LONGITUDINAL VIBRATIONS OF A NON-UNIFORM RODS COUPLED BY DOUBLE SPRING-MASS SYSTEMS

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Abstract. The purpose of this paper is free longitudinal vibration problem of a non-uniform rods coupled by spring-mass systems. The solution of the boundary problem is obtained by the use of the Green's function method. In the paper examples of the Green's functions corresponding to the second order differential operator are presented.

Introduction

The longitudinal vibration problem of a system composed of rods coupled by discrete elements has been presented, for example, in the references [1-4].

In the reference [2] authors presented the solution in a closed form of the problem of longitudinal vibrations of two uniform rods coupled by translational springs. The papers [1] and [3] present symilar problem of vibration of uniform rods coupled by spring-mass systems. The paper [1] presents rods vibration problem using alternative formulation of the frequency equation. The formulation is based on the discretization of the elastic rods by their first *n* eigenfunctions. To find a solution of vibration problem the Green's function method is applied in references [2-4].

The purpose of this paper is solution of vibration problem of two rods with variable geometrical parameters. It is assumed that rods are connected by an arbitrary number of discrete spring-mass systems. The formulation of the problem establish the differential equations of motion of rods and the boundary conditions corresponding to attachements of the rods' ends. Exact solution of the boudary problem is obtained by using of the Green's function metod.

In the paper are also presented examples of Green's functions corresponding to a second order differential operators occurring in differential description od the rods vibration. This Green's functions are obtained for chosen functions describing varrying cross-section of the rod.

1. Formulation of the problem

Consider a system of two rods of length L_i which are coupled by n spring-mass systems with the masses m_j and stiffness module $k_{ij}(x)$ (Fig. 1) where i = 1, 2; j = 1,..., n. Discrete elements are attached at points $\overline{x}_{ij}$ of the rods.

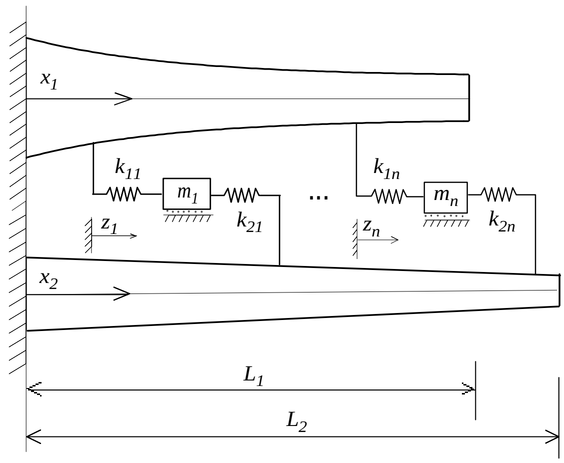

Fig. 1. A sketch of non-uniform rods coupled by n spring-mass systems

Assuming that the non-uniform rods are characterized by $\tilde{A}_i\left(x_i\right)$ - functions describing cross sections of the rods, ρ_i, E_i - densities of rods material and Young modulus', respectively, the differential equations of longitudinal vibrations of the rods are:

$$\tilde{\Lambda}_i\left[u_i\left(x_i,t\right)\right]=-\sum_{j=1}^{n}k_{ij}\cdot\left[z_j\left(t\right)-u_i\left(\overline{x}_{ij},t\right)\right]\cdot\delta\left(x_i-\overline{x}_{ij}\right) \tag{1}$$

i = 1, 2; j = 1, 2,…, n

where $\delta(\cdot)$ denotes the Dirac delta function, $x_i\in\left[0,L_i\right]$ and $\tilde{\Lambda}_i$ (i = 1, 2) are differential operators in the form:

$$\tilde{\Lambda}_i\equiv\frac{\partial}{\partial x_i}\left[E_i\tilde{A}_i\left(x_i\right)\frac{\partial}{\partial x_i}\right]-\rho_i\tilde{A}_i\left(x_i\right)\frac{\partial^2}{\partial t^2} \tag{2}$$

