APPLICATION OF THE BEM FOR NUMERICAL SOLUTION OF NONLINEAR DIFFUSION EQUATION

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Abstract. In the paper the nonlinear diffusion equation is considered, this means the volumetric specific heat and thermal conductivity are temperature dependent. To solve the problem by means of the boundary element method the Kirchhoff transformation is introduced and for each time step the mean values of these parameters are taken into account. In the final part of the paper the results of computations are shown.

1. Formulation of the problem

Non-steady temperature field in the plate (1D problem) is described by the following energy equation

$$0 < x < L \quad : \quad c(T)\frac{\partial\, T(x,\ t)}{\partial\, t} = \frac{\partial}{\partial\, x}\left[\lambda(T)\frac{\partial\, T(x,\ t)}{\partial\, x}\right] \tag{1}$$

where $c(T)$ is the volumetric specific heat, $\lambda(T)$ is the thermal conductivity, T, x, t denote temperature, spatial co-ordinate and time, respectively.

Equation (1) is supplemented by the boundary initial conditions

$$\begin{aligned} x = 0 \quad &: \quad q(x,\ t) = -\lambda(T)\frac{\partial\, T(x,\ t)}{\partial\, x} = q_b \\ x = L \quad &: \quad T(x,\ t) = T_b \\ t = 0 \quad &: \quad T(x,\ t) = T_0 \end{aligned} \tag{2}$$

where T_b, q_b are the known boundary temperature and boundary heat flux, while T_0 is the initial temperature.

The Kirchhoff transformation is introduced

$$U(T) = \int_0^T \lambda(\mu)\ \mathrm{d}\mu \tag{3}$$

and then the equation (1) takes a form [1, 2]

$$0 < x < L \quad : \quad c(T)\frac{\partial U(x, t)}{\partial t} = \lambda(T)\frac{\partial^2 U(x, t)}{\partial x^2} \tag{4}$$

or

$$0 < x < L \quad : \quad \frac{\partial U(x, t)}{\partial t} = a(T)\frac{\partial^2 U(x, t)}{\partial x^2} \tag{5}$$

where $a(T) = \lambda(T)/c(T)$. The boundary initial conditions (2) are also transformed using definition (3) and then

$$\begin{aligned} x = 0 \quad &: \quad q(x, t) = -\frac{\partial U(x, t)}{\partial x} = q_b \\ x = L \quad &: \quad U(x, t) = U_b \\ t = 0 \quad &: \quad U(x, t) = U_0 \end{aligned} \tag{6}$$

where $U_b = U(T_b)$ and $U_0 = U(T_0)$.

2. First scheme of the boundary element method

At first the time grid

$$0 = t^0 < t^1 < t^2 < \ldots < t^{f-1} < t^f < \ldots < t^F < \infty \tag{7}$$

with constant step $\Delta t = t^f - t^{f-1}$ is introduced.

Using the weighted residual criterion [2, 3] for equation (5) one obtains

$$\int_{t^{f-1}}^{t^f} \int_0^L \left[a_f \frac{\partial^2 U(x, t)}{\partial x^2} - \frac{\partial U(x, t)}{\partial t} \right] U^*(\xi, x, t^f, t)\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{8}$$

where a_f is the mean value of thermal diffussivity for interval time $[t^{f-1}, t^f]$.

In equation (8) $\xi \in (0, L)$ is the observation point, $U^*(\xi, x, t^f, t)$ is the fundamental solution [2-4]

$$U^*(\xi, x, t^f, t) = \frac{1}{2\sqrt{\pi\, a_f (t^f - t)}} \exp\left[-\frac{(x - \xi)^2}{4\, a_f (t^f - t)} \right] \tag{9}$$

Heat flux resulting from the fundamental solution is defined as follows

$$q^*(\xi, x, t^f, t) = -\frac{\partial U^*(\xi, x, t^f, t)}{\partial x} \tag{10}$$

this means

$$q^*(\xi,\ x,\ t^f,t) = \frac{x-\xi}{4\sqrt{\pi}\ \left[a_f(t^f-t)\right]^{3/2}}\ \exp\left[-\frac{(x-\xi)^2}{4\,a_f(t^f-t)}\right] \tag{11}$$

