M U L T I P L E C R I T E R I A D E C I S I O N M A K I N G
Vol. 10 2015

Maciej Wolny[*]

THE CONCEPT OF RISK DOMINANCE IN MADM WITH NO INTER-CRITERIA INFORMATION

Abstract

This paper deals with the analysis of a multiple attribute decision making problem with no inter-criteria information. The problem is studied as a multiplayer, non-cooperative coordination game. Each equilibrium in the game corresponds to a decision variant. To choose a variant the general theory of equilibrium selection in games is used. The relation of risk dominance, introduced by Harsanyi and Selten (1988), is applied. In the method proposed a key element is to determine the reference point (status quo situation) – the least desirable situation with respect to each criterion separately. The method proposed supports decision making as regards selection and ordering.

Keywords: risk dominance, inter-criteria information, MADM.

1 Introduction

In this paper we analyze the issues in which both the set of decision variants and the set of criteria are finite. Therefore, we deal here with Multiple Attribute Decision Making (MADM) problems (Hwang and Yoon, 1981, p. 4). Such decision making problems are treated in this paper as multiple criteria decision making (MCDM) problems with a finite set of feasible solutions. Additionally, no information on inter-criteria preferences is known – the decision-maker does not want or cannot determine them.

The multiple attribute decision making problem will be treated as a game. Multiple attribute problems as games have been formulated as two-person zero-sum games (Kofler, 1967) and in the form of games with nature have been used

[*] Silesian University of Technology, Faculty of Organisation and Management, Poland, e-mail: maciej.wolny@polsl.pl.

for solving problems with no information on preferences (Hwang and Yoon, 1981). An analysis of the multiple attribute decision making problem as a multi-player non-cooperative game with non-zero sum was presented in the paper by Madani and Lund (2011) and earlier in the papers by Wolny (2007, 2008). The starting point for building a model in the form of a multiplayer game is to identify the correspondences between the elements of the multiple attribute problem and the game.

In general, the player is identified with the decision-maker who considers the problem from the point of view of one criterion (player-criterion). The single strategy of the player consists in the choice of a decision variant (strategy--variant). The payoff of the player is the estimate of the decision variant with respect to a given criterion. Therefore, the game is an abstraction analyzed "in the decision-maker's mind". The essence of the problem (that is, the selection of one variant) is the choice by all players of a strategy connected with the same decision variant. In order to determine the game fully it is necessary to establish the payoffs in the situation when the players-criteria choose the strategies corresponding to different decision variants, taking into account the consequences of this action (Wolny, 2013). Analysis of the game defined in this way may involve cooperation among the players, in which case the key element is to determine the tradeoff for the player's payoff. It may also deal with the situation when there is no cooperation (incomparability of the estimates of variants with respect to criteria). The approach in the first case involves the aggregation of estimates and requires additional information needed to determine the tradeoffs for the payoffs or to construct the characteristic function of the game. This last issue was raised in the paper by Wolny (2007). At the same time it should be taken into account that methods based on the scalarization of the problem and on various notions of aggregation have been developed for many years in many theoretical and practical areas (Brans and Vincke, 1985; Greco et al., 2005; Nowak, 2008; Trzaskalik, 2014a; Trzaskalik, 2014b), and many methods and ideas for solving such problems have been suggested in the literature. In the second approach, based on lack of cooperation, it is assumed that the estimates of decision variants with respect to different criteria are not comparable. This is especially important in the situations when the preferences of the decision-maker are not revealed (lack of inter-criteria information).

The investigated game may be approached in two ways:
– the game is played only once (between player-criteria) with perfect information of strategies and payoffs,
– the game is played in many stages until a stable solution (equilibrium) is reached, also with perfect information.

This paper is focused on the analysis of the non-cooperative game only, that is, on the analysis of problems with no information on the relationships between the criteria. Furthermore, only the game played once is considered[1]. It is assumed that the estimates for the individual criteria are expressed on an interval scale at least and that they reflect the utility of the variants considered only as regards each criterion separately – the payoff for the player-criterion in each situation reflects the utility of the variants for the player (the higher the payoff, the higher the utility). However, there are no assumptions or information on the possibility of determining the collective utility for all players-criteria in a particular situation in the game.