Motion z_j of masses m_j are governed by:

$$m_j\frac{d^2z_j}{dt^2}+k_{1j}\left[z_j\left(t\right)-u_1\left(\overline{x}_{1j},t\right)\right]+k_{2j}\left[z_j\left(t\right)-u_2\left(\overline{x}_{2j},t\right)\right]=0 \tag{3}$$

The function describing the longitudinal displacements of the first and second rods satysfies homogeneous boundary conditions, which can be written symbolically in the form:

$$\tilde{\mathbf{B}}_{0i}\left[u_i\left(x_i,t\right)\right]\Big|_{x_i=0}=0\,,\ \tilde{\mathbf{B}}_{1i}\left[u_i\left(x_i,t\right)\right]\Big|_{x_i=L_i}=0 \tag{4}$$

Longitudinal vibrations of the rods are harmonic with the frequency ω. Using separation of variables according to:

$$u_i\left(x_i,t\right)=\bar{U}_i\left(x_i\right)cos\,\omega t\,,\ z_j\left(t\right)=Z_j\,cos\,\omega t$$

where $\bar{U}_i\left(x_i\right)$ and Z_j are the corresponding amplitude functions and putting them into equations (1), (3) and (4), the eqations of motion and boudary conditions can be reformulated as:

$$\bar{\Lambda}_1\left[\bar{U}_1\left(x_1\right)\right]=-\sum_{j=1}^{n}k_{1j}\cdot\frac{k_{2j}\left[\bar{U}_2\left(\bar{x}_{2j}\right)-\bar{U}_1\left(\bar{x}_{1j}\right)\right]+\omega^2 m_j\bar{U}_1\left(\bar{x}_{1j}\right)}{k_{1j}+k_{2j}-\omega^2 m_j}\cdot\delta\left(x_1-\bar{x}_{1j}\right) \tag{5a}$$

$$\bar{\Lambda}_2\left[\bar{U}_2\left(x_2\right)\right]=-\sum_{j=1}^{n}k_{2j}\,\frac{-k_{1j}\left[\bar{U}_2\left(\bar{x}_{2j}\right)-\bar{U}_1\left(\bar{x}_{1j}\right)\right]+\omega^2 m_j\bar{U}_2\left(\bar{x}_{2j}\right)}{k_{1j}+k_{2j}-\omega^2 m_j}\,\delta\left(x_2-\bar{x}_{2j}\right) \tag{5b}$$

$$\bar{\mathbf{B}}_{0i}\left[\bar{U}_i\left(x_i\right)\right]\Big|_{x_i=0}=0\,,\ \bar{\mathbf{B}}_{1i}\left[\bar{U}_i\left(x_i\right)\right]\Big|_{x_i=L_i}=0 \tag{6}$$

where $\bar{\Lambda}_i\equiv\dfrac{\partial}{\partial x_i}\left[E_i\tilde{A}_i\left(x_i\right)\dfrac{\partial}{\partial x_i}\right]+\rho_i\omega^2\tilde{A}_i\left(x_i\right)$.

Introducing non-dimentional coordinates and non-dimensional values: $\xi_i=\dfrac{x_i}{L_i}$, $\zeta_{ij}=\dfrac{\bar{x}_{ij}}{L_i}$, $\xi_i,\zeta_{ij}\in\left[0,1\right]$, $\mathbf{\Omega}_i^2=\dfrac{\omega^2\rho_i L_i^2}{E_i}$, $K_j=\dfrac{k_{2j}}{k_{ij}}$, $K_{ij}=\dfrac{k_{ij}L_i}{E_i A_i\left(0\right)}$, $M_{ij}=\dfrac{m_j}{\rho_i L_i A_i\left(0\right)}$ into diferentional equations (5a,b) and boudary conditions (6) yields following non-dimensional boundary problem:

$$\Lambda_1\left[U_1\left(\xi_1\right)\right]=-\sum_{j=1}^{n}K_{1j}\cdot\frac{K_j\left[U_2\left(\zeta_{2j}\right)-U_1\left(\zeta_{1j}\right)\right]+\Omega_1^2\dfrac{M_{1j}}{K_{1j}}U_1\left(\zeta_{1j}\right)}{1+K_j-\Omega_1^2\dfrac{M_{1j}}{K_{1j}}}\cdot\delta\left(\xi_1-\zeta_{1j}\right) \tag{7a}$$

$$\Lambda_2\left[U_2\left(\xi_2\right)\right]=-\sum_{j=1}^{n}K_{2j}\cdot\frac{-\left[U_2\left(\zeta_{2j}\right)-U_1\left(\zeta_{1j}\right)\right]+\Omega_2^2\frac{M_{2j}}{K_{2j}}K_jU_2\left(\zeta_{2j}\right)}{1+K_j-\Omega_2^2\frac{M_{2j}}{K_{2j}}K_j}\cdot\delta\left(\xi_2-\zeta_{2j}\right) \quad (7b)$$

$$\mathbf{B}_{0i}\left[U_i\left(\xi_i\right)\right]\Big|_{\xi_i=0}=0\,,\ \mathbf{B}_{1i}\left[U_i\left(\xi_i\right)\right]\Big|_{\xi_i=1}=0 \quad (8)$$

Diferrentional operators Λ_i , apearing on left side of equations (7), have the form:

$$\Lambda_i\equiv\frac{\partial}{\partial\xi_i}\left[A_i\left(\xi_i\right)\frac{\partial}{\partial\xi_i}\right]+\Omega_i^2A_i\left(\xi_i\right)$$

2. Solution of the problem

The solution of the boundary problem (7)-(8) is obtained with the use of the Greens function method. Using boundary conditions and properties of the Green's functions, the solution of the considered problem can be written as:

$$U_1\left(\xi_1\right)=\sum_{j=1}^{n}\left\{-K_{1j}\cdot\frac{K_j\left[U_2\left(\zeta_{2j}\right)-U_1\left(\zeta_{1j}\right)\right]+\Omega_1^2\frac{M_{1j}}{K_{1j}}U_1\left(\zeta_{1j}\right)}{1+K_j-\Omega_1^2\frac{M_{1j}}{K_{1j}}}\right\}\cdot G_1\left(\xi_1,\zeta_{1j}\right) \quad (9a)$$

$$U_2\left(\xi_2\right)=\sum_{j=1}^{n}\left\{-K_{2j}\cdot\frac{-\left[U_2\left(\zeta_{2j}\right)-U_1\left(\zeta_{1j}\right)\right]+\Omega_2^2\frac{M_{2j}}{K_{2j}}K_jU_2\left(\zeta_{2j}\right)}{1+K_j-\Omega_2^2\frac{M_{2j}}{K_{2j}}K_j}\right\}\cdot G_2\left(\xi_2,\zeta_{2j}\right) \quad (9b)$$

or introducing following coefficients:

$$A_{1j}=\frac{K_{1j}\left(K_j-\mathbf{\Omega}_1^2\frac{M_{1j}}{K_{1j}}\right)}{1+K_j-\mathbf{\Omega}_1^2\frac{M_{1j}}{K_{1j}}},\ A_{2j}=\frac{-K_{1j}K_j}{1+K_j-\mathbf{\Omega}_1^2\frac{M_{1j}}{K_{1j}}}$$

$$B_{1j}=\frac{-K_{2j}}{1+K_j-\mathbf{\Omega}_2^2\frac{M_{2j}}{K_{2j}}K_j},\ B_{2j}=\frac{K_{2j}\left[1-\mathbf{\Omega}_2^2\frac{M_{2j}}{K_{2j}}K_j\right]}{1+K_j-\mathbf{\Omega}_2^2\frac{M_{2j}}{K_{2j}}K_j},\ j=1,2,\ldots,n$$

as:

$$U_1\left(\xi_1\right)=\sum_{j=1}^{n}\left\{A_{1j}U_1\left(\zeta_{1j}\right)+A_{2j}U_2\left(\zeta_{2j}\right)\right\}\cdot G_1\left(\xi_1,\zeta_{1j}\right) \tag{10a}$$

$$U_2\left(\xi_2\right)=\sum_{j=1}^{n}\left\{B_{1j}U_1\left(\zeta_{1j}\right)+B_{2j}U_2\left(\zeta_{2j}\right)\right\}\cdot G_2\left(\xi_2,\zeta_{2j}\right) \tag{10b}$$

