FINDING OF EXPECTED INCOMES IN OPEN EXPONENTIAL NETWORKS OF ARBITRARY ARCHITECTURE

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Abstract. Investigation of the open exponential queueing network of arbitrary architecture with incomes is carry out in paper. The incomes of transitions between network's states are random variables with specified moments of two first orders. The expressions for expected incomes and variances of incomes are received.

Introduction

Exponential queueing networks with incomes in case, when incomes of transitions between stations of network were values that depended from network's state and time were examined in [1, 2]. The expressions for expected incomes and variances of incomes in systems of such networks were obtained in [3, 4] in the case, when incomes of transitions between network's stations were random variables (RV) with specified moments of the first and the second degrees. However, only networks with central system were examined in these works. This work is devoted to finding of expected incomes in exponential networks of arbitrary architecture.

Let's consider open exponential queueing network with one-type messages that consists of *n* queueing systems $S_1, S_2, ..., S_n$. State of such network could be discribed by vector $k(t) = (k_1, k_2, ..., k_n, t)$, where k_i - number of messages in system S_i in the moment *t*, $i = \overline{1,n}$. The incoming flow arrives in the network with rate λ. Let's denote the service rate in system S_i (when there are k_i messages in it) as $\mu_i(k_i)$; p_{0j} - probability that the message comes in system S_j from the outside, $\sum_{j=1}^{n} p_{0j} = 1$; p_{ij} - probability that the message after service in system S_i comes to S_j, $\sum_{j=1}^{n} p_{ij} = 1$, $i = \overline{1,n}$. Message that comes from one system to another brings the last one some random income, so the income of the first system descened on this random value.

1. Expressions for expected incomes of network's systems

Let's consider variation dynamics of incomes in system S_i. Let's denote its income in the moment t as $V_n(t)$. Let at any moment t_0 income of this system equals to v_{i0}. We are interesting in incomes $V_i(t_0+t)$ of system S_i at the moment t_0+t. We will broke segment $[t_0, t_0+t]$ by m equal parts with length $\Delta t = t/m$, where m is rather large. Let's give out all events which can occur at *l*-th range. Following situations are possible.

1. Message arrives to system S_i from the outside with the probability $\lambda p_{0i}\Delta t + o(\Delta t)$ and increases its income by r_{0i}, where r_{0i} – RV with distribution function (DF). $F_{0j}(x)$.
2. Message goes from system S_i to the outside with probability $\mu_i(k_i)u(k_i)p_{i0}\Delta t + o(\Delta t)$, where $u(x) = \begin{cases} 1, x > 0, \\ 0, x \le 0 \end{cases}$ - Heaviside's function, and decreases an income by R_{i0}, where R_{i0} - RV with DF $F_{i0}(x)$.
3. Message arrives from S_j to S_i with probability $\mu_j(k_j)u(k_j)p_{ji}\Delta t + o(\Delta t)$ and income of system S_i increses by r_{ji}, and income of system S_j decreses by the same value, $j = \overline{1,n}, j \ne i$, where r_{ji} – RV with DF $F_{1ji}(x)$.
4. Message arrives from S_i to S_j with probability $\mu_i(k_i)u(k_i)p_{ij}\Delta t + o(\Delta t)$ and income of system S_i decreses by R_{ij}, and income of system S_j increases by the same value, where R_{ij} – RV with DF $F_{2ij}(x)$, $i, j = \overline{1,n}$.
5. No changes of network's state are happened during time period Δt with probability $1 - (\lambda p_{0i} + \mu_i(k_i)u(k_i))\Delta t + o(\Delta t)$. It is evident that $r_{ji} = R_{ji}$ with probability 1, $i, j = \overline{1,n}$, that is

$$F_{1ij}(x) = F_{2ij}(x),\ i, j = \overline{1,n} \tag{1}$$

Besides, system S_i increases its income by $r_i\Delta t$ during any short time period Δt at the cost of money percents it stores. Let r_i is RV with DF $F_i(x), i = \overline{1,n}$.

