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PREFERENCE-DRIVEN MULTIOBJECTIVE OPTIMIZATION USING ROBUST ORDINAL REGRESSION FOR CONE CONTRACTION

Abstract

We present a new interactive procedure for multiobjective optimization problems (MOO), which involves robust ordinal regression in contraction of the preference cone in the objective space. The most preferred solution is achieved by means of a systematic dialogue with the decision maker (DM) during which (s)he specifies pairwise comparisons of some non-dominated solutions from a current sample. The origin of the cone is located at a reference point chosen by the DM. It is formed by all directions of isoquants of the achievement scalarizing functions compatible with the pairwise comparisons of non-dominated solutions provided by the DM. The compatibility is assured by robust ordinal regression, i.e. the DM's statements concerning strict or weak preference relations for pairs of compared solutions are represented by all compatible sets of weights of the achievement scalarizing function. In successive iterations, when new pairwise comparisons of solutions are provided, the cone is contracted and gradually focused on a sub-region of the Pareto optimal set of greatest interest. The DM is allowed to change the reference point and the set of pairwise comparisons at any stage of the

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method. Such preference information does not need much cognitive effort on the part of the DM. The phases of preference elicitation and cone contraction alternate until the DM finds at least one satisfactory solution, or there is no such solution for the current problem setting.

Keywords: Multiobjective optimization, robust ordinal regression, interactive procedure, preference elicitation, cone contraction

1 Introduction

In multiobjective optimization (MOO), several objectives compete for the best compromise. Identification of a small subset of non-dominated solutions (sometimes reduced to a singleton) that, according to the preferences of the decision maker (DM) yield the best compromise among the conflicting objectives, is the main task of interactive multiple objective optimization (IMO). IMO procedures are composed of two alternating stages: optimization and decision making (see, e.g. Vanderpooten and Vincke, 1997). The stage of decision making, or, more precisely, preference elicitation, consists in the exchange of information between the method and the DM. The method provides the DM with a sample of candidate solutions and the DM returns some critiques of these solutions, which permits to generate in the next optimization stage a new sample that better fits the DM's preferences. One of the major advantages of the IMO is that it aids the DM in improving her/his knowledge about the problem statement, its potential solutions, possible tradeoffs and existing limitations.

A review of interactive procedures shows that reference point methods (RPMs) are gaining importance. In the recent years, one has been able to observe a growing interest in the development of theoretical foundations of the RPMs (see, e.g. Branke et al., 2008; Ogryczak, 2001; Wierzbicki, 1999) as well as a large variety of real-world applications (see, e.g. Granat and Guerriero, 2003)). A reference point is a vector composed of desirable or acceptable values of the objective functions, so-called aspiration levels, represented by a point in the objective space. Given a set of non-dominated solutions, which, in the objective space, are called non-dominated points or the Pareto frontier, the DM is interested in getting a non-dominated point located either as close as possible to the reference point (when the reference point appears infeasible) or as far as possible from the reference point (when the reference point appears feasible). Thus, the reference point is projected onto the set of non-dominated points with the aim of producing solutions which are most preferred to the DM. The result of this projection depends on the weights of the achievement scalarizing function that measures the distance in the objective space between a reference point and non-dominated points. The direction in which the distance is measured depends on the weights assigned to the objective functions. As some projection directions may lead to more desirable non-dominated points to the DM than others, the most straightforward way for browsing interesting regions of the Pareto frontier consists in incorporation of preference information into weights of the achievement function. As far as interaction with the DM is concerned, the recently proposed RPMs present to the DM a sample of non-dominated points at each decision making stage, and expect her/him to state some crucial evaluation of the proposed points, e.g., multiple objective comparisons of some pairs of non-dominated points. Assessment of a preference model reflecting such holistic preferences necessitates looking for the rational basis through which the desired pairwise comparisons were made.