Subsituting $\xi_i := \zeta_{ij}$ for i = 1, 2; j = 1, 2,…, n into equations (10a,b), displacements of the points ζ_{ij} can be presented in the form:

$$U_1\left(\zeta_{1k}\right)=\sum_{j=1}^{n}\left\{A_{1j}U_1\left(\zeta_{1j}\right)+A_{2j}U_2\left(\zeta_{2j}\right)\right\}\cdot G_1\left(\zeta_{1k},\zeta_{1j}\right) \tag{11a}$$

$$U_2\left(\zeta_{2k}\right)=\sum_{j=1}^{n}\left\{B_{1j}U_1\left(\zeta_{1j}\right)+B_{2j}U_2\left(\zeta_{2j}\right)\right\}\cdot G_2\left(\zeta_{2k},\zeta_{2j}\right) \tag{11b}$$

This equations represent a set of 2n homogeneous equations:

$$\mathbf{A}\cdot\mathbf{U}=\mathbf{0} \tag{12}$$

with unknown $\mathbf{U}=\left[U_1\left(\zeta_{11}\right),U_1\left(\zeta_{12}\right),\ldots,U_1\left(\zeta_{1n}\right),U_2\left(\zeta_{21}\right),U_2\left(\zeta_{22}\right),\ldots,U_2\left(\zeta_{2n}\right)\right]^T$.
The non-trivial solution of equation (12) exists when the determinant of the coefficient matrix **A** vanishes (δ_{jk} is the Kronecker delta):

$$\mathbf{A}=\begin{bmatrix}\left[-\delta_{jk}+A_{1k}G_1\left(\zeta_{1k},\zeta_{1j}\right)\right]_{1\le j,k\le n} & \left[A_{2k}G_1\left(\zeta_{1k},\zeta_{1j}\right)\right]_{1\le j,k\le n}\\ \left[B_{1k}G_2\left(\zeta_{2k},\zeta_{2j}\right)\right]_{1\le j,k\le n} & \left[-\delta_{jk}+B_{2k}G_2\left(\zeta_{2k},\zeta_{2j}\right)\right]_{1\le j,k\le n}\end{bmatrix} \tag{13}$$

It yields the frequency equation:

$$det\,\mathbf{A}=0 \tag{14}$$

which is numerically solved with respect to non-dimensional frequency parameters Ω_i^2, where $\Omega_1^2=\dfrac{\rho_1 L_1^2 E_2}{\rho_2 L_2^2 E_1}\Omega_2^2$. With the determined eigenfrequency Ω_{im}^2 correspond the mode shapes:

$$\begin{aligned}U_{1n}\left(\xi_1\right)=&\sum_{j=1}^{n-1}\left\{A_{1j}U_1\left(\zeta_{1j}\right)+A_{2j}U_2\left(\zeta_{2j}\right)\right\}\cdot G_1\left(\xi_1,\zeta_{1j},\Omega_{1n}\right)\\&+\left[A_{1n}C+A_{2n}U_2\left(\zeta_{2n}\right)\right]\cdot G_1\left(\xi_1,\zeta_{1n},\Omega_{1n}\right)\end{aligned} \tag{15}$$

$$U_2(\xi_2)=\sum_{j=1}^{n-1}\left\{B_{1j}U_1(\zeta_{1j})+B_{2j}U_2(\zeta_{2j})\right\}\cdot G_2(\xi_2,\zeta_{2j},\Omega_{2n})$$
$$+\left[B_{1n}C+B_{2n}U_2(\zeta_{2n})\right]\cdot G_2(\xi_2,\zeta_{2n},\Omega_{2n}) \tag{16}$$

where is $C=U_1(\zeta_{1n})$.