Let $\Delta V_{il}(t) = V_{il}(t+\Delta t) - V_{il}(t)$ is a change of income of *i*-th system at *l*-th time segment. It follows from above that

$$\Delta V_{il}(t)=\begin{cases} r_{0i}+r_i\Delta t & \text{with probability} \quad \lambda p_{0j}\Delta t+o(\Delta t), \\ -R_{i0}+r_i\Delta t & \text{with probability} \quad \mu_i(k_i)u(k_i)p_{i0}\Delta t+o(\Delta t), \\ r_{ji}+r_i\Delta t & \text{with probability} \quad \begin{array}{l}\mu_j(k_j)u(k_j)p_{ji}\Delta t+o(\Delta t), \\ j=\overline{1,n},\ j\neq i,\end{array} \\ -R_{ij}+r_i\Delta t & \text{with probability} \quad \begin{array}{l}\mu_i(k_i)u(k_i)p_{ij}\Delta t+o(\Delta t), \\ j=\overline{1,n},\ j\neq i,\end{array} \\ r_i\Delta t & \text{with probability} \quad 1-\left(\lambda p_{0i}+\mu_i(k_i)u(k_i)\right)\Delta t+o(\Delta t). \end{cases}$$

Income of S_i equals

$$V_i(t)=v_{i0}+\sum_{l=1}^{m}\Delta V_{il}(t) \tag{2}$$

Let's introduce denotation for corresponding expectation values:

$$M\{r_{ji}\}=\int_0^\infty xdF_{1ji}(x)=a_{ji}\,,\ M\{R_{ij}\}=\int_0^\infty xdF_{2ji}(x)=b_{ij,}\,,\ M\{r_i\}=\int_0^\infty xdF_i(x)=c_i$$

$$M\{r_{0i}\}=\int_0^\infty xdF_{0i}(x)=a_{0i}\,,\quad M\{R_{i0}\}=\int_0^\infty xdF_{i0}(x)=b_{i0}\,,\ i,j=\overline{1,n}$$

in view of (1)

$$a_{ji}=b_{ji}\,,\ i,j=\overline{1,n} \tag{3}$$

Let's get a relationship for expectation value of income of system S_i at time moment *t*. With fixed member function $k(t)$ we get:

$$M\{V_{il}(t)/k(t)\}=a_{0i}\lambda p_{0i}\Delta t-b_{i0}\mu_i(k_i)u(k_i)p_{i0}\Delta t+\sum_{j=1}^{n}a_{ji}\mu_j(k_j)u(k_j)p_{ji}\Delta t-$$

$$-\sum_{j=1}^{n}b_{ij}\mu_i(k_i)u(k_i)p_{ij}\Delta t+c_i\Delta t\left(1-\lambda p_{0i}\Delta t-\mu_i(k_i)u(k_i)\Delta t\right)+o(\Delta t)$$

that is

$$M\{V_{il}(t)/k(t)\}=a_{0i}\lambda p_{0i}\Delta t+\sum_{j=1}^{n}a_{ji}\mu_j(k_j)u(k_j)p_{ji}\Delta t-$$

$$-\sum_{j=0}^{n}b_{ij}\mu_i(k_i)u(k_i)p_{ij}\Delta t+c_i\Delta t+o(\Delta t)\,,\ i=\overline{1,n} \tag{4}$$

Therefore, taking into account that $\Delta t = \dfrac{t}{m}$, we get from (2), (4)

$$M\{V_i(t)/k(t)\} = v_{i0} + \sum_{l=1}^{m} M\{\Delta V_{il}(t)/k(t)\} =$$

$$= v_{i0} + \left[a_{0i}\lambda p_{0i} + \sum_{j=1}^{n} a_{ji}\mu_j(k_j)u(k_j)p_{ji} - \sum_{j=0}^{n} b_{ij}\mu_i(k_i)u(k_i)p_{ij} + c_i \right]\Delta t =$$

$$= v_{i0} + \left[a_{0i}\lambda p_{0i} + \sum_{j=1}^{n} a_{ji}\mu_j(k_j)u(k_j)p_{ji} - \sum_{j=0}^{n} b_{ij}\mu_i(k_i)u(k_i)p_{ij} + c_i \right] t + o(\Delta t)$$