A method that would combine the aforementioned features, i.e. interactive elicitation of preferences consisting of co-ordinates of a reference point, pairwise comparisons of some non-dominated points from a current sample, and incorporation of the DM's preferences into the weights in the achievement scalarizing function, would have many desirable properties of MOO techniques. This motivation has driven our work on a new interactive method designed for the exploitation of the Pareto frontier (PF) in view of searching for the best compromise non-dominated point (Pareto-optimal solution in the decision space). The first version of our method has appeared recently (see Kadziński and Słowiński, 2012). In this method, the identification of the most preferred solution is achieved by means of a systematic dialogue with the DM during which (s)he specifies pairwise comparisons of some non-dominated points from a current sample. Within the method, statements concerning strict or weak preference relations for pairs of points are represented by a compatible form of the achievement scalarizing function (ASF). The preferences are translated into inequalities between distances of compared points from the current reference point. Subsequently, a corresponding set of constraints on the weights of objectives in the ASF is formulated, which ensures that points compared by the DM are compared by the function in the same way. The directions of the isoquants of all compatible ASFs create a cone in the objective space. The origin of the cone is located at the current reference point specified by the DM. Consequently, the preference model used in the method is a set of ASFs compatible with the currently available preference information, rather than only a single compatible ASF. Since we are considering all ASFs compatible with the pairwise comparisons provided, and not just a single ASF as in the traditional methods, our approach can be seen as an inherent part of the robust ordinal regression paradigm (see, e.g. Greco et al., 2008, 2011). In successive iterations, when still new pairwise comparisons of non-dominated points are provided, the cone is contracted and gradually focused on a subregion of the Pareto frontier of greatest interest. The DM is allowed to change the reference point at any decision making stage of the method. The phases of preference elicitation and cone contraction alternate until the DM is satisfied by the compromise yielded by the values of objective functions of at least one non-dominated solution, or until the DM states that there is no such compromise solution for the current problem setting, or until some other stopping criterion is satisfied. The idea of "cone contraction" comes from IMO procedures originally proposed by Steuer (1978), Steuer and Choo (1983), Jaszkiewicz and Słowiński (1992), and Kaliszewski (1994), however, in our method, the preference information provided by the DM, and the way of translating it into constraints contracting the cone, are very different from the previous methods – the preference information has the form of holistic pairwise comparisons of some non-dominated points, and the cone contraction proceeds via robust ordinal regression.

This paper adapts the original proposal of Kadziński and Słowiński (2012) to the conference presentation, omitting many technical details and focusing on the methodological aspect of the procedure. The paper is organized as follows. In Section 2, we introduce notation and concepts used in the paper, including a formal statement of the problem, definition of the non-dominated solutions and points, and characteristics of the ASF. In Section 3, we describe the IMO procedure based on cone contraction via robust ordinal regression. In Section 4, we illustrate this procedure using an exemplary three-objective optimization problem. The final section contains conclusions.

2 Concepts: Definitions and Notation

The *general multiple-objective programming problem* is formulated as:

$$Minimize\ \{f_1(x), f_2(x), \ldots, f_k(x)\},\ \text{subject to}\ x \in S,$$

where $x = [x_1, \ldots, x_n]$ is a vector of decision variables from the nonempty feasible region $S \subset \mathbb{R}^n$, and $f_1, \ldots, f_k$, with $k \geq 2$ are conflicting objective functions $f_i : \mathbb{R}^n \to \mathbb{R}$, that we want to minimize simultaneously. We assume, without loss of generality, that all objective functions are characterized by decreasing directions of preference, i.e., less is preferred to more. Let us denote the set of indices of the considered objectives by $I = \{1, 2, \ldots, k\}$. This problem can also be formulated as:

$$Minimize: \; z, \text{ subject to } z \in Z,$$

where $z = [z_1 = f_1(x), \ldots, z_k = f_k(x)]$ is a vector of objective function values, and Z is an image of the set S in the objective space $\mathbb{R}^k$, $Z = f(S)$, $f : \mathbb{R}^n \to \mathbb{R}^k$.

To avoid switching between x and z when speaking about solutions of a MOO problem, we will use x to design a solution (non-dominated, dominated, feasible, etc.) understood either as a vector in the decision space, or as its image vector (point) in the objective space. The context in which x is used makes it clear whether we mean a solution in the decision space or a point in the objective space; e.g., when speaking about a distance between a non-dominated solution x and the reference point $\bar{z}$, we mean a distance in the objective space, or when speaking about preferential comparison of non-dominated solutions x^1 and x^2, we mean comparison of their images in the objective space, as ASFs and the preference cone are considered in this space.

Non-dominated solutions

In multiple objective optimization no unique optimal solution usually exists, but a set of options with different trade-offs, i.e. such that none of their components can be improved without deterioration of some other components. Formally, a decision vector $x \in S$ is called *non-dominated* (Pareto-optimal, efficient) if and only if there is no other $y \in S$ such that y is at least as good as x with respect to all objectives, and strictly better for at least one objective, i.e. $f_i(y) \leq f_i(x)$, for all $i \in I$, and there exists $j \in I$, for which $f_j(y) < f_j(x)$. The set of all non-dominated solutions is called the *non-dominated set* and denoted by $P(S)$. In the objective space, $P(S)$ is also called *Pareto frontier*.

Reference point

To measure the quality of non-dominated points, the DM may define some desired objective function values, which constitute a *reference point* denoted by $\bar{z} = \{\bar{z}_1, \ldots, \bar{z}_k\}$. Most often, reference points correspond to objective values that the DM would like to achieve (*aspiration levels*), or that should at least be achieved, according to the DM (*reservation levels*). The reference point may be feasible or not.

