THE *n*-DIMENSIONAL FOURIER EQUATION WITH THE ROBIN'S BOUNDARY CONDITION USING THE FINITE DIFFERENCE METHOD

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Abstract. The article is a continuation of the considerations on the three-dimensional Fourier equation supplemented by the third boundary condition (the Robin's condition). The paper is based on the Finite Difference Method.

Keywords: *block matrices, determinant, Fourier equation, boundary condition*

Introduction

The subject of our discussion is the heat flow in the *n*-dimensional area described by the Fourier equation with Robin's boundary condition. In this paper we present the exact solution to this problem.

Solution of the problem

We consider the *n*-dimensional Fourier equation [1, 2]

$$\lambda\left(\frac{\partial^2 T\left(x_1, x_2,..., x_n, t\right)}{\partial x_1^2}+\frac{\partial^2 T\left(x_1, x_2,..., x_n, t\right)}{\partial x_2^2}+...+\frac{\partial^2 T\left(x_1, x_2,..., x_n, t\right)}{\partial x_n^2}\right)= \\ =\rho c\frac{\partial T\left(x_1, x_2,..., x_n, t\right)}{\partial t} \tag{1}$$

where λ is a thermal conductivity [W/mK], c is a specific heat [J/kgK], ρ is a mass density [kg/m^3], T is temperature [K], $x_1, x_2, ..., x_n$ denote the geometrical co-ordinates and t is time [s].

Assuming the following difference quotients we obtain the differential approximation of the second derivatives appearing in equation (1)

$$
\begin{aligned}
&\frac{\Delta^2 T}{\Delta x_1^2}=\frac{T_{i_1-1,i_2,\dots,i_n,l}-2T_{i_1,i_2,\dots,i_n,l}+T_{i_1+1,i_2,\dots,i_n,l}}{\left(\Delta x_1\right)^2}, \qquad 1\le i_1\le m_1-1\\
&\frac{\Delta^2 T}{\Delta x_2^2}=\frac{T_{i_1,i_2-1,\dots,i_n,l}-2T_{i_1,i_2,\dots,i_n,l}+T_{i_1,i_2+1,\dots,i_n,l}}{\left(\Delta x_2\right)^2}, \qquad 1\le i_2\le m_2-1\\
&\vdots\\
&\frac{\Delta^2 T}{\Delta x_n^2}=\frac{T_{i_1,i_2,\dots,i_n-1,l}-2T_{i_1,i_2,\dots,i_n,l}+T_{i_1,i_2,\dots,i_n+1,l}}{\left(\Delta x_n\right)^2}, \qquad 1\le i_n\le m_n-1
\end{aligned}
\tag{2}
$$

and the approximation of the first derivative of the time

$$\frac{\Delta T}{\Delta t}=\frac{T_{i_1,i_2,\dots,i_n,l}-T_{i_1,i_2,\dots,i_n,l-1}}{\Delta t}, \quad 1\le l\le q \tag{3}$$

The initial condition has the form

$$T_{i_1,i_2,\dots,i_n,l}=T_{ini}, \quad 0\le l\le q \tag{4}$$

where T_{ini} is the initial temperature.

The system of equations containing the Fourier equation with the Robin conditions [2-4] is presented

$$
\begin{cases}
\lambda\dfrac{T_{1,i_2,\dots,i_n,l}-T_{env}}{2\Delta x_1}=\alpha\left(T_{T_{0,\,i_2,\dots,i_n,l}}-T_{env}\right)\\
\lambda\dfrac{T_{T_{i_1,1,\dots,i_n,l}}-T_{env}}{2\Delta x_2}=\alpha\left(T_{T_{i_1,0,\dots,i_n,l}}-T_{env}\right)\\
\vdots\\
\lambda\dfrac{T_{T_{i_1,\,i_2,\dots,1,l}}-T_{env}}{2\Delta x_n}=\alpha\left(T_{T_{i_1,\,i_2,\dots,0,l}}-T_{env}\right)\\
\lambda\left(\dfrac{\partial^2 T}{\partial x_1^2}+\dfrac{\partial^2 T}{\partial x_2^2}+\dots+\dfrac{\partial^2 T}{\partial x_n^2}\right)=\rho c\dfrac{\Delta T}{\Delta t}
\end{cases}
\tag{5}
$$

where α is the heat transfer coefficient and T_{env} is the ambient temperature of environment.