So, when $\Delta t \to 0$, we will have

$$M\{V_i(k)/k(t)\} = v_{i0} + \left[c_i + a_{0i}\lambda p_{0i} + \sum_{j=1}^{n} a_{ji}\mu_j(k_j)u(k_j)p_{ji} - \sum_{j=0}^{n} b_{ij}\mu_i(k_i)u(k_i)p_{ij} \right] t$$

Considering $\sum_{k} P(k(t) = k) = 1$ we get average by $k(t)$

$$v_i(t) = M\{V_i(t)\} = v_{i0} + \sum_{k} P(k(t) = k) M\{V_i(t)/k(t)\} =$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\dots\sum_{k_n=0}^{\infty} P(k(t) = (k_1, k_2, \dots, k_n, t)) M\{V_i(t)/k(t) = (k_1, k_2, \dots, k_n, t)\} =$$

$$= v_{i0} + \left[c_i + a_{0i}\lambda p_{0i} + \sum_{j=1}^{n} a_{ji}p_{ji}\sum_{k} P(k(t) = k)\mu_j(k_j)u(k_j) - \right.$$

$$\left. - \sum_{j=0}^{n} b_{ij}p_{ij}\sum_{k} P(k(t) = k)\mu_i(k_i)u(k_i) \right] t$$

Then we suppose that all network systems are multi-line, any system S_i has m_i identical service lines, service time of messages have exponential distribution with μ_i at any line, $i = \overline{1, n}$. In this case

$$\mu_i(k_i) = \begin{cases} \mu_i k_i, & k_i \le m_i, \\ \mu_i m_i, & k_i > m_i, \end{cases} = \mu_i \min(k_i, m_i),\ i = \overline{1, n}$$

and it's normally to suppose that average of $\mu_i(k_i)u(k_i)$ will be $\mu_i \min(N_i(t), m_i)$, where $N_i(t)$ - mean number of messages (being waiting and servicing) in system S_i at time interval $[t_0, t_0+t]$, $i=\overline{1,n}$. So finally we will have

$$v_i(t) = v_{i0} + \left[c_i + a_{0i}\lambda p_{0i} + \sum_{j=1}^{n} a_{ji}\mu_j \min\left(N_j(t), m_j\right) p_{ji} - \right.$$

$$\left. - \mu_i \min\left(N_i(t), m_i\right) \sum_{j=0}^{n} b_{ij} p_{ij} \right] t \qquad (5)$$

For expected income of the whole queueing network we have

$$W(t) = \sum_{i=1}^{n} V_i(t)$$

$$M\{W(t)\} = \sum_{i=1}^{n} v_{i0} + \sum_{i=1}^{n} \left[c_i + a_{0i}\lambda p_{0i} + \sum_{j=1}^{n} a_{ji}\mu_j \min\left(N_j(t), m_j\right) p_{ji} - \right.$$

$$\left. - \mu_i \min\left(N_i(t), m_i\right) \sum_{j=1}^{n} b_{ij} p_{ij} \right] t \qquad (6)$$

but, taking into acount (3),

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}\mu_j \min\left(N_j(t), m_j\right) p_{ji} = \sum_{i=1}^{n} \mu_i \min\left(N_i(t), m_i\right) \sum_{j=1}^{n} b_{ij} p_{ij}$$

so

$$M\{W(t)\} = \sum_{i=1}^{n} v_{i0} + \sum_{i=1}^{n} \left[c_i + a_{0i}\lambda p_{0i} - \mu_i b_{i0} p_{i0} \right] t \qquad (7)$$

Note, that general expected income of network depends on r_{ij}, R_{ij}, $i, j=\overline{1,n}$, as these incomes extinguish each other (if message from one system of network arrivals to another then income of the last one increases by some value and the first system's income decreases by the same value).