Achievement scalarizing function

Achievement scalarizing function is used to project a reference point onto the set of non-dominated solutions. ASF is often defined as (see Wierzbicki, 1982):

$$s(x, \lambda, f) = max_i\{\lambda_i(f_i(x) - \bar{z}_i)\} + \rho \sum_{i=1}^{k}(f_i(x) - \bar{z}_i), \tag{1}$$

where $\lambda = [\lambda_1, \ldots, \lambda_k]$ is a weighting vector, $\lambda_i > 0,\ i = 1, \ldots, k$, and $\rho > 0$ is an augmentation multiplier (sufficiently small positive number). By giving a slight slope to the contours of the scalarizing function, one avoids weakly non-dominated solutions. Without this slope, the contours (isoquants of the scalarizing function) have the shape of orthogonal cones (see Figure 1). Note that if the scales of objectives differ substantially, to avoid problems with significantly different weights λ_i, $i = 1, \ldots, k$, one should use ASF defined as (see Wierzbicki, 1986):

$$s(x, \lambda, f) = max_i\{\lambda_i(f_i(x) - \bar{z}_i)\} + \rho \sum_{i=1}^{k}\lambda_i(f_i(x) - \bar{z}_i). \tag{2}$$

Note that in RPMs, an ASF is switching from minimization to maximization of the distance between non-dominated solutions and the reference point when the reference point changes from an infeasible one to a feasible one. Thus, e.g., for a infeasible reference point, the smaller the value of the ASF for a given weighting vector, the smaller the distance between a feasible solution and the reference point, i.e. the more this solution is preferred to the DM.

3 Interactive Robust Cone Contraction Method

In this section, we present the IMO procedure based on cone contraction via robust ordinal regression. It is designed for preference-driven exploration of the non-dominated set $P(S)$ of the MOO problem. Thus, we assume that this set, its proper representation or approximation, is generated prior to the right procedure, using some non-interactive parametric or evolutionary (EMO) technique.

In the course of the interactive procedure, the DM specifies pairwise comparisons of some non-dominated solutions from a current sample. More precisely, in the *q-th* iteration the preference information concerns the direction of a strict $\succ$ or weak $\succsim$ preference relation between two solutions x^1

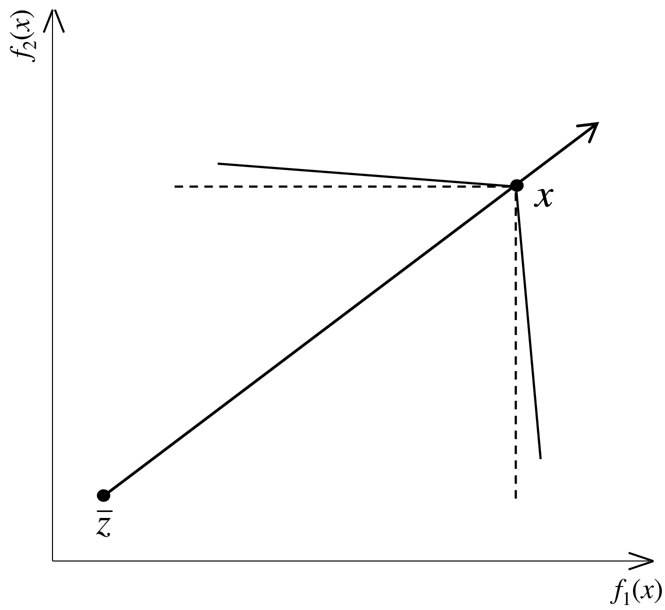

Figure 1. Direction of the isoquant of an achievement scalarizing function

and x^2 chosen from the subset $P(S)_q \subseteq P(S)$ delimited in the previous step $(P(S)_q \subset P(S)_{q-1})$. In this way, the DM specifies some examples of holistic judgments, which requires a relatively small cognitive effort from her/him.

Within the method, the preference information provided is represented by a compatible form of the ASF. The incorporation of the DM's preferences into weights in the achievement scalarizing function is achieved by the formulation of the suitable inequalities. The directions of the isoquants of all compatible ASFs create a convex polyhedral cone in the objective space, with the origin at the current reference point. When new pairwise comparisons are performed in the subsequent iterations, the cone is contracted, and, consequently, the region of the non-dominated solutions which are supposed to better fit the DM's preferences is constrained. The desired effect is to reduce the set of compatible ASFs with each new piece of preference information, and in this way to focus on a subregion of the non-dominated set that better corresponds to the DM's preferences. The phases of preference elicitation and contraction of the cone alternate until the DM has found the most preferred solution, or until (s)he concludes that there is no satisfactory solution for the current problem setting.

The steps of the proposed interactive robust cone contraction method are summarized below:

S1. Compute a representation of the non-dominated set $P(S)_0$ of the considered multiple objective optimization problem.

S2. Ask the DM to specify the starting reference point $\bar{z}_0$.