Now we consider the systems of equations corresponding to the exemplary vertices, edges and lateral planes of our n-cube.

In the first vertex of the *n*-cube (for $i_1, i_2, ..., i_n$) we have

$$
\begin{aligned}
&\lambda \frac{T_{1,0,...,0,l} - T_{env}}{2\Delta x_1} = \alpha\left(T_{0,0,...,0,l} - T_{env}\right) \\
&\lambda \frac{T_{0,1,0,...,l} - T_{env}}{2\Delta x_2} = \alpha\left(T_{0,0,...,0,l} - T_{env}\right) \\
&\vdots \\
&\lambda \frac{T_{0,0,...,1,l} - T_{env}}{2\Delta x_n} = \alpha\left(T_{0,0,...,0,l} - T_{env}\right) \\
&\frac{\lambda}{\left(\Delta x_1\right)^2} T_{env} - \frac{2\lambda}{\left(\Delta x_1\right)^2} T_{0,0,0,...,l} + \frac{\lambda}{\left(\Delta x_1\right)^2} T_{1,0,0,...0,l} + \\
&+ \frac{\lambda}{\left(\Delta x_2\right)^2} T_{env} - \frac{2\lambda}{\left(\Delta x_2\right)^2} T_{0,0,0,...,l} + \frac{\lambda}{\left(\Delta x_2\right)^2} T_{0,1,0,...0,l} + ... + \\
&+ \frac{\lambda}{\left(\Delta x_n\right)^2} T_{env} - \frac{2\lambda}{\left(\Delta z\right)^2} T_{0,0,0,...,l} + \frac{\lambda}{\left(\Delta z\right)^2} T_{0,0,0,...1,l} = \rho c \frac{T_{0,0,0,...0,l} - T_{0,0,0,...0,l-1}}{\Delta t}
\end{aligned} \tag{6}
$$

Thus, the system of equations is as follows:

$$
\begin{cases}
\alpha\, T_{0,0,...,0,l} - \dfrac{\lambda}{2\Delta x_1} T_{1,0,...,0,l} = \left(\alpha - \dfrac{\lambda}{2\Delta x_1}\right) T_{env} \\
\alpha\, T_{0,0,...,0,l} - \dfrac{\lambda}{2\Delta x_2} T_{0,1,...0,l} = \left(\alpha - \dfrac{\lambda}{2\Delta x_2}\right) T_{env} \\
\vdots \\
\alpha\, T_{0,0,...,0,l} - \dfrac{\lambda}{2\Delta x_n} T_{0,0,...,1,l} = \left(\alpha - \dfrac{\lambda}{2\Delta x_n}\right) T_{env} \\
\left[\displaystyle\sum_{p=1}^{n} \frac{2\lambda}{\left(\Delta x_p\right)^2} + \frac{\rho c}{\Delta t}\right] T_{0,0,...,0,l} - \dfrac{\lambda}{\left(\Delta x_1\right)^2} T_{1,0,...,0,l} - \dfrac{\lambda}{\left(\Delta x_2\right)^2} T_{0,1,...,0,l} - ... - \dfrac{\lambda}{\left(\Delta x_n\right)^2} T_{0,0,...,1,l} = \\
= \displaystyle\sum_{p=1}^{n} \frac{\lambda}{\left(\Delta x_p\right)^2} T_{env} + \frac{\rho c}{\Delta t} T_{0,0,...,0,l-1}
\end{cases} \tag{7}
$$