Example. Let's consider close queueing network with central system, that consists of central system S_n and peripheral systems $S_1, S_2, ..., S_{n-1}$. Let $\lambda = 0$, $p_{in} = 1$, $p_{ni} \neq 0$, other $p_{ij} = 0, p_{0i} = p_{i0} = 0$, $i=\overline{1,n}$. In this case, from (5), (6) we recieve expressions for total expected income of the network

$$M\{W(t)\} = \sum_{i=1}^{n} v_{i0} + \sum_{i=1}^{n} c_i t \qquad (8)$$

expected income of central system is

$$v_n(t) = v_{n0} + \left[c_n + \sum_{j=1}^{n-1} a_{jn}\mu_j \min\left(N_j(t), m_j\right) - b_{nj}\mu_n p_{nj} \min\left(N_n(t), m_n\right) \right] t \qquad (9)$$

and expected incomes of peripheral systems taking into (3) are

$$v_i(t) = v_{i0} + \left[c_i + a_{ni}\mu_n \min\left(N_n(t), m_n\right) p_{ni} - b_{in}\mu_i \min\left(N_i(t), m_i\right)\right] t =$$
$$= v_{i0} + \left[c_i - a_{in}\mu_i \min\left(N_i(t), m_i\right) + b_{ni}\mu_n \min\left(N_n(t), m_n\right) p_{ni}\right] t \qquad (10)$$

Note that exprassions (8)-(10) concur with expressions for corresponding expected incomes in network with central system [5].

2. Recurrent mean-value analysis method with respect to the moments of time for open queueing network

From relations (5), (6) follows that we have know expressions of average number of messages $N_i(t)$, $i = \overline{1,n}$, for finding of expected system's incomes and the whole network. Finding of these values in analytic form for network that functions in transient behaviour is rather complicated problem. Before recurrent mean-value analysis method with respect to the moments of time was developed for finding of series of average characteristics for arbitrary closed queueing networks with one-type messages and multi-line systems [6]. In this paper we suggest substantiation of such method for open queueing network with arbitrary distributions of service times of messages in network's system lines.

Let's consider open queueing network with one-type messages and multi-line systems. Let $M_i(t)$ is average rate of massage's flow from the *i*-th system, $\rho_i(t)$, $\tau_i(t)$ - average number of busy lines and average sojourn time (include wating) at *i*-th system at time period $[t_0, t_0 + t]$, $i = \overline{1,n}$, correspondently. Then expression is correct

$$M_i(t)\tau_i(t) = N_i(t),\ i = \overline{1,n} \qquad (11)$$

For proving it we can use technique [7]. Let $Z_i(t)$ - total number of messages wich have left i-th system during time period $[t_0, t_0 + t]$, $\Theta_i(t)$ - total sojourn time in *i*-th system during time period $[t_0, t_0 + t]$. So

$$\tau_i(t)=\frac{M\Theta_i(t)}{MZ_i(t)},\ N_i(t)=\frac{M\Theta_i(t)}{t},\ M_i(t)=\frac{MZ_i(t)}{t},\ i=\overline{1,n}$$

therefore, formula (11) is correct.

As $\rho_i(t)$ we can approximately take value $\min(N_i(t),m_i)$ because at any time moment *t*

$$\overline{\rho}_i(t)=\min(k_i(t),m_i)$$

where $\overline{\rho}_i(t)$ - number of busy lines in *i*-th system at the moment *t*.

Let λ_i - arrival rate of messages in *i*-th system, $i=\overline{1,n}$. It is known [8] that

$$\lambda_i=\lambda p_{0i}+\lambda\sum_{j=1}^{n}e_j p_{ji},\ i=\overline{1,n}, \tag{12}$$

where λ - arrival rate of messages in network, p_{ji} - probability of message's transition between *j*-th and *i*-th systems, $i,j=\overline{1,n}$, p_{0i} - probability that message comes from outside to the *i*-th system, values e_j, $j=\overline{1,n}$, meet the system of linear equations

$$e_j=p_{0j}+\sum_{j=1}^{n}e_i p_{ij},\ j=\overline{1,n}$$

It follows from (11)

$$\lambda_i\tau_i(t)=\frac{\lambda_i N_i(t)}{\mu_i\rho_i(t)} \tag{13}$$

but

$$\lambda_i\tau_i(t)\approx N_i(t) \tag{14}$$

what follows from formula that is similar to Littl's formula [3].