S3. Update the index of the current step $q := q + 1$ (at the beginning $q = 0$).

S4. Present the set $P(S)_q$ to the DM.

S5. **If** the DM feels satisfied with at least one solution found in the set $P(S)_q$, **then** the procedure stops. **If** (s)he concludes that no compromise point exists, or some other stopping criteria are satisfied, **then** the procedure stops without finding the satisfactory solution. **Otherwise**, continue.

S6. **If** the DM wants to backtrack to one of the previous iterations and continue from this point, **then** go to **S4** of the chosen iteration.

S7. **If** the DM wants to change the reference point, **then** ask her/him to provide a new one, $\bar{z}_q$. **Otherwise** $\bar{z}_q = \bar{z}_{q-1}$

S8. Ask the DM to provide preference information in the form of pairwise comparisons of two solutions chosen from $P(S)_q$ (let us assume that in each iteration x^1 will represent a solution preferred to x^2, i.e., $x^1 \succ x^2$ or $x^1 \succsim x^2$).

S9. Formulate constraints on the weights of the compatible ASFs, which compare the solutions x^1 and x^2 in the same way as the DM.

S10. Form a set $P(S)_{q+1}$ by leaving only those solutions from $P(S)_q$ that are inside the area delimited by the cone formed by all the directions of isoquants of the compatible achievement scalarizing functions.

S11. **If** $P(S)_{q+1}$ is empty, or $x^1 \notin P(S)_{q+1}$, or $x^2 \in P(S)_{q+1}$ (in case $x^1 \succ x^2$), **then** inform the DM about inconsistency and go back to **S4**.

S12. Go to **S3**.

Three points of the procedure need to be commented in more detail. The first point concerns some restrictions on the location of the reference point (for discussion of S7, see Subsection 3.1). The second point concerns the way we obtain the weights of the compatible ASFs (for a discussion of

S9, see Subsection 3.2). The third one deals with checking which solutions should still be considered as potential "best choices" in the next iteration (for discussion of S10, see Subsection 3.3).

3.1 Location of a Reference Point

In each iteration, the DM may specify the reference point which constitutes the origin of the cone indicating the non-dominated solutions that correspond to the DM's preferences. However, its location is subject to some restrictions. In particular, at the initial stages of the interaction, when the DM's knowledge about the shape of the Pareto frontier is rather poor, the reference point should be at least as good as the utopia point. This guarantees that all solutions are included within the considered cone, and thus, each of them can become the best compromise. This is reasonable because all non-dominated solutions are incomparable when no preference information is provided.

In the subsequent stages, when the DM's knowledge about the existing solutions improves, the DM may move the reference point. In this way, (s)he could indicate a more promising subregion and eliminate from further consideration the non-dominated solutions situated outside the new cone. Hence, the desired aspiration or reference objective levels which form the reference point should be selected so that a subregion of non-dominated set covered by the new cone is non-empty. In fact, when considering a finite set of non-dominated solutions representing the Pareto frontier, a rational DM needs to indicate the reference point which is not worse than some non-dominated solutions. Thus, the specified levels should correspond to the best objective values in the promising subregion.

3.2 Inferring Achievement Scalarizing Functions Compatible with Preference Information

Consider the pairwise comparison of solutions $x^1 \succsim x^2$. In this section, we will show how to represent this comparison by constraints on the weights of the compatible ASFs. These constraints contract the cone, which represents the currently available preference information. In this way, we are able to indicate a subset of non-dominated solutions which satisfy the preferences expressed by the DM.

Pairwise comparison $x^1 \succsim x^2$ implicates that the distance from the reference point $\bar{z}$ to the solution x^1 is not greater than the distance from $\bar{z}$ to the solution x^2, i.e., $s(x^1, \lambda, f) \leq s(x^2, \lambda, f)$. Considering ASF in form (2), this inequality leads to the following alternative of k systems of linear inequalities:

$$[\lambda_1(f_1(x^1) - \bar{z}_1) + \rho\sum_{i=1}^{k}\lambda_i(f_i(x^1) - f_i(x^2)) \leq \lambda_1(f_1(x^2) - \bar{z}_1) \vee$$

$$\vee\ \lambda_2(f_2(x^1) - \bar{z}_2) + \rho\sum_{i=1}^{k}\lambda_i(f_i(x^1) - f_i(x^2)) \leq \lambda_1(f_1(x^2) - \bar{z}_1)\ \vee\ \ldots$$

$$\ldots\ \vee\ \lambda_k(f_k(x^1) - \bar{z}_k) + \rho\sum_{i=1}^{k}\lambda_i(f_i(x^1) - f_i(x^2)) \leq \lambda_1(f_1(x^2) - \bar{z}_1)]\ \wedge$$

$$\wedge\ [\lambda_j(f_j(x^1) - \bar{z}_j) + \rho\sum_{i=1}^{k}\lambda_i(f_i(x^1) - f_i(x^2)) \leq \lambda_2(f_2(x^2) - \bar{z}_2), \quad \text{for some } j = 1, \ldots, k]\ \wedge\ \ldots$$

$$\ldots\ \wedge [\lambda_j(f_j(x^1) - \bar{z}_j) + \rho\sum_{i=1}^{k}\lambda_i(f_i(x^1) - f_i(x^2)) \leq \lambda_k(f_k(x^2) - \bar{z}_k), \quad \text{for some } j = 1, \ldots, k].$$