The notion of risk dominance was introduced by Harsanyi and Selten (1988) in order to choose equilibrium in the game. These authors propose to choose the risk dominance equilibrium in the situation when there is no payoff dominance equilibrium. The relation of risk dominance will be presented further in the paper on the example of a two-criteria problem.

The main objective of the paper is to present the possibilities of using risk dominance for multiple criteria decision making support with no information on inter-criteria preferences. The starting point is to present the multiple criteria problem in the form of a multiplayer, non-cooperative non-zero sum game in which each equilibrium from the set of pure strategies corresponds to a decision variant. This is a typical coordination game with the problem of equilibrium selection. The application of the risk dominance relation will be presented on a numerical example.

2 Multiple criteria problem as a game

Let a multiple criteria decision problem of the following form be given:

$$\max_{x \in X} F(x) = \max_{x \in X}\left[f_1(x), f_2(x),..., f_k(x)\right], \tag{1}$$

where X is a finite set of feasible decision variants, $X = \{x_1, x_2,\ldots, x_n\}$, x is an element of this set, f_j is the j-th criterion-function defined on the set X ($j = 1,2,\ldots,k$), $F(x)$ is a vector grouping together all the objective functions, $f_j(x)$ is the estimate of the decision variant with respect to the j-th criterion. Furthermore, all estimates of the decision variants with respect to all the criteria are given. The solution of the problem of vector optimization (1) is the set of effective solutions.

[1] The game played in many stages is discussed in the papers by Madani and Lund (2011); Wolny (2013).

Using the correspondence between the multiple criteria problem and the game presented in the introduction, we can transform the problem (1) into a k-person non-cooperative non-zero sum game in the standard form:

$$G = (\Phi, H) \tag{2}$$

where $\Phi = X^k$ is the set of all possible situations in the game while H is the function of the players' payoffs defined on Φ. Each situation in the game is uniquely defined by the vector of pure strategies chosen by each player. Elements of the set Φ are vectors $\phi = (x_{i1}^1, x_{i2}^2, ..., x_{ik}^k), x_{ij}^j \in X$, whose components are the strategies of the individual players chosen in the given situation – the ij-th strategy is chosen by the j-th player (i, j = 1,2,…,n). The situation in which all players select a strategy related to the same i-th decision variant is:

$$\phi_i = (x_i^1, x_i^2, ..., x_i^n), x_i^1 = x_i^2 = ... = x_i^n \tag{3}$$

The motivation of the players-criteria to achieve a situation ϕ_i, i.e. a unique determination of the decision variant, is the reference point. Its payoffs reflect the situation in which the players-criteria achieve a situation different from ϕ_i. Achieving coordination between the players in order to reach the situation ϕ_i is possible if the analyzed game is a coordination game (Wolny, 2008). The situation described as the reference point (status quo) should generate the lowest possible payoffs for the players; this will create motivation to achieve any situation ϕ_i, i.e., the choice of the same variant by all players-criteria. In view of this, the payoff function will have the following form:

$$H(\phi) = \begin{cases} (f_1(x_i), f_2(x_i), ..., f_k(x_i)) & \text{in situation } \varphi_i, \\ (\min\limits_{i=1,2,..n} f_1(x_i), \min\limits_{i=1,2,..n} f_2(x_i), ..., \min\limits_{i=1,2,..n} f_k(x_i)) & \text{in other situation.} \end{cases} \tag{4}$$

The model of the multiple criteria problem in the form of game (2) with the payoff function (4) is a coordination game with n equilibriums in the set of pure strategies. The determination of an equilibrium is equivalent to the choice of a decision variant.

In a situation when domination occurs with respect to the payoffs, rational players, having perfect information about the payoffs, will use the strategies implying risk dominance equilibrium, though the risk is tied to subjective probability.