The main matrix system above has the form

$$A=\begin{bmatrix} \alpha & \dfrac{-\lambda}{2\Delta x_1} & 0 & \cdots & 0 \\ \alpha & 0 & \dfrac{-\lambda}{2\Delta x_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha & 0 & 0 & \cdots & \dfrac{-\lambda}{2\Delta x_n} \\ \sum_{p=1}^{n}\dfrac{2\lambda}{\left(\Delta x_p\right)^2}+\dfrac{\rho c}{\Delta t} & -\dfrac{\lambda}{\left(\Delta x_1\right)^2} & -\dfrac{\lambda}{\left(\Delta x_2\right)^2} & \cdots & -\dfrac{\lambda}{\left(\Delta x_n\right)^2} \end{bmatrix} \tag{8}$$

Using the determinant of the matrix A we obtain the limitations on the steps of the differential grid [5]

$$\begin{cases} \alpha-\dfrac{\lambda}{2\Delta x_1}>0 \\ \alpha-\dfrac{\lambda}{2\Delta x_2}>0 \\ \vdots \\ \alpha-\dfrac{\lambda}{2\Delta x_n}>0 \\ \det A>0 \end{cases} \tag{9}$$

So

$$\begin{cases} \dfrac{\lambda}{2\alpha}<\Delta x_1<\dfrac{\lambda}{\alpha} \\ \dfrac{\lambda}{2\alpha}<\Delta x_2<\dfrac{\lambda}{\alpha} \\ \vdots \\ \dfrac{\lambda}{2\alpha}<\Delta x_n<\dfrac{\lambda}{\alpha} \end{cases} \tag{10}$$

The determinant of the matrix A can be written in the following form:

$$\det A = \frac{\lambda^3}{4\prod_{p=1}^{n}\Delta x_p}\cdot\left[\sum_{p=1}^{n}\frac{\lambda}{\left(\Delta x_p\right)^2}+\frac{\rho c}{2\Delta t}-\alpha\sum_{p=1}^{n}\frac{1}{\Delta x_p}\right] \tag{11}$$

Now, we calculate the determinant corresponding to the variable $T_{0,0,\dots,0,l}$

$$\det A_{T_{0,0,\dots,0,l}} = \frac{\lambda^3}{4\prod_{p=1}^{n}\Delta x_p}\cdot\left[\left(\sum_{p=1}^{n}\frac{\lambda}{\left(\Delta x_p\right)^2}-\alpha\sum_{p=1}^{n}\frac{1}{\Delta x_p}\right)T_{env}+\frac{\rho c}{2\Delta t}T_{0,0,\dots,0,l-1}\right] \tag{12}$$

The temperature of node (0, 0, …,0) at the time l is as follows:

$$T_{0,0,\dots,0,l} = \frac{\left(\sum_{p=1}^{n}\frac{\lambda}{\left(\Delta x_p\right)^2}-\alpha\sum_{p=1}^{n}\frac{1}{\Delta x_p}\right)T_{env}+\frac{\rho c}{2\Delta t}T_{0,0,\dots,0,l-1}}{\sum_{p=1}^{n}\frac{\lambda}{\left(\Delta x_p\right)^2}+\frac{\rho c}{2\Delta t}-\alpha\sum_{p=1}^{n}\frac{1}{\Delta x_p}} \tag{13}$$

Assuming the following markings

$$a_p = \frac{\lambda}{\left(\Delta x_p\right)^2},\ p=1,\dots,n, \quad b=\alpha\sum_{p=1}^{n}\frac{1}{\Delta x_p}, \quad d=\frac{\rho c}{2\Delta t} \tag{14}$$

we get

$$T_{0,0,\dots,0,l} = \frac{\left(\sum_{p=1}^{n}a_p-b\right)T_{env}+dT_{0,0,\dots,0,l-1}}{\sum_{p=1}^{n}a_p-b+d} \tag{15}$$

or in another notation

$$T_{0,0,\dots,0,l} = \frac{\sum_{p=1}^{n}a_p-b}{\sum_{p=1}^{n}a_p-b+d}T_{env}+\frac{d}{\sum_{p=1}^{n}a_p-b+d}T_{0,0,\dots,0,l-1} \tag{16}$$

Similarly, we proceed for the remaining vertices.

