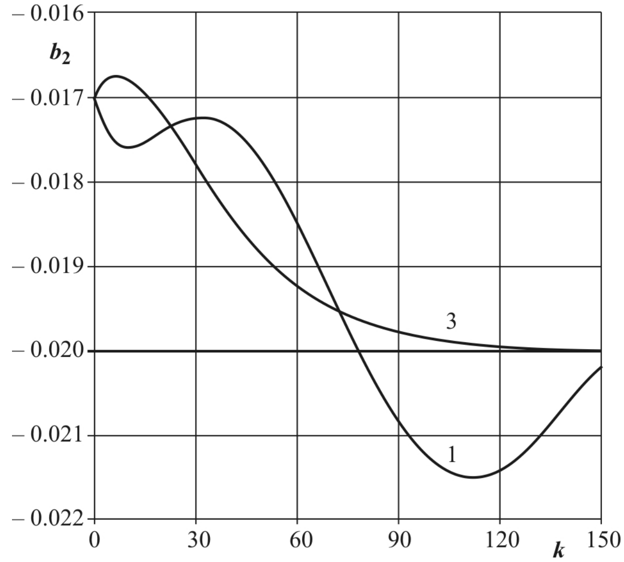

Fig. 6. Identification of b_2

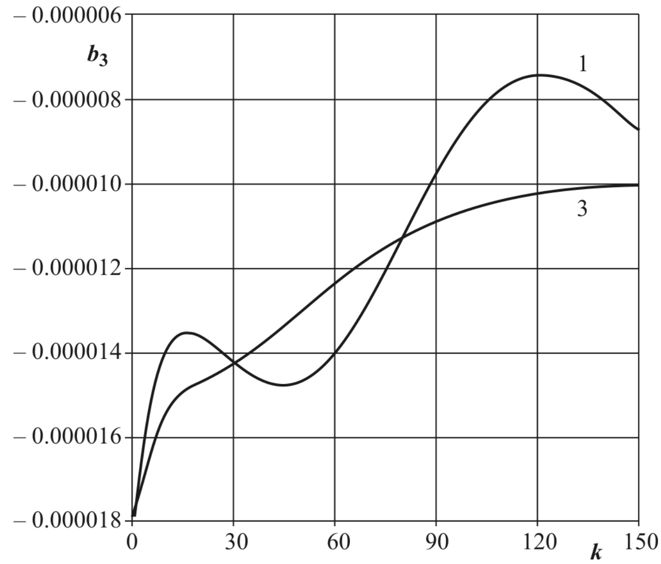

Fig. 7. Identification of b_3

It is visible, that the iteration process is convergent, but even for the initial values of parameters b_1^0, b_2^0 and b_3^0 close to the exact solution the number of iterations is very big. The effectiveness of the algorithm proposed considerably improves when three heating curves have been taken into account.

Summing up, the algorithm proposed constitutes the effective tool of such inverse problem solution but the number of points for which the temperature course is known has an essential effect on the number of iterations.

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