3 Utility of risk dominance

The concept of risk dominance will be presented using the example of a two-player game with two non-payoff-dominant strategies, which will be then compared. Furthermore, we assume that in the multiple criteria decision problem there are at least three strategies-variants; for simplicity, only the set of effective solutions is considered. The comparison of a pair of strategies can be presented as a game in normal form using the following matrix:

$$\begin{bmatrix} (f_1(x_1), f_2(x_1) & (\min_i f_1(x_i), \min_i f_2(x_i)) \\ (\min_i f_1(x_i), \min_i f_2(x_i)) & (f_1(x_2), f_2(x_2) \end{bmatrix} \tag{5}$$

and in order to meet the condition of non-payoff-dominance the following conditions have to be met simultaneously:

$$\begin{matrix} f_1(x_1) > f_1(x_2) \\ f_2(x_1) < f_2(x_2) \end{matrix}. \tag{6}$$

In other words, the first strategy-variant (x_1) is better than the second one (x_2) for the first player-criterion (f_1), while for the second player-criterion (f_2) the converse is true: x_2 is better than x_1.

Player-criterion f_1 will select his better strategy if the expected value of his payoff when using this strategy is greater than that resulting from the application of strategy x_2, that is:

$$(1-q)\cdot f_1(x_1) + q\cdot \min_i f_1(x_i) > (1-q)\cdot \min_i f_1(x_i) + q\cdot f_1(x_2), \tag{7}$$

where q is the probability of player-criterion f_2 applying his better strategy--variant (x_2). As a consequence, the first player will select the first strategy if the following condition is met:

$$q < \frac{f_1(x_1) - \min_i f_1(x_i)}{f_1(x_1) + f_1(x_2) - 2\cdot \min_i f_1(x_i)}, \tag{8}$$

which means that he will expect the probability of the second player using his better strategy to be lower than:

$$q_0 = \frac{f_1(x_1) - \min_i f_1(x_i)}{f_1(x_1) + f_1(x_2) - 2\cdot \min_i f_1(x_i)}. \tag{9}$$

Similarly, player f_2 will select his better strategy-variant x_2 if the expected value of the payoff resulting from x_2 is greater than the payoff from using x_1, that is:

$$p\cdot \min_i f_2(x_i) + (1-p)\cdot f_2(x_2) > p\cdot f_2(x_1) + (1-p)\cdot \min_i f_2(x_i), \tag{10}$$

where p is the probability of player-criterion f_1 using his better strategy-variant (x_1). Consequently, the second player, similarly to the first player, will select his better strategy, that is x_2, if the following condition is met:

$$p < \frac{f_2(x_2) - \min_i f_2(x_i)}{f_2(x_1) + f_2(x_2) - 2\cdot \min_i f_2(x_i)}, \tag{10}$$

so he will expect the probability of the first player using his better strategy to be lower than:

$$p_0 = \frac{f_2(x_2) - \min_i f_2(x_i)}{f_2(x_1) + f_2(x_2) - 2\cdot \min_i f_2(x_i)}. \tag{11}$$

The players' expectations are subjective, but both of them have perfect information about the payoffs. Therefore, if they approach the game in a similar way, they will both select the variant which is better for the first player, if the first player has stronger indications to select his better strategy than the second one has to select his own better strategy, that is:

$$p_0 < q_0 , \tag{12}$$

therefore borderline, subjective probability causing the first player to select his own better strategy is greater than the borderline, subjective probability causing the second player to select his better strategy. In this case strategy-variant x_1 is risk dominant over variant x_2, and therefore x_1 will be preferred over x_2.

When $p_0 > q_0$, variant x_2 is preferable over variant x_1 and for $p_0 = q_0$ both variants are equivalent or impossible to compare.

It can be observed that when only two decision variants are considered they are always equivalent in terms of the suggested approach. This is a consequence of adopting a minimal estimate of the decision variant as the reference point: in the case of two non-dominant variants we compare the best one and the worst one with respect to each criterion, taking into account that the best variant with respect to one criterion is the worst one with respect to the other criterion. The goal of considering such a situation is to show that the comparison of two variants such that for one of them the estimate with respect to a given criterion is minimal, will generate a borderline value of the probability equal to one – with respect to this criterion the better variant will never be risk dominant[2]. To sum up, the reference point is of significant importance in forming the relationships of risk dominance.