Subsequently we repeat the procedure for each edge.
And so, on the edge (i_1,0,…,0), where $1 \le i_1 \le m_1 - 1,\ i_2 = 0,\ ...,\ i_n = 0$ we receive

$$
\begin{aligned}
&\lambda \frac{T_{i_1+1,0,...,0,l} - T_{i_1-1,0,...,0,l}}{2\Delta x_1} = \alpha\left(T_{i_1,0,,...,0,l} - T_{env}\right) \\
&\lambda \frac{T_{i_1,1,0,...,0,l} - T_{env}}{2\Delta x_2} = \alpha\left(T_{i_1,0,..., 0,l} - T_{env}\right) \\
&\vdots \\
&\lambda \frac{T_{i_1,0,..., 1,l} - T_{env}}{2\Delta x_n} = \alpha\left(T_{i_1,0,...,0,l} - T_{env}\right) \\
&\frac{\lambda}{\left(\Delta x_1\right)^2} T_{i_1-1,0,...,0,l} - \frac{2\lambda}{\left(\Delta x_1\right)^2} T_{i_1\ ,0,...,0,l} + \frac{\lambda}{\left(\Delta x_1\right)^2} T_{i_1+1,0,...,0,l} + \\
&+ \frac{\lambda}{\left(\Delta x_2\right)^2} T_{env} - \frac{2\lambda}{\left(\Delta x_2\right)^2} T_{i_1,0,...,0,l} + \frac{\lambda}{\left(\Delta x_2\right)^2} T_{i_1,1,...,0,l} + ... + \\
&+ \frac{\lambda}{\left(\Delta x_n\right)^2} T_{env} - \frac{2\lambda}{\left(\Delta x_n\right)^2} T_{i_1,0,...,0,l} + \frac{\lambda}{\left(\Delta x_n\right)^2} T_{i_1,0,..., 1,l} = \rho c \frac{T_{i_1,0,...,0,l} - T_{i_1,0,...,0,l-1}}{\Delta t}
\end{aligned}
\tag{17}
$$

Thus, the respective determinants take the following forms:

$$
\det A = \frac{\lambda^n}{4\prod_{p=1}^{n} \Delta x_p} \cdot \left(\sum_{p=1}^{n} \frac{\lambda}{\left(\Delta x_p\right)^2} + \frac{\rho c}{2\Delta t} - \alpha \sum_{p=1}^{n} \frac{1}{\Delta x_p} \right) \tag{18}
$$

$$
\begin{aligned}
&\det A_{T_{i,0,...,0,l}} = \\
&= \frac{\lambda^n}{4\prod_{p=1}^{n} \Delta x_p} \cdot \left[\left(\sum_{p=2}^{n} \frac{\lambda}{\left(\Delta x_p\right)^2} - \alpha \sum_{p=1}^{n} \frac{1}{\Delta x_p} \right) T_{env} + \frac{\lambda}{\left(\Delta x_1\right)^2} T_{i-1,0,...,0,\,l} + \frac{\rho c}{2\Delta t} T_{i,0,...,0,l-1} \right]
\end{aligned}
\tag{19}
$$

Ultimately, the temperature on the edge (i, 0, 0) $1 \le i_1 \le m_1 - 1,\ i_2 = 0,\ ...,\ i_n = 0$ at the moment l is given by the formula

$$T_{i,0,\ldots,0,l}=\frac{\left(\sum_{p=2}^{n}\frac{\lambda}{\left(\Delta x_p\right)^2}-\alpha\sum_{p=1}^{n}\frac{1}{\Delta x_p}\right)T_{env}+\frac{\lambda}{\left(\Delta x_1\right)^2}T_{i-1,0,\ldots,0,l}+\frac{\rho c}{2\Delta t}T_{i,0,\ldots,0,l-1}}{\sum_{p=1}^{n}\frac{\lambda}{\left(\Delta x_p\right)^2}-\alpha\sum_{p=1}^{n}\frac{1}{\Delta x_p}+\frac{\rho c}{2\Delta t}} \tag{20}$$

