A CONTRIBUTION TO THE HEAT CONDUCTION PROBLEM IN A LAMINATED MEDIUM

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Abstract. In this contribution we are to show that on the boundaries of a laminated medium, which are perpendicular to the laminae interfaces, some near-boundary phenomena related to boundary conditions take place. The aim of this note is to describe and discuss those phenomena. The analysis is carried out in the framework of the tolerance averaging technique, [1]. The obtained model is confronted with the homogenized model. General considerations are illustrated by a certain special problem.

Introduction

It is known that the direct approach to the analysis of heat conduction processes in composites with a dense periodic structure (i.e. the approach satisfying Fourier heat conduction equation in every component) leads to ill-conditioned and complicated computational problems, [2]. That is why some averaged (macroscopic) mathematical models for obtaining solutions to special problems have been formulated. This situation is typical for both the stationary and nonstationary processes in periodic solids and structures.

As a rule averaged models of periodic structures are based on the homogenized method [3]. A certain drawback of homogenized models is that they do not describe the effect of the period lengths on the overall behaviour of a periodically inhomogeneous solid. Hence some effects taking place in boundary and initial layers cannot be investigated in the framework of homogenization, [4]. That is why some alternative approaches to the modelling of periodic materials and structures have been formulated. The overview of these approaches can be found in [1]. In this monograph foundations of what was called *the tolerance averaging technique* have been summarized. It has to be emphasized that models derived using this technique in contrast to homogenized models describe the effect of periodicity cell size on the overall behaviour of solid under consideration (a length scale effect).

In this contribution we formulates certain simplified tolerance averaging model equations which can be applied to analyse some boundary layer-phenomena. Moreover some numerical results obtained for an benchmark boundary problem using proposed model are compared with those obtained in the framework of homogenized model and verified by an exact solution.

1. Preliminary

Let the physical space be parametrized by the orthogonal Cartesian coordinate system $Ox_1x_2x_3$ with the x_3-axis perpendicular to the laminae interfaces, and let t stand for the time coordinate. A laminated rigid conductor is assumed to occupy the unbounded region $\langle 0, L\rangle \times R \times \langle 0, H\rangle$ in the physical space. The scheme of this conductor is shown in Figure 1, where l is a period of inhomogeneity, $l << L$, l', l'' are laminae thicknesses.

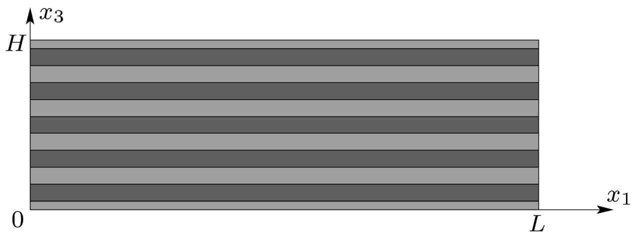

Fig. 1. Scheme of a laminated rigid conductor

A specific heat and heat conduction tensor components in pertinent laminae are denoted by c', $K'_{ij} = K'_{ji}$ and c'', $K''_{ij} = K''_{ji}$ $K'_{\alpha 3} = K''_{\alpha 3} = 0$ due to the assumed material symmetry of the conductor. Hence $c(\cdot)$ and $K_{ij}(\cdot)$ are l-periodic piecewise constant functions of argument x_3. Let $\theta = \theta(\mathrm{x}, t)$ be a temperature field at time t. We assume that in every lamina this field $\theta(\cdot)$ satisfies the well-known linearized Fourier heat transfer equation

$$c(x_3)\dot{\theta}(\mathrm{x},t) - \partial_i\left(K_{ij}(x_3)\partial_j\theta(\mathrm{x},t)\right) = 0 \tag{1}$$

On the interfaces between laminae we deal with the heat flux continuity conditions

$$K^+_{i3}\partial^+_3\theta(\mathrm{x},t) = K^-_{i3}\partial^-_3\theta(\mathrm{x},t) \tag{2}$$

where ∂^+_3, ∂^-_3 stand for the right-hand side and left-hand side derivatives, respectively, and K^+_{i3}, K^-_{i3} are the values K_{i3} in the pertinent adjacent laminae. Equations (1), (2) have to be satisfied together with the appropriate initial and boundary conditions.

Bearing in mind the remarks mentioned in Introduction, we shall replace equations (1), (2) by certain model equations which have constant coefficients. Following the approach applied