ON DETERMINANT OF CERTAIN PENTADIAGONAL MATRIX

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Abstract. In this paper, using the LU factorization, the relation between the determinant of a certain pentadiagonal matrix and the determinant of a corresponding tridiagonal matrix will be derived. Moreover, it will be shown that determinant of this special pentadiagonal matrix can be calculated by applying the fourth order homogeneous linear difference equation.

Keywords: *tridiagonal matrix, pentadiagonal matrix, determinant, LU factorization*

Introduction

The subject of consideration is a special case of pentadiagonal matrix which has the form

$$\mathbf{P} = \begin{bmatrix} a & 0 & c & & & & \\ 0 & a & 0 & c & & & \\ b & 0 & a & 0 & c & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c \\ & & & b & 0 & a & 0 \\ & & & & b & 0 & a \end{bmatrix}_{n\times n} \tag{1}$$

Pentadiagonal matrices play an important role in many areas of science and engineering, for instance in numerical solution of differential equations, interpolation problems, boundary value problems, parallel computing and the finite element method, e.g. [1].

The aim of this paper is to show the relation between the determinant of a considered pentadiagonal matrix and the determinant of corresponding tridiagonal matrix of the form

$$\mathbf{T}=\begin{bmatrix} a & c & & & & & \\ b & a & c & & & & \\ & b & a & c & & & \\ & & b & a & c & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & b & a & c \\ & & & & & b & a \end{bmatrix}_{n\times n} \tag{2}$$

Determinant W_n of matrix **T** was considered in [2]. It was shown that determinant W_n is the particular solution of second order homogeneous linear difference equation

$$W_{n+2}-aW_{n+1}+bcW_n=0,\ n\geq 1 \tag{3}$$

with initial conditions of the form

$$W_1=a,\ W_2=a^2-bc \tag{4}$$

On the basis of linear difference equations theory it was obtained two cases of solutions of equation (3) depending on relation between elements of matrix **T**.
For $a^2-4bc\neq 0$ determinant of matrix **T** has the form

$$W_n=\frac{1}{\sqrt{a^2-4bc}}\left[\left(\frac{a+\sqrt{a^2-4bc}}{2}\right)^{n+1}-\left(\frac{a-\sqrt{a^2-4bc}}{2}\right)^{n+1}\right] \tag{5}$$

Whilst for $a^2-4bc=0$ it is equal to

$$W_n=\frac{n+1}{2^n}a^n \tag{6}$$

1. The mains results

In this section we are to show the relation between determinants of matrices **P** and **T**. To this end we begin with LU factorization, [3], of the matrix **P** in the form

$$\mathbf{P}=\mathbf{LU} \tag{7}$$

in which we have

$$\mathbf{L}=\begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ \frac{b}{x_1} & 0 & 1 & & & \\ 0 & \frac{b}{x_2} & 0 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & \frac{b}{x_{n-2}} & 0 & 1 \end{bmatrix}_{n\times n} \tag{8}$$

and

$$\mathbf{U}=\begin{bmatrix} x_1 & 0 & c & 0 & \dots & 0 \\ & x_2 & 0 & \ddots & & \vdots \\ & & x_3 & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & c \\ & & & & \ddots & 0 \\ & & & & & x_n \end{bmatrix}_{n\times n} \tag{9}$$

where

$$x_i=\begin{cases} a & i=1,2 \\ a-\dfrac{bc}{x_{i-2}} & i=3,\dots,n \end{cases}, \qquad x_i\neq 0 \tag{10}$$

Bearing in mind (7)-(10), under denotation $F_n=\det\mathbf{P}$, we get

$$F_n=\prod_{i=1}^{n} x_i \tag{11}$$

Theorem 1. *Let* **U** *be the matrix of order n given by* (9). *Then the following statements hold*

1) *If* $n=2k$ *then*

$$x_{2k-1}=x_{2k},\ k=1,2,\dots,\frac{n}{2}$$

2) *If* $n=2k+1$ *then*

$$x_{2k-1}=x_{2k},\ k=1,2,\dots,\frac{n-1}{2}$$

Proof by induction

Case 1. $n = 2k$, $k = 1,2,\ldots,\frac{n}{2}$

For $k = 1$ we have $x_1 = x_2 = a$.