The first component of equation (8) is integrated twice by parts with respect to x and then

$$\begin{aligned}
&\int_{t^{f-1}}^{t^f}\int_0^L a_f\,\frac{\partial^2 U(x,\ t)}{\partial x^2}\ U^*(\xi,\ x,\ t^f,t)\,\mathrm{d}x\,\mathrm{d}\,t \ = \\
&\left[\int_{t^{f-1}}^{t^f} a_f\ U^*(\xi,\ x,\ t^f,t)\frac{\partial\, U(x,\ t)}{\partial\, x}\ \mathrm{d}\,t\right]_{x=0}^{x=L} - \\
&\left[\int_{t^{f-1}}^{t^f} a_f\ U(x,\ t)\ \frac{\partial\, U^*(\xi,\ x,\ t^f,t)}{\partial\, x}\ \mathrm{d}\,t\right]_{x=0}^{x=L} + \\
&\int_{t^{f-1}}^{t^f}\int_0^L a_f\,\frac{\partial^2 U^*(\xi,\ x,\ t^f,t)}{\partial x^2}\,U(x,\ t)\,\mathrm{d}x\,\mathrm{d}\,t
\end{aligned} \tag{12}$$

The second component of equation (8) is integrated by parts with respect to t

$$\begin{aligned}
&\int_{t^{f-1}}^{t^f}\int_0^L \frac{\partial\, U(x,t)}{\partial\, t}\ U^*(\xi,\ x,\ t^f,t)\,\mathrm{d}x\,\mathrm{d}\,t = \\
&\left[\int_0^L U^*(\xi,\ x,\ t^f,t)\ U(x,t)\,\mathrm{d}x\ \right]_{t=t^{f-1}}^{t=t^f} - \int_{t^{f-1}}^{t^f}\int_0^L \frac{\partial\, U^*(\xi,\ x,\ t^f,t)}{\partial\, t}\ U(x,t)\,\mathrm{d}x\,\mathrm{d}\,t
\end{aligned} \tag{13}$$

Putting (12), (13) into (8) one obtains

$$\begin{aligned}
&\int_{t^{f-1}}^{t^f}\int_0^L \left[a_f\,\frac{\partial^2 U^*(\xi,\ x,\ t^f,t)}{\partial x^2} + \frac{\partial\, U^*(\xi,\ x,\ t^f,t)}{\partial\, t}\right] U(x,\ t)\,\mathrm{d}x\,\mathrm{d}\,t \ + \\
&\left[\int_{t^{f-1}}^{t^f} a_f\ U(x,\ t)\,q^*(\xi,\ x,\ t^f,t)\,\mathrm{d}\,t\right]_{x=0}^{x=L} - \\
&\left[\int_{t^{f-1}}^{t^f} a_f\ U^*(\xi,\ x,\ t^f,t)\,q(x,\ t)\,\mathrm{d}\,t\right]_{x=0}^{x=L} - \\
&\left[\int_0^L U^*(\xi,\ x,\ t^f,t)\ U(x,t)\,\mathrm{d}x\ \right]_{t=t^{f-1}}^{t=t^f} \quad = 0
\end{aligned} \tag{14}$$

where $q(x,\ t) = -\partial\ U(x,\ t)/\partial x$. Taking into account the properties [2-4] of fundamental solution (9) one obtains

$$U(\xi,\ t^f) + \left[a_f \int_{t^{f-1}}^{t^f} U^*(\xi,\ x,\ t^f,\ t)\, q(x,\ t)\, \mathrm{d}\,t \right]_{x=0}^{x=L} = \left[a_f \int_{t^{f-1}}^{t^f} q^*(\xi,\ x,\ t^f,\ t)\, U(x,\ t)\, \mathrm{d}\,t \right]_{x=0}^{x=L} + \int_0^L U^*(\xi,\ x,\ t^f,\ t^{f-1})\, U(x, t^{f-1})\, \mathrm{d}x \tag{15}$$

3. Numerical realization of boundary element method

In numerical realization of the BEM the constant elements with respect to time are considered

$$t \in \left[t^{f-1},\ t^f \right] \ : \quad \begin{cases} U(x,\ t) = U(x,\ t^f) \\ q(x,\ t) = q(x,\ t^f) \end{cases} \tag{16}$$