Knowing $f_i(x^1), f_i(x^2),\ \bar{z}_i,\ i = 1, \ldots, k$, and ρ, we obtain the set of constraints on the weights that contract the cone. Note that weights which satisfy the above set of constraints need to be nonnegative, i.e. $\lambda_i \geq 0,\ i = 1, \ldots, k$. For the strict preference ($x^1 \succ x^2$), we replace weak inequalities with strict inequalities. Since all weights $\lambda_i,\ i = 1, \ldots, k$, are used in each inequality, it is impossible, in general, to reduce the system above by indicating that some inequalities hold for all possible vectors of weights or none of them. Such an analysis is possible for the ASF having the form (1). In this case, the considered alternative of k systems of linear inequalities has the following form:

$$\lambda_j(f_j(x^1) - \bar{z}_j) + \rho\sum_{i=1}^{k}(f_i(x^1) - f_i(x^2)) \leq \lambda_p(f_p(x^2) - \bar{z}_p), \quad \text{for some } j = 1, \ldots, k,$$

for all $p = 1, \ldots, k$ and $\lambda_i \geq 0,\ i = 1, \ldots, k$. Thus, unlike the case of an ASF in the form (2), here each inequality involves only one pair of weights since the augmentation factor ($\rho\sum_{i=1}^{k}(f_i(x^1) - f_i(x^2))$) is constant.

3.3 Passing Solutions to the Next Iteration

There are two equivalent ways of checking whether a solution x from $P(S)_q$ should be left in $P(S)_{q+1}$ and still considered to be the potential best compromise. One of them consists in checking whether the weights of the ASF corresponding to the direction determined by x satisfy the set of conditions defined in Subsection 3.2. These conditions delimit the cone so that ASFs with the isoquants going in the directions of the solutions which are inside the cone compare reference solutions in the same way as the DM does. The other way consists in a direct verification that an ASF with the set of weights λ^x compares solutions in the same way as the DM does. Thus, it is sufficient to check whether $s(x^1, \lambda^x, f) < s(x^2, \lambda^x, f)$, if the DM stated that $x^1 \succ x^2$, or $s(x^1, \lambda^x, f) \leq s(x^2, \lambda^x, f)$, if (s)he claimed $x^1 \succsim x^2$. If it is the case, $x \in P(S)_q$ is left in $P(S)_{q+1}$. Otherwise, x is excluded from the set of solutions which are still considered to be the potential best compromise.

4 Illustrative Example

In this section, we illustrate the way our method supports the DM in solving a MOO problem, and we give examples of possible interactions. We consider a MOO problem that involves three objectives to be minimized. The non-dominated solutions satisfy the following condition $f_1(x) + f_2(x) + f_3(x) = 0.5$ (like in Three-Objective Test Problem DTLZ1 (Zitzler et al., 2000)). We consider the subset $P(S)_0$ composed of 66 non-dominated solutions (see Table 1 and Figure 2). The initial reference point is situated at the point $[0.0, 0.0, 0.0]$. Since the scales of the objectives are the same, we will use the ASF in the form (1).

Obviously, solutions in $P(S)_0$ are incomparable, unless preference information is expressed by the DM. In this perspective, (s)he provides a first comparison: $x^{33} = [0.15, 0.10, 0.25] \succ x^{55} = [0.30, 0.15, 0.05]$. Note that x^{33} is evaluated better than x^{55} on objectives f_1 and f_2, whereas it is worse on the third objective f_3. Therefore, the cone formed by the directions of isoquants of all ASFs compatible with the statement $x^{33} \succ x^{55}$, is a sum of the following two cones. The first is formed by the directions of ASFs which ensure that solutions included in this cone would be evaluated better on objective f_1 to recompense for weakness on objective f_3, whereas the other cone is formed by the directions of ASFs which guarantee that the advantage of evaluation on f_2 would allow to recompense for a relatively worse evaluation on f_3. To be precise, the constraints on the weights of