From the account above we recieve recurrent mean-value analysis method with respect to the moments of time for open queueing networks:

$$\rho_i(t)=\min\left(N_i(t),m_i\right),\ i=\overline{1,n} \tag{15}$$

$$\tau_i(t)=\frac{N_i(t)}{\mu_i\rho_i(t)},\ i=\overline{1,n} \tag{16}$$

$$N_i(t+\Delta t)=\lambda_i\tau_i(t),\ i=\overline{1,n} \tag{17}$$

where λ_i, $i=\overline{1,n}$, meet the system of equation (12). Initial conditions can be choosen in following way: $N_i(t_0)\neq 0,\ i=\overline{1,n}$.

3. Variation of incomes of network's systems

Taking into account (3) we will rewrite expression of expected income of system S_i as

$$v_i(t)=v_{i0}+\left[c_i+a_{0i}\lambda p_{0i}+\sum_{j=1}^{n}a_{ji}\mu_j\min\left(N_j(t),m_j\right)p_{ji}-\right.$$

$$\left.-\mu_i\min\left(N_i(t),m_i\right)\left(b_{i0}p_{i0}+\sum_{j=1}^{n}a_{ij}p_{ij}\right)\right]t=$$

$$=v_{i0}+\left[c_i+a_{0i}\lambda p_{0i}-b_{i0}\mu_i\min\left(N_i(t),m_i\right)p_{i0}+\right.$$

$$\left.+\sum_{j=1}^{n}\left(a_{ji}\mu_j\min\left(N_j(t),m_j\right)p_{ji}-a_{ij}\mu_i\min\left(N_i(t),m_i\right)p_{ij}\right)\right]t \tag{18}$$

Let's introduce denotations for finding of variation of system's incomes:

$$M\left\{r_{ji}^{2}\right\}=a_{2ji},\ M\left\{R_{ij}^{2}\right\}=b_{2ij},\ i,j=\overline{1,n}$$

$$M\left\{r_{0i}^{2}\right\}=a_{20i},\ M\left\{r_{i}^{2}\right\}=c_{2i},\ M\left\{R_{i0}^{2}\right\}=b_{2i0},\ i=\overline{1,n}$$

Let's consider square of difference $\left(V_i(t)-v_{i0}\right)^2$:

$$\left(V_i(t)-v_{i0}\right)^2=\left(v_{i0}+\sum_{l=1}^{m}\Delta V_{il}(t)-v_{i0}\right)^2=\left(\sum_{l=1}^{m}\Delta V_{il}(t)\right)^2=$$

$$=\sum_{l=1}^{m}\Delta V_{il}^{2}(t)+\sum_{l=1}^{m}\sum_{\substack{j=1\\ l\neq j}}^{m}\Delta V_{il}(t)\Delta V_{ij}(t),i=\overline{1,n} \tag{19}$$

Let's find an expectation value of summands in the right part of the last equation. First we will write additional equations, supposed that RV r_{ji}, R_{ij}, r_{0i}, R_{i0} are independent in couple by r_i, $i,j=\overline{1,n}$. So

$$M\left\{\left(r_{0i}+r_i\Delta t\right)^2\right\}=a_{20i}+2a_{0i}c_i\Delta t+c_{2i}(\Delta t)^2 \tag{20}$$

$$M\left\{\left(-R_{i0}+r_i\Delta t\right)^2\right\}=b_{2i0}-2b_{i0}c_i\Delta t+c_{2i}(\Delta t)^2 \tag{21}$$

$$M\left\{\left(r_{ji}+r_i\Delta t\right)^2\right\}=a_{2ji}+2a_{ji}c_i\Delta t+c_{2i}(\Delta t)^2 \tag{22}$$

$$M\left\{\left(-R_{ij}+r_i\Delta t\right)^2\right\}=b_{2ij}-2b_{ii}c_i\Delta t+c_{2i}(\Delta t)^2 \tag{23}$$