The equation (15) takes a form

$$U(\xi,\ t^f) + \left[a_f\ q(x,\ t^f) \int_{t^{f-1}}^{t^f} U^*(\xi,\ x,\ t^f,\ t)\, \mathrm{d}\,t \right]_{x=0}^{x=L} = \left[a_f U\ (x,\ t^f) \int_{t^{f-1}}^{t^f} q^*(\xi,\ x,\ t^f,\ t)\, \mathrm{d}\,t \right]_{x=0}^{x=L} + \int_0^L U^*(\xi,\ x,\ t^f,\ t^{f-1})\, U(x, t^{f-1})\, \mathrm{d}x \tag{17}$$

or

$$\begin{gathered} U(\xi,\ t^f) + g(\xi,\ L)\, q(L,\ t^f) - g(\xi,\ 0)\, q(0,\ t^f) = \\ h(\xi,\ L)\, U(L,\ t^f) - h(\xi,\ 0)\, U(0,\ t^f) + P(\xi) \end{gathered} \tag{18}$$

where

$$g(\xi,\ x)\ =\ a_f \int_{t^{f-1}}^{t^f} U^*(\xi,\ x,\ t^f,\ t)\,\mathrm{d}t \tag{19}$$

and

$$h(\xi,\ x) = a_f \int_{t^{f-1}}^{t^f} q^*(\xi,\ x,\ t^f,\ t)\,\mathrm{d}t \tag{20}$$

while

$$P(\xi)\ =\ \int_0^L U^*(\xi,\ x,\ t^f,\ t^{f-1})\,U(x, t^{f-1})\,\mathrm{d}x \tag{21}$$

The integrals (19), (20) are calculated in analytical way [2] and then

$$g(\xi,\ x) = \sqrt{\frac{a_f \Delta t}{\pi}}\exp\left[-\frac{(x-\xi)^2}{4a_f \Delta t}\right] - \frac{|x-\xi|}{2}\operatorname{erfc}\left(\frac{|x-\xi|}{2\sqrt{a_f \Delta t}}\right) \tag{22}$$

and

$$h(\xi,\ x) = \frac{\operatorname{sgn}(x-\xi)}{2}\operatorname{erfc}\left(\frac{|x-\xi|}{2\sqrt{a_f \Delta t}}\right) \tag{23}$$

For $\xi \to 0^+$ and $\xi \to L^-$ one obtains the following system of equations

$$\begin{cases} U(0,\ t^f) + g(0,\ L)\ q(L,\ t^f) - g(0,\ 0)\ q(0,\ t^f) = \\ h(0^+,\ L)\ U(L,\ t^f) - h(0^+,\ 0)\ U(0,\ t^f) + P(0) \\ U(L,\ t^f) + g(L,\ L)\ q(L,\ t^f) - g(L,\ 0)\ q(0,\ t^f) = \\ h(L^-,\ L)\ U(L,\ t^f) - h(L^-,\ 0)\ U(0,\ t^f) + P(L) \end{cases} \tag{24}$$

which can be written in the matrix form

$$\begin{gathered} \begin{bmatrix} -g(0,0) & g(0,L) \\ -g(L,0) & g(L,L) \end{bmatrix} \begin{bmatrix} q(0,t^f) \\ q(L,t^f) \end{bmatrix} = \\ \begin{bmatrix} -h(0^+,0)\text{-}1 & h(0^+,L) \\ -h(L^-,0) & h(L^-,L)-1 \end{bmatrix} \begin{bmatrix} U(0,t^f) \\ U(L,t^f) \end{bmatrix} + \begin{bmatrix} P(0) \\ P(L) \end{bmatrix} \end{gathered} \tag{25}$$

or

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} q(0, t^f) \\ q(L, t^f) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} U\,(0, t^f) \\ U\,(L, t^f) \end{bmatrix} + \begin{bmatrix} P(0) \\ P(L) \end{bmatrix} \tag{26}$$

It is easy to check that

$$\begin{gathered} H_{11} = H_{22} = -0.5 \\ H_{12} = H_{21} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{L}{2\sqrt{a_f \Delta t}}\right) \end{gathered} \tag{27}$$

and

$$\begin{gathered} G_{11} = -G_{22} = -\sqrt{\frac{a_f \Delta t}{\pi}} \\ G_{12} = -G_{21} = \sqrt{\frac{a_f \Delta t}{\pi}}\exp\left(-\frac{L^2}{4a\Delta t}\right) - \frac{L}{2}\,\mathrm{erfc}\left(\frac{L}{2\sqrt{a_f \Delta t}}\right) \end{gathered} \tag{28}$$