Suppose that for $k = l$, $k > 1$ the following induction assumption holds

$$x_{2l-1} = x_{2l}$$

It has to be shown that for $k = l + 1$ we have

$$x_{2l+1} = x_{2l+2}$$

Bearing in mind (10) we get

$$x_{2l+1} = a - \frac{bc}{x_{2l-1}}$$

Using the induction assumption we obtain

$$x_{2l+1} = a - \frac{bc}{x_{2l}}$$

at the same time from (10) we have

$$a - \frac{bc}{x_{2l}} = x_{2l+2}$$

and finally we get $x_{2l+1} = x_{2l+2}$ which ends the proof.

Case 2. $n = 2k + 1$, $k = 1,2,\ldots,\frac{n-1}{2}$

It can be observed that for this case the induction steps are the same.

Theorem 2. *Let* **U** *be the matrix of order n given by* (9) *and* W_n, $n \geq 1$ *be the determinant of a tridiagonal matrix of the form* (2). *Moreover we assume that* $W_0 = 1$. *Then the following statements hold*

1) *If* $n = 2k$ *then*

$$x_{2k-1} = \frac{W_k}{W_{k-1}},\ k = 1,2,\ldots,\frac{n}{2}$$

2) *If* $n = 2k + 1$ *then*

$$x_{2k-1} = \frac{W_k}{W_{k-1}},\ k = 1,2,\ldots,\frac{n+1}{2}$$

Proof by induction

Case 1. $n=2k$, $k=1,2,\ldots,\frac{n}{2}$

For $k=1$ we have $x_1=\frac{W_1}{W_0}=W_1=a$.

Suppose that for $k=l$, $k>1$ the following induction assumption holds

$$x_{2l-1}=\frac{W_l}{W_{l-1}}$$

It has to be shown that for $k=l+1$ we have

$$x_{2l+1}=\frac{W_{l+1}}{W_l}$$

Bearing in mind (10) we get

$$x_{2l+1}=a-\frac{bc}{x_{2l-1}}$$

Using the induction assumption we obtain

$$x_{2l+1}=a-bc\frac{W_{l-1}}{W_l}=\frac{aW_l-bcW_{l-1}}{W_l}$$

hence from (3) we have $x_{2l+1}=\frac{W_{l+1}}{W_l}$ which ends the proof.

Case 2. $n=2k+1$, $k=1,2,\ldots,\frac{n-1}{2}$

It can be observed that for this case the induction steps are the same.

Corollary

If F_n is the determinant of pentadiagonal matrix **P** of the form (1) and W_n is the determinant of tridiagonal matrix **T** of the form (2) then

1) For $n=2k$

$$F_n=F_{2k}=(W_k)^2$$

2) For $n=2k+1$

$$F_n=F_{2k+1}=W_k\cdot W_{k+1}$$

2. Fourth order difference equation for a certain pentadiagonal matrix

Now we are to show that determinant F_n of matrix **P** under consideration can be calculated by applying a corresponding difference equation.
Let us observe that

$$F_1 = a\,,\ F_2 = a^2\,,\ F_3 = a^3 - abc\,,\ F_4 = a^4 - 2a^2bc + b^2c^2 \tag{12}$$

Using the method of Laplace expansion four times: firstly with respect to the first column, then with respect to the first row, subsequently with respect to the first column and finally with respect to the first row of matrix **P**, we obtain

$$F_{n+4} - aF_{n+3} + abcF_{n+1} - b^2c^2F_n = 0 \tag{13}$$

Hence the value of determinant F_n is the particular solution of equation (13) fulfilling initial conditions (12). It can be easily observed, [4], that the direct solution to equation (13) has a rather complicated form and is not useful from the practical point of view. On the other hand bearing in mind the results obtained in section 2 we can express solution of equation (13) with the initial condition (12) by a solution of equation (3) with initial conditions (4).

Conclusions

It was shown that the determinant of the pentadiagonal matrix which has only three non-zero diagonals can be expressed by a determinant of the corresponding tridiagonal matrix. The question arises whether there exists an analogous relation in the case of multidiagonal matrices with only three non-zero diagonals. This problem will be studied in the forthcoming paper.

References

[1] Gladwell G.M.L., Lawrence K.P., Siegel D., Isospectral finite element membranes, Mechanical Systems and Signal Processing 2009, 23, 1986-1999.

[2] Borowska J., Łacińska L., Rychlewska J., Application of difference equation to certain tridiagonal matrices, Scientific Research of the Institute of Mathematics and Computer Science 2012, 3(11), 15-20.

[3] Zhao X., Huang T., On the inverse of a general pentadiagonal matrix, Applied Mathematics and Computation 2008, 202, 639-646.

[4] Elaydi S., An Introduction to Difference Equations, Springer, 2005.