Table 1

The representative set of non-dominated solutions $P(S)_0$

	$f_1(x)$	$f_2(x)$	$f_3(x)$		$f_1(x)$	$f_2(x)$	$f_3(x)$
x^1	0.00	0.00	0.50	x^{34}	0.15	0.15	0.20
x^2	0.00	0.05	0.45	x^{35}	0.15	0.20	0.15
x^3	0.00	0.10	0.40	x^{36}	0.15	0.25	0.10
x^4	0.00	0.15	0.35	x^{37}	0.15	0.30	0.05
x^5	0.00	0.20	0.30	x^{38}	0.15	0.35	0.00
x^6	0.00	0.25	0.25	x^{39}	0.20	0.00	0.30
x^7	0.00	0.30	0.20	x^{40}	0.20	0.05	0.25
x^8	0.00	0.35	0.15	x^{41}	0.20	0.10	0.20
x^9	0.00	0.40	0.10	x^{42}	0.20	0.15	0.15
x^{10}	0.00	0.45	0.05	x^{43}	0.20	0.20	0.10
x^{11}	0.00	0.50	0.00	x^{44}	0.20	0.25	0.05
x^{12}	0.05	0.00	0.45	x^{45}	0.20	0.30	0.00
x^{13}	0.05	0.05	0.40	x^{46}	0.25	0.00	0.25
x^{14}	0.05	0.10	0.35	x^{47}	0.25	0.05	0.20
x^{15}	0.05	0.15	0.30	x^{48}	0.25	0.10	0.15
x^{16}	0.05	0.20	0.25	x^{49}	0.25	0.15	0.10
x^{17}	0.05	0.25	0.20	x^{50}	0.25	0.20	0.05
x^{18}	0.05	0.30	0.15	x^{51}	0.25	0.25	0.00
x^{19}	0.05	0.35	0.10	x^{52}	0.30	0.00	0.20
x^{20}	0.05	0.40	0.05	x^{53}	0.30	0.05	0.15
x^{21}	0.05	0.45	0.00	x^{54}	0.30	0.10	0.10
x^{22}	0.10	0.00	0.40	x^{55}	0.30	0.15	0.05
x^{23}	0.10	0.05	0.35	x^{56}	0.30	0.20	0.00
x^{24}	0.10	0.10	0.30	x^{57}	0.35	0.00	0.15
x^{25}	0.10	0.15	0.25	x^{58}	0.35	0.05	0.10
x^{26}	0.10	0.20	0.20	x^{59}	0.35	0.10	0.05
x^{27}	0.10	0.25	0.15	x^{60}	0.35	0.15	0.00
x^{28}	0.10	0.30	0.10	x^{61}	0.40	0.00	0.10
x^{29}	0.10	0.35	0.05	x^{62}	0.40	0.05	0.05
x^{30}	0.10	0.40	0.00	x^{63}	0.40	0.10	0.00
x^{31}	0.15	0.00	0.35	x^{64}	0.45	0.00	0.05
x^{32}	0.15	0.05	0.30	x^{65}	0.45	0.05	0.00
x^{33}	0.15	0.10	0.25	x^{66}	0.50	0.00	0.00

compatible ASFs in the first iteration are the following:

$$\{[\lambda_1 > 5/6 \cdot \lambda_3] \;\vee\; [\lambda_2 > 5/3 \cdot \lambda_3]\} \;\wedge\; \{\lambda_i \geq 0,\; i = 1,2,3\}.$$

The transition from the formulated inequalities to the cone formed by the directions of all compatible ASFs in the three-dimensional objective space is presented in Figure 3. The set of solutions which are inside the cone is:

$$P(S)_1 = \{x^1, \ldots, x^{20}, x^{22}, \ldots, x^{28}, x^{31}, \ldots, x^{35}, x^{39}, x^{40}, x^{41},$$