Taking into account the boundary conditions (6) the system of equations (26) takes a form

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} q_b \\ q(L, t^f) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} U(0, t^f) \\ U_b \end{bmatrix} + \begin{bmatrix} P(0) \\ P(L) \end{bmatrix} \tag{29}$$

This system of equations allows to find the values $U(0, t^f)$, $q(L, t^f)$. The values of function U at the internal points ξ are calculated using the formula (c.f. equation (18))

$$\begin{gathered} U(\xi,\ t^f) = h(\xi,\ L)\, U_b - h(\xi,\ 0)\, U(0,\ t^f) - \\ g(\xi,\ L)\, q(L,\ t^f) + g(\xi,\ 0)\, q_b + P(\xi) \end{gathered} \tag{30}$$

4. Results of computations

The plate of thickness L = 0.05 m made of steel is considered. On the left surface x = 0 the boundary heat flux $q_b = 9\cdot 10^5$ W/m^2 is assumed, on the right surface $x = L$ the temperature $T_b = 100^{\circ}$C is accepted. The initial condition $T_0 = 100^{\circ}$C is given. It is assumed that

$$c(T) = c_1 + c_2 T + c_3 T^2 \qquad [\mathrm{J/(m^3\ K)}] \tag{31}$$

and

$$\lambda(T) = b_1 + b_2 T + b_3 T^2 \qquad [\mathrm{W/(m\,K)}] \tag{32}$$

where c_1 = 3.7733·10^6, c_2 = 407.8587, c_3 = 1.43313, b_1 = 52.266, b_2 = –0.016, b_3 = –0.00002. These coefficients have been obtained on the basis of experimental data for steel of chemical component 0.23% C, 0.11% Si, 0.635% Mn, 0.034% S, 0.034% P, 0.074% Ni, 0.13% Cu, 0.01% Al, 0.036% A [5].

Using the definition (3) one has

$$U(T) = b_1 T + \frac{b_2}{2} T^2 + \frac{b_3}{3} T^3 \tag{33}$$

The inverse function, this means $T(U)$ is the solution of following equation

$$\frac{b_3}{3} T^3 + \frac{b_2}{2} T^2 + b_1 T - U = 0 \tag{34}$$

In Figure 1 the course of function $a\,(T) = \lambda\,(T)/c(T)$ is shown.

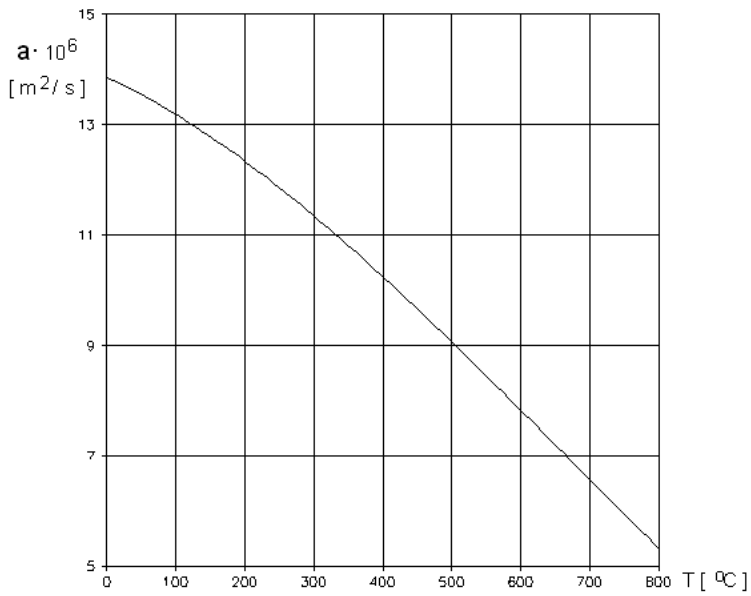

Fig. 1. Course of function $a\,(T\,)$

The problem has been solved using the boundary element method. The domain considered has been divided into N = 100 internal cells, this means

$$0 = x_0 < x_1 < \ldots < x_{j-1} < x_j < \ldots < x_N = L \tag{35}$$

where $h = x_j - x_{j-1}$ = const. Time step was equal $\Delta t = 1$ s. The following approximation of the thermal diffussivity has been taken into account

$$a_f = \frac{1}{N+1}\sum_{j=0}^{N} a\left[T(x_j,\, t^{f-1})\right] \tag{36}$$

In Figure 2 the temperature distribution in the domain considered is shown, while Figure 3 illustrates the heating curves at the points $x_1 = 0$, $x_2 = 0.01$ m, $x_3 =$ $= 0.02$ m, $x_4 = 0.03$ m, $x_5 = 0.04$ m.