3. Examples of Green's functions

The Greens functions corresponding to the consided second order differential operator

$$\Lambda\equiv\frac{d}{d\xi}\left(A(\xi)\frac{d}{d\xi}\right)+\Omega^2A(\xi)$$

are derived. The cross section functions $A(\xi)$ are assumed in the form:

a) $A(\xi)=sin^2(\alpha\xi+1)$,

b) $A(\xi)=e^{-\alpha\xi}$,

c) $A(\xi)=cosh^2(\alpha\xi+1)$

(α = *const.*). The Greens functions corresponding to thefunctions can be written in the form [4]:

$$G(\xi,\eta)=c_1\frac{\cos\kappa\xi}{f(\xi)}+c_2\frac{\sin\kappa\xi}{f(\xi)}+\frac{\sin\kappa(\xi-\eta)}{\kappa f(\xi)f(\eta)}H(\xi-\eta) \tag{17}$$

where $H(\cdot)$ denotes Heaviside function and function $f(\xi)$ for the considered cases are:

a) $f(\xi)=sin(\alpha\xi+1)$, $\kappa^2=\Omega^2+\alpha^2$

b) $f(\xi)=e^{-\frac{\alpha\xi}{2}}$, $\kappa^2=\Omega^2-\frac{\alpha^2}{4}$

c) $f(\xi)=cosh(\alpha\xi+1)$, $\kappa^2=\Omega^2-\alpha^2$.

Coefficients c_1 and c_2 in formula (17) are determined on basis of the boundary conditions (Table 1).

Table 1

Values of the coefficients c_1 and c_2 for different boundary conditions
$M_1(\xi,\eta) = \sin\kappa(\xi - \eta)$, $M(\xi,\eta) = \kappa f(\xi)\cos\kappa(\xi - \eta) - f'(\xi)\sin\kappa(\xi - \eta)$

Boundary conditions	$c = [c_1, c_2]^T$
$u\big\vert_{\xi=0} = 0,\ u'\big\vert_{\xi=1} = 0$	$c_1 = 0,\ \ c_2 = \dfrac{-M(1,\eta)}{\kappa f(\eta) M(1,0)}$
$u\big\vert_{\xi=0} = 0,\ u\big\vert_{\xi=1} = 0$	$c_1 = 0,\ \ c_2 = \dfrac{-M_1(1,\eta)}{\kappa f(\eta) M_1(1,0)}$
$u'\big\vert_{\xi=0} = 0,\ u'\big\vert_{\xi=1} = 0$	$c_1 = \dfrac{-f(0)M(1,\eta)}{f(\eta)\left[f(1)\,M'_\eta(0,1) - f'(1)\,M(0,1)\right]}$ $c_2 = \dfrac{-f'(0)M(1,\eta)}{\kappa f(\eta)\left[f(1)\,M'_\eta(0,1) - f'(1)\,M(0,1)\right]}$
$u'\big\vert_{\xi=0} = 0,\ u\big\vert_{\xi=1} = 0$	$c_1 = \dfrac{-f(0)\,M_1(1,\eta)}{f(\eta)M(0,1)},\ \ c_2 = \dfrac{-f'(0)\,M_1(1,\eta)}{\kappa f(\eta)M(0,1)}$

Next example concerns the rod which cross-section area is described by function $A(\xi) = (\alpha\xi + 1)^n$. In this case the Green's function has the form [4]:

$$G(\xi,\eta) = A^\gamma(\xi)\left[c_1 J_\upsilon(z_\xi) + c_2 Y_\upsilon(z_\xi)\right] + G_S(\xi,\eta) \qquad \text{when } \upsilon \in \boldsymbol{Z} \tag{18a}$$

$$G(\xi,\eta) = p^\gamma(\xi)\left[c_1 J_\upsilon(z_\xi) + c_2 J_{-\upsilon}(z_\xi)\right] + G_S(\xi,\eta) \qquad \text{when } \upsilon \notin \boldsymbol{Z} \tag{18b}$$