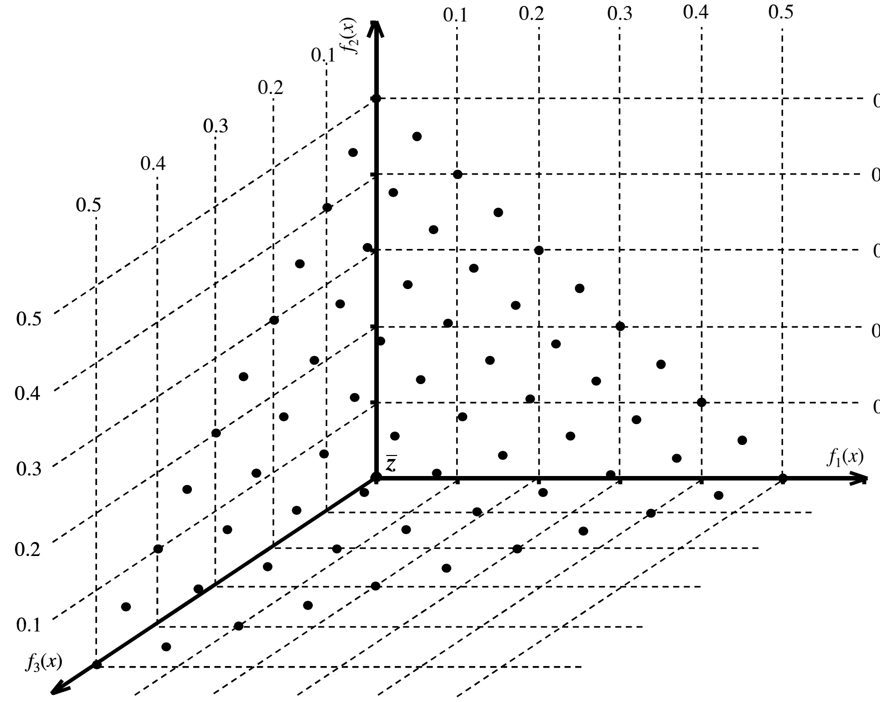

Figure 2. The set of non-dominated solutions $P(S)_0$ before the first iteration

$$x^{46}, x^{47}, x^{52}, x^{53}, x^{57}, x^{58}, x^{61}, x^{64}\}.$$

Thus, in the second iteration, the DM needs to consider 43 solutions out of the initial 66 ones.

To make the remaining solutions more comparable, the DM states that $x^{41} = [0.20, 0.10, 0.20] \succ x^{13} = [0.05, 0.05, 0.40]$. Note that x^{41} is better than x^{13} only on the third objective, while being worse on the other two. Consequently, the constraints on the weights of the ASFs compatible with the pairwise comparison provided in the second iteration are the following:

$$\{[\lambda_3 > 1/2 \cdot \lambda_1] \ \wedge \ [\lambda_3 > 1/4 \cdot \lambda_2]\} \ \wedge \ \{\lambda_i \geq 0, \ i = 1, 2, 3\}.$$

Taking into account the outcomes of the previous iteration, we could present the cone formed by the directions of compatible ASFs which guarantee that $x^{33} \succ x^{55}$ and $x^{41} \succ x^{13}$ as in Figure 4. The set of non-dominated solutions situated inside the contracted cone consists of 10 solutions:

$$P(S)_2 = \{x^{11}, x^{20}, x^{27}, x^{28}, x^{33}, x^{34}, x^{35}, x^{41}, x^{53}, x^{58}\}.$$

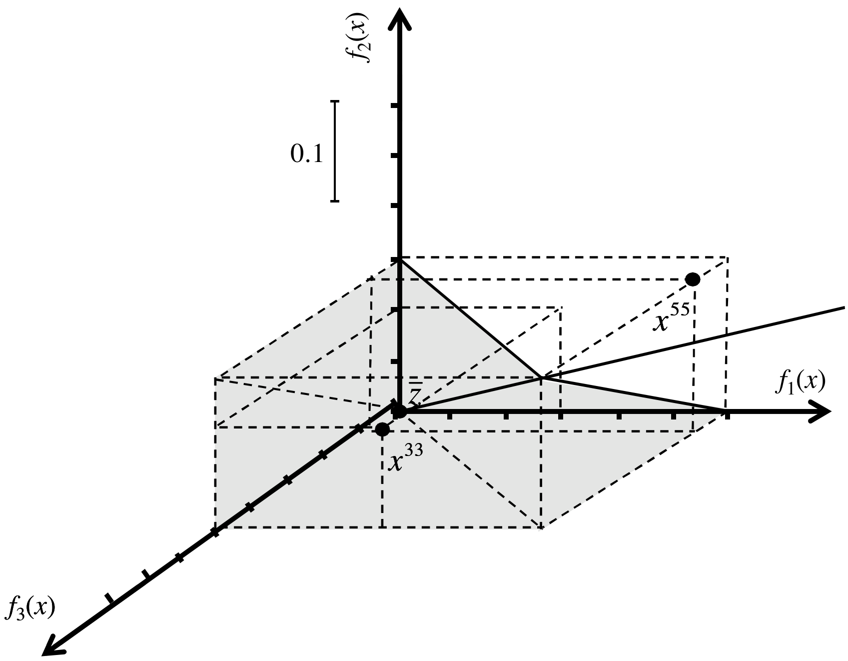

Figure 3. The directions of ASFs compatible with a pairwise comparison provided in the first iteration

Knowing the evaluations of the solutions which are still perceived as the potential best compromise solutions, the DM decides to change the reference point to $\bar{z} = [0.15, 0.10, 0.10]$. Consequently, the set of considered solutions is limited to $\{x^{33}, x^{34}, x^{35}, x^{41}\}$ (see Figure 5). The DM states that $x^{35} = [0.15, 0.20, 0.15] \succ x^{34} = [0.15, 0.15, 0.20]$. In this way, (s)he prefers a solution with a slightly better evaluation on f_3 than a solution with a slightly better evaluation on f_2. Since within the contracted cone there is only one solution (see Figure 5), it is presented to the DM as the one that best satisfies her/his indirectly provided preferences.