Taking into account (20)-(23), we have:

$$\begin{aligned}
M\left\{\Delta V_{il}^{\,2}(t)/k(t)\right\} &= c_{2i}(\Delta t)^2\left[1-\left(\lambda p_{0i}+\mu_i(k_i)u(k_i)\right)\Delta t\right]+\\
&+\left[b_{2i0}-2b_{i0}c_i\Delta t+c_{2i}(\Delta t)^2\right]\mu_i(k_i)u(k_i)p_{i0}\Delta t+\\
&+\left[a_{20i}+2a_{0i}c_i\Delta t+c_{2i}(\Delta t)^2\right]\lambda p_{0i}\Delta t+\\
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[a_{2ji}+2a_{ji}c_i\Delta t+c_{2i}(\Delta t)^2\right]\mu_j(k_j)u(k_j)p_{ji}\Delta t+\\
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[b_{2ij}-2b_{ij}c_i\Delta t+c_{2i}(\Delta t)^2\right]\mu_i(k_i)u(k_i)p_{ij}\Delta t+o(\Delta t)=\\
&=\Bigg[\lambda a_{20i}p_{0i}+b_{2i0}\mu_i(k_i)u(k_i)p_{i0}+\\
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\left(a_{2ji}\mu_j(k_j)u(k_j)p_{ji}+b_{2ij}\mu_i(k_i)u(k_i)p_{ij}\right)\Bigg]\Delta t+o(\Delta t)
\end{aligned} \tag{24}$$

Taking into account that with fixed member function $k(t)$ values $\Delta V_{il}(t)$ and $\Delta V_{ij}(t)$ independent when $l\neq j$, using (3), (4) it can be fined

$$\begin{aligned}
M\left\{\Delta V_{il}(t)\Delta V_{ij}(t)/k(t)\right\} &= \Bigg\{\Bigg[a_{0i}\lambda p_{0i}-a_{i0}\mu_i(k_i)u(k_i)p_{i0}+c_i+\\
&+\sum_{j=1}^{n}\left(a_{ji}\mu_j(k_j)u(k_j)p_{ji}-a_{ij}\mu_i(k_i)u(k_i)p_{ij}\right)\Bigg]\Delta t+c(\Delta t)\Bigg\}^2=o(\Delta t)
\end{aligned} \tag{25}$$

Then, making extreme transition by $\Delta t\to 0$, it follows from (19), (24), (25)

$$M\left\{(V_i(t)-v_{i0})^2 \middle| k(t)\right\}=\sum_{l=1}^{m} M\left\{\Delta V_{il}{}^2(t)/k(t)\right\}+$$

$$+\sum_{l=1}^{m}\sum_{\substack{j=1\\ l\neq j}}^{m} M\left\{\Delta V_{il}(t)\Delta V_{ij}(t)/k(t)\right\}=\left[\lambda a_{20i}p_{0i}+b_{2i0}\mu_i(k_i)u(k_i)p_{i0}+\right.$$

$$\left.+\sum_{\substack{j=1\\ j\neq i}}^{n}\left(a_{2ji}\mu_j(k_j)u(k_j)p_{ji}+b_{2ij}\mu_i(k_i)u(k_i)p_{ij}\right)\right]t,\ i=\overline{1,n}$$

Get average by $k(t)$ we will have

$$M\left\{(V_i(t)-v_{i0})^2 \middle| k(t)\right\}=\left[\lambda a_{20i}p_{0i}+b_{2i0}\min\left(N_i(t),m_i\right)p_{i0}+\right.$$

$$\left.+\sum_{\substack{j=1\\ j\neq i}}^{n}\left(a_{2ji}\min\left(N_i(t),m_i\right)p_{ji}+b_{2ij}\min\left(N_i(t),m_i\right)p_{ij}\right)\right]t=$$

$$=\left[\lambda a_{20i}p_{0i}\sum_{\substack{j=1\\ j\neq i}}^{n}a_{2ji}\min\left(N_i(t),m_i\right)p_{ji}+\min\left(N_i(t),m_i\right)\sum_{\substack{j=0\\ j\neq i}}^{n}b_{2ij}p_{ij}\right]t \quad (26)$$