In the case of more than two criteria when the variants are compared pairwise the criteria are gathered into two concordant coalitions (groups). Each coalition prefers a different decision variant[3]. Each coalition is represented by a player who has the strongest indications to select a variant which is better for the coalition. The choice of the variant is made among the players representing consistent coalitions playing a game.

The suggested approach will be illustrated using a simple numerical example.

4 Numerical example

In this problem nine decision variants are being considered with respect to three criteria. All criteria are maximized. The estimates of the decision variants are presented in Table 1.

---

[2] In this situation the equilibrium in the game is related to the variant with the minimal estimate, with respect to the player-criterion, is a weak equilibrium, because whatever other strategy--variant he chooses he obtains the same result regardless of the action of other players.

[3] In the case of payoff-dominance one coalition is created.

Table 1: The assessments of decision variants

Decision variants	f_1	f_2	f_3
x_1	411	55252	19
x_2	469	58251	11
x_3	297	82739	29
x_4	1581	89022	20
x_5	1092	99118	22
x_6	966	78119	25
x_7	650	84084	38
x_8	414	68300	10
x_9	737	85071	39

The problem will be treated as a game. It can be observed that variant--strategy x_9 payoff-dominates variants x_8, x_7, x_3, x_2 and x_1. Payoff dominance relationships existing between all the variants are presented in Table 2.

Table 2: Payoff dominance

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1		0	0	0	0	0	0	0	0
x_2	0		0	0	0	0	0	0	0
x_3	0	0		0	0	0	0	0	0
x_4	1	1	0		0	0	0	1	0
x_5	1	1	0	0		0	0	1	0
x_6	1	1	0	0	0		0	1	0
x_7	0	0	0	0	0	0		1	0
x_8	0	0	0	0	0	0	0		0
x_9	1	1	1	0	0	0	1	1	

'0' means that a relation does not exist, '1' that it exists between the variant in the row and the variant in the column of the table: e.g., x_4 payoff-dominates x_1.

The use of payoff dominance does not allow a single variant to be chosen in this case. According to the suggested approach, in further analysis risk dominance will be used.

For the pair of variants (x_4, x_9), x_4 is a better variant for criteria f_1 and f_2, while x_9 is better for f_3. The borderline probability values, expressed by formulas (9) and (11) and condition (12), make it possible to determine the relationship of risk dominance for this pair of variants. Those values for the consecutive criteria are: for f_1 – 0.745, for f_2 – 0.531, for f_3 – 0.744. Therefore, player-criterion f_1 has the strongest indications to select his better strategy (variant). It implies that x_4 risk dominates x_9.

The remaining relationships of risk dominance existing between the variants are presented in Table 3.

Table 3: Risk dominance

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1		0	0	0	0	0	0	0	0
x_2	1		1	0	0	0	0	1	0
x_3	0	0		0	0	0	0	0	0
x_4	1	1	1		1	1	1	1	1
x_5	1	1	1	0		**1**	**0**	1	0
x_6	1	1	1	0	**0**		**1**	1	0
x_7	1	1	1	0	**1**	**0**		1	0
x_8	0	0	0	0	0	0	0		0
x_9	1	1	1	0	1	1	1	1	

On the basis of the information in table 3[4] it may be stated that the best decision variant in the sense of the suggested approach is x_4.

Analyzing the relationships for variants x_5, x_6, x_7 we can observe that this relation is not transitive, because it is impossible to determine the preferences between those variants. In general, risk dominance may not sort the set of decision variants[5].

In the final sorting (Table 4) the variants for which the relation is not transitive are on the same preference level.

Table 4: Ranking of decision variants

Rank	Decision variants		
1	x_4		
2	x_9		
3	x_5	x_6	x_7
4	x_2		
5	x_1	x_3	x_8

5 Summary

In this paper we have proposed a game-theoretic approach to the discrete multiple criteria problems with no information on inter-criteria preferences. The multiple criteria problem is treated as a multiplayer (k-person), non-cooperative non-zero sum game.

4 When there is a relationship of payoff dominance, there is also a relationship of risk dominance. In general, this regularity does not occur in game theory (Harsanyi and Selten, 1988).