5 Conclusions

The major advantage of the presented interactive robust cone contraction method is the organization of the search over the non-dominated set through pairwise comparisons of solutions from the current sample and suitable moving of the reference point by the DM, which may be inspired by the knowledge gained by her/him in the course of the interactive process. The

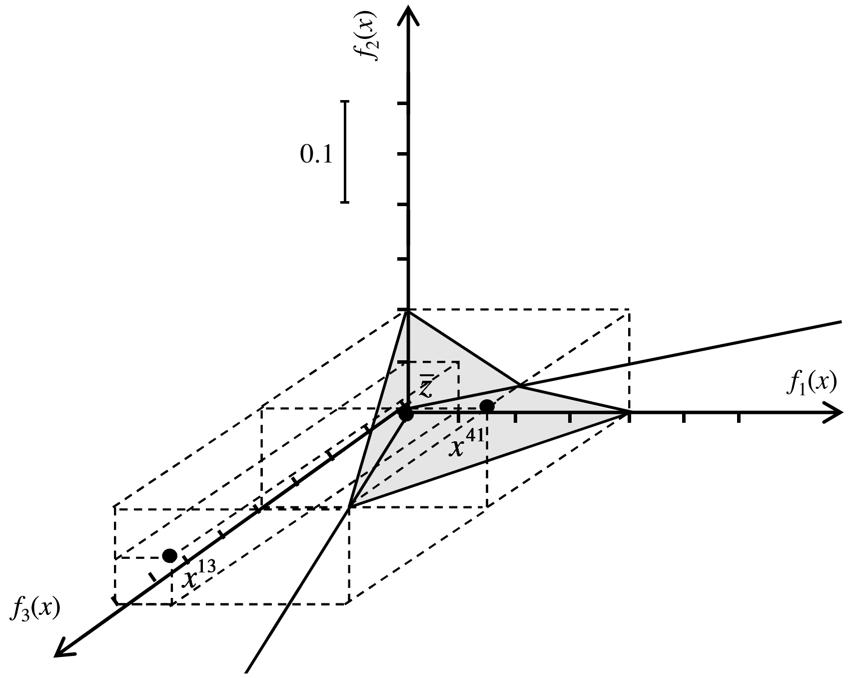

Figure 4. The directions of ASFs compatible with a pairwise comparison provided in the second iteration

motivation for employing the achievement scalarizing function came from its suitability for producing different solutions by weighting the marginal differences between attainable values of objective functions and respective co-ordinates of the current reference point. This permits to get control over the process of solving a MOO problem through an appropriate formulation of constraints on the weights.

Within the presented procedure, the DM is required to provide preferences composed of understandable and not very demanding holistic judgments. According to psychologists, people feel more confident exercising their decisions rather than explaining them directly in terms of values of some preference model parameters. Since the process of selecting a single, most preferred solution is organized by contraction of a cone in the objective space, the DM can easily observe the consequences of one's decisions and learn about the nature of the problem. Moreover, as in every iteration the set of still considered solutions is being delimited and its intuitive representation is presented to the DM, (s)he is able to build a conviction about what is possible in this psychologically convergent process.

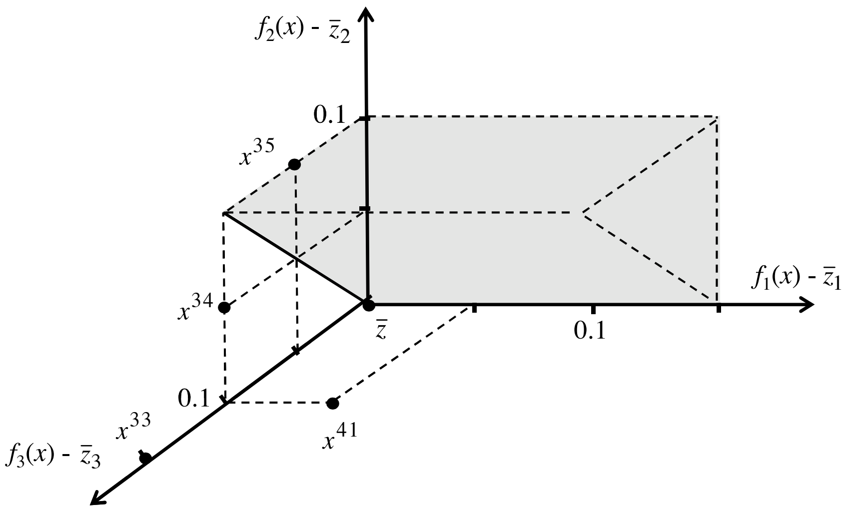

Figure 5. The directions of compatible ASFs after changing the reference point to $\bar{z} = [0.15, 0.10, 0.10]$ and accounting for a pairwise comparison provided in the third iteration

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