Let's find $M^2\{(V_i(t)-v_{i0})\}$ with help of (3), (4) for calculation of the variation using standard formula:

$$M^2\{(V_i(t)-v_{i0})\}=M^2\left\{\sum_{l=1}^{m}\Delta V_{il}(t)\middle|k(t)\right\}=\left[\sum_{l=1}^{m}M\left\{\Delta V_{il}(t)\middle|k(t)\right\}\right]^2=$$

$$=\left\{\left[c_i+a_{0i}\lambda p_{0i}-b_{i0}\mu_i(k_i)u(k_i)p_{i0}+\right.\right.$$

$$\left.\left.+\sum_{j=1}^{n}\left(a_{ji}\mu_j(k_j)u(k_j)p_{ji}-a_{ij}\mu_i(k_i)u(k_i)p_{ij}\right)\right]t+o(\Delta t)\right\}^2=$$

$$=\left[c_i+a_{0i}\lambda p_{0i}-b_{i0}\mu_i(k_i)u(k_i)p_{i0}+\right.$$

$$\left.+\sum_{j=1}^{n}\left(a_{ji}\mu_j(k_j)u(k_j)p_{ji}-a_{ij}\mu_i(k_i)u(k_i)p_{ij}\right)\right]^2t^2,\ \text{for } \Delta t\to 0$$

that is

$$M^2\{(V_i(t)-v_{i0})|k(t)\}=\{[a_{0i}\lambda p_{0i}-b_{i0}\mu_i(k_i)u(k_i)p_{i0}+$$

$$+\sum_{j=1}^{n}\left(a_{ji}\mu_j(k_j)u(k_j)p_{ji}-a_{ij}\mu_i(k_i)u(k_i)p_{ij}\right)\Bigg]^2+2c_i\,[a_{0i}\lambda p_{0i}-b_{i0}\mu_i(k_i)u(k_i)p_{i0}+$$

$$+\sum_{j=1}^{n}\left(a_{ji}\mu_j(k_j)u(k_j)p_{ji}-a_{ij}\mu_i(k_i)u(k_i)p_{ij}\right)\Bigg]+c_n{}^2\Bigg\}t^2$$

Average by $k(t)$ with taking in account (18) give

$$M^2\{(V_i(t)-v_{i0})\}=\left\{\sum_k P(k(t)=k)[a_{0i}\lambda p_{0i}-b_{i0}\mu_i(k_i)u(k_i)p_{i0}+\right.$$

$$\left.+\sum_{j=1}^{n}\left(a_{ji}\mu_j(k_j)u(k_j)p_{ji}-a_{ij}\mu_i(k_i)u(k_i)p_{ij}\right)\Bigg]^2+2c_i\left(\frac{V_i(t)-v_{i0}}{t}-c_i\right)+c_i\right\}t^2=$$

$$=\left\{\sum_k P(k(t)=k)[a_{0i}\lambda p_{0i}-b_{i0}\mu_i(k_i)u(k_i)p_{i0}+\right.$$

$$\left.+\sum_{j=1}^{n}\left(a_{ji}\mu_j(k_j)u(k_j)p_{ji}-a_{ij}\mu_i(k_i)u(k_i)p_{ij}\right)\Bigg]-c_i{}^2\right\}t^2+2c_i\left(v_i(t)-v_{i0}\right)t \qquad (27)$$

where $v_i(t)$, $i=\overline{1,n}$, is finding by formula (18).

Thus, taking in account (26), (27) variation of income i-th system of network can be written as

$$DV_i(t)=D(V_i(t)-v_{i0})=M\left\{(V_i(t)-v_{i0})^2\right\}-M^2\{(V_i(t)-v_{i0})\}=$$

$$=\Bigg[\lambda a_{20i}p_{0i}+\sum_{\substack{j=1\\ j\neq i}}^{n}a_{2ji}\min(N_i(t),m_i)p_{ji}+\min(N_i(t),m_i)\sum_{\substack{j=0\\ j\neq i}}^{n}b_{2ij}p_{ij}-$$

$$-2c_i\left(v_i(t)-v_{i0}\right)\Big]t+\left\{c_i{}^2-\sum_k P(k(t)=k)[a_{0i}\lambda p_{0i}-b_{i0}\mu_i(k_i)u(k_i)p_{i0}+\right.$$

$$\left.+\sum_{j=1}^{n}\left(a_{ji}\mu_j(k_j)u(k_j)p_{ji}-a_{ij}\mu_i(k_i)u(k_i)p_{ij}\right)\Bigg]^2\right\}t^2,\ i=\overline{1,n}$$

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