5 For two-criteria problems it was shown that risk dominance may sort the set of decision variants (Wolny, 2014).

The determination of the status quo situation (Wolny, 2013) which corresponds to the least desirable situation is the key element of the suggested approach[6]. The solution to this problem depends to a large extent on the selected reference point. Therefore, it is recommended to acquire information on the values related to status quo from the decision-maker. If the status quo situation cannot be explicitly determined it is suggested that the lowest possible values of the maximized criterion-functions be adopted. As a result, the variants with the lowest estimate with respect to any criterion are discriminated against. The equilibrium in the game corresponding to such a variant is weak. The player-criterion achieves the least possible payoff, similarly to any other situation. For this reason he has no 'motivation' to achieve the equilibrium, other than the indications from other players-criteria.

The application of risk dominance to solving the multiple criteria problem is based on the comparison of the probabilities expressing the strength of the indications for the selection of a given decision variant. The suggested approach originates in the construction of the model of the multiple criteria problem in the form of non-cooperative non-zero sum game. The choice of the equilibrium is based on the general theory of equilibrium selection in games.

An important feature of the suggested method is that the estimates of the decision variants do not have to be normalized. The presentation of the multiple criteria problem as a game can assist in the interaction with the decision-maker and make the structuring of the problem possible, particularly when it is not possible to acquire information on inter-criteria preferences.

References

Brans J.P., Vincke P. (1985), *A Preference Ranking Organization Method (the Promethee Method for Multiple Criteria Decision-making)*, Management Science, 31.

Greco S., Ehrgott M., Figueira J. (2005), *Multiple Criteria Decision Analysis. State of the Art Surveys*, Springer Science + Business Media Inc., Boston.

Harsanyi J.C., Selten R. (1988), *A General Theory of Equilibrium Selections in Games*, MIT Press, Cambridge.

Hwang C.L., Yoon K. (1981), *Multiple Attribute Decision Making: Methods and Applications. State of the Art Surveys*, Springer-Verlag, Berlin-Heidelberg-New York.

Kofler E. (1967), *O zagadnieniu optymalizacji wielocelowej*, Przegląd Statystyczny, 1.

Madani K., Lund J.R. (2011), *A Monte-Carlo Game Theoretic Approach for Multi-Criteria Decision Making under Uncertainty*, Advances in Water Resources, 34.

Nowak M. (2008), *Interaktywne wielokryterialne wspomaganie decyzji w warunkach ryzyka. Metody i zastosowania*, Wydawnictwo Akademii Ekonomicznej, Katowice.

Roy B. (1985), *Methodologie multicritère d'aide à la décision* (*Wielokryterialne wspomaganie decyzji*, WNT, Warszawa 1990), Editions Economica, Paris.

[6] Similarly as a negative-ideal solution in the TOPSIS method (Hwang, Yoon, 1981).

Trzaskalik T. (2014a), *Wielokryterialne wspomaganie decyzji. Przegląd metod i zastosowań*, Organizacja i Zarządzanie, Zeszyty Naukowe Politechniki Śląskiej, nr 74.

Trzaskalik T. (ed.) (2014b), *Wielokryterialne wspomaganie decyzji. Metody i zastosowania*, PWE, Warszawa.

Wolny M. (2007), *Wspomaganie decyzji kierowniczych w przedsiębiorstwie przemysłowym. Wieloatrybutowe wspomaganie organizacji przestrzennej komórek produkcyjnych z zastosowaniem teorii gier*, Wydawnictwo Politechniki Śląskiej, Gliwice.

Wolny M. (2008), *Decision Making Problem with Two Incomparable Criteria – Solution Based on Game Theory*, Multiple Criteria Decision Making (MCDM'07).

Wolny M. (2013), *Aspekt sytuacji status quo we wspomaganiu wielokryterialnego wyboru bazującego na teorii gier*, Studia Ekonomiczne. Zeszyty Naukowe Uniwersytetu Ekonomicznego w Katowicach, nr 132.

Wolny M. (2014), *Wykorzystanie dominacji ze względu na ryzyko do porządkowania wariantów w zagadnieniach dwukryterialnych przy nieporównywalności kryteriów*, Organizacja i Zarządzanie. Zeszyty Naukowe Politechniki Śląskiej, nr 68.