Function $G_S(\zeta,\eta)$ can be written as:

$$G_S(\xi,\eta) = M^{-1}\left[2J_\upsilon(z_{\xi-\eta})Y_\upsilon(z_0) - p^\gamma(\xi-\eta)J_\upsilon(z_0)Y_\upsilon(z_{\xi-\eta})\right]H(\xi-\eta)$$
$$\text{when } \upsilon \in \boldsymbol{Z} \tag{19a}$$

$$G_S(\xi,\eta) = M^{-1}\left[2J_\upsilon(z_{\xi-\eta})J_{-\upsilon}(z_0) - p^\gamma(\xi-\eta)J_\upsilon(z_0)J_{-\upsilon}(z_{\xi-\eta})\right]H(\xi-\eta)$$
$$\text{when } \upsilon \notin \boldsymbol{Z} \tag{19b}$$

where: $z_\xi = z(\xi)$, $z_{\xi\text{-}\eta} = z(\xi - \eta)$, $z_0 = z(0)$, $z_1 = z(1)$, $\alpha = const.$ and $M = \dfrac{4\alpha}{\pi}sin(\upsilon\pi)$. Values of coefficients c_1 and c_2 depend on the boundary conditions. For example, coefficients for rod clamped at left end ($G\big\vert_{\xi=0} = 0$) and free at the right end ($G'_{\boldsymbol{\xi}}\big\vert_{\xi=1} = 0$) for $\upsilon \notin \boldsymbol{Z}$ are:

$$c_1 = \frac{J_{-\upsilon}(z_0)N_3}{D_1}, \qquad c_2 = \frac{-J_{\upsilon}(z_0)N_3}{D_1}$$

For a free-free rod ($G'_{\xi}\big|_{\xi=0} = G'_{\xi}\big|_{\xi=1} = 0$), c_1 and c_2 have the form:

$$c_1 = \frac{N_2(z_0)N_3}{D_2}, \qquad c_2 = \frac{-N_1(z_0)N_3}{D_2}$$

where:

$$D_1 = z_0^{-2\upsilon} M\left[J_{\upsilon}(z_0)N_1(z_1) - J_{-\upsilon}(z_0)N_2(z_1)\right],$$

$$D_2 = M\left[N_1(z_0)N_2(z_1) - N_1(z_1)N_2(z_0)\right],$$

$$N_3 = z_1^{-\upsilon+1} z_{1-\eta}^{-\upsilon-1}\left[2 z_{1-\eta}^{-\upsilon+1} z_0^{\upsilon} J_{-\upsilon}(z_0)\Big(J_{\upsilon-1}(z_{1-\eta}) - J_{\upsilon+1}(z_{1-\eta})\Big) - J_{\upsilon}(z_0)N_1(z_{1-\eta})\right],$$

$$N_1(\varepsilon) = 2\upsilon J_{-\upsilon}(\varepsilon) + \varepsilon\left[J_{-\upsilon-1}(\varepsilon) - J_{-\upsilon+1}(\varepsilon)\right],$$

$$N_2(\varepsilon) = 2\upsilon J_{\upsilon}(\varepsilon) + \varepsilon\left[J_{\upsilon-1}(\varepsilon) - J_{\upsilon+1}(\varepsilon)\right] \quad \text{for } \varepsilon \in \left\{z_0, z_1, z_{1-\eta}\right\}.$$

Conclusions

The paper presents exact solution to the problem of longitudinal vibrations of a system of non-uniform rods connected by double spring-mass elements. The solution can be used in numerical investigation of vibration of the considered systems. The Greens functions corresponding to the differential operators occuring in differential description of the rods vibration are presented. The presented method can be applied to solve the vibration problems of systems consisting many non-uniform rods coupled by many discrete elements.

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