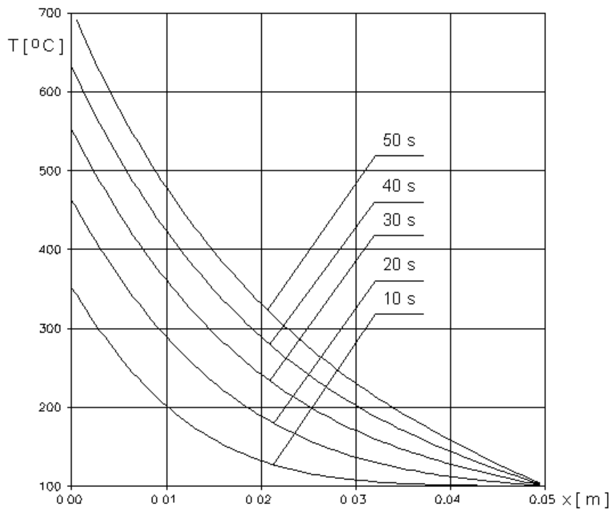

Fig. 2. Temperature distribution

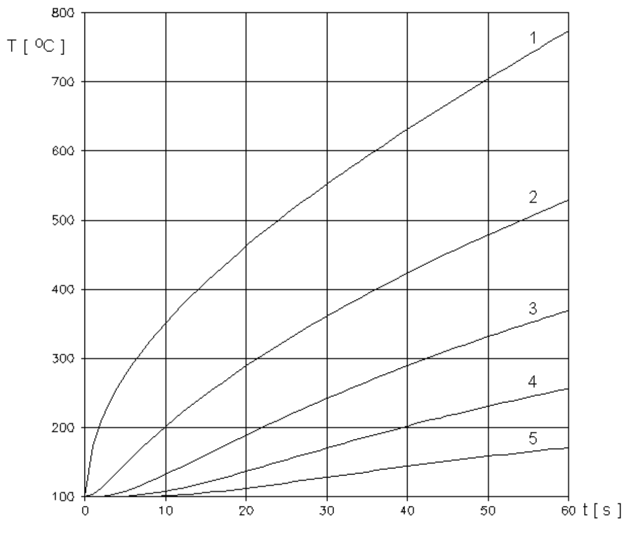

Fig. 3. Heating curves

In Figure 4 the heating curves at the points 1 and 3 for temperature dependent thermal diffussivity $a(T)$ and the mean value of this parameter, this means $a_m = 9.92 \cdot 10^{-6}$ W/ m^2 (marked by 1a and 3a) are shown.

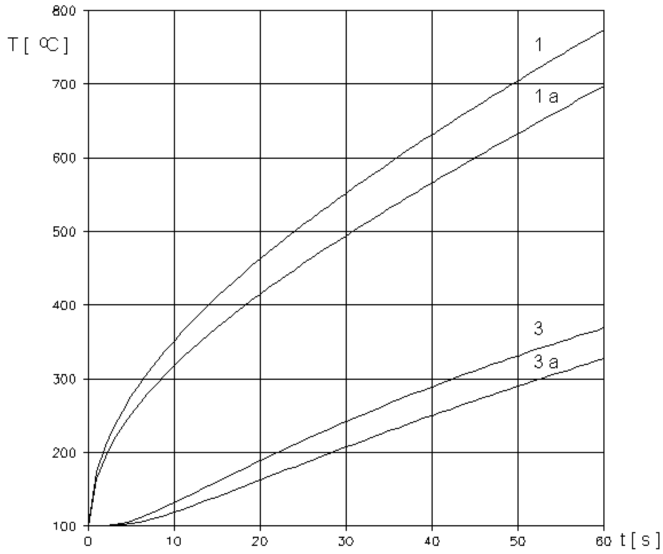

Fig. 4. Heating curves for temperature dependent (1, 3) and mean value of *a* (1a, 3a)

References

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