Using designations (14), we obtain

$$T_{i,0,\ldots,0,l}=\frac{\sum_{p=2}^{n}a_p-b}{\sum_{p=1}^{n}a_p-b+d}T_{env}+\frac{a_1}{\sum_{p=1}^{n}a_p-b+d}T_{i-1,0,\ldots,,l}+\frac{d}{\sum_{p=1}^{n}a_p-b+d}T_{i,0,\ldots,0,l-1} \tag{21}$$

Analogously we calculate the temperature for the other edge points.
On the plane (i_1, i_2, …, 0), $1\le i_1\le m_1-1,\ 1\le i_2\le m_2-1,\ i_3=0,\ldots,\ i_n=0$ we receive

$$\begin{aligned}&\frac{\lambda}{\left(\Delta x_1\right)^2}T_{i_1-1,i_2,0,\ldots,0,l}-\frac{2\lambda}{\left(\Delta x_1\right)^2}T_{i_1,i_2,0,\ldots,0,l}+\frac{\lambda}{\left(\Delta x_1\right)^2}T_{i_1+1,i_2,0,\ldots,0,l}+\\&+\frac{\lambda}{\left(\Delta x_2\right)^2}T_{i_1,i_2-1,0,\ldots,0,l}-\frac{2\lambda}{\left(\Delta x_2\right)^2}T_{i_1,i_2,0,\ldots,0,l}+\frac{\lambda}{\left(\Delta x_2\right)^2}T_{i_1,i_2+1,0,\ldots,0,l}+\ldots+\\&+\frac{\lambda}{\left(\Delta x_n\right)^2}T_{env}-\frac{2\lambda}{\left(\Delta x_n\right)^2}T_{i_1,i_2,0,\ldots,0,l}+\frac{\lambda}{\left(\Delta x_n\right)^2}T_{i_1,i_2,0,\ldots,1,l}=\frac{\rho c}{\Delta t}\left(T_{i_1,i_2,0,\ldots,0,l}-T_{i_1,i_2,0,\ldots,0,l-1}\right)\end{aligned} \tag{22}$$

In the case of other planes of the cube we proceed similarly.
The system of equations in the internal points of the area takes the following form:

$$\begin{aligned}&\frac{\lambda}{\left(\Delta x_1\right)^2}T_{i_1-1,i_2,\ldots,i_n,l}-\frac{2\lambda}{\left(\Delta x_1\right)^2}T_{i_1,i_2,\ldots,i_n,l}+\frac{\lambda}{\left(\Delta x_1\right)^2}T_{i_1+1,i_2,\ldots,i_n,l}+\\&+\frac{\lambda}{\left(\Delta x_2\right)^2}T_{i_1,i_2-1,\ldots,i_n,l}-\frac{2\lambda}{\left(\Delta x_2\right)^2}T_{i_1,i_2,\ldots,i_n,l}+\frac{\lambda}{\left(\Delta x_2\right)^2}T_{i_1,i_2+1,\ldots,i_n,l}+\ldots+\\&+\frac{\lambda}{\left(\Delta x_n\right)^2}T_{i_1,i_2,\ldots,i_n-1,l}-\frac{2\lambda}{\left(\Delta x_n\right)^2}T_{i_1,i_2,\ldots,i_n,l}+\frac{\lambda}{\left(\Delta x_n\right)^2}T_{i_1,i_2,\ldots,i_n+1,l}=\\&=\frac{\rho c}{\Delta t}\left(T_{i_1,i_2,\ldots,i_n,l}-T_{i_1,i_2,\ldots,i_n,l-1}\right)\end{aligned} \tag{23}$$

Conclusion

This article is the continuation of the discussion on the boundary-initial problem for the heat flow described by the Fourier equation [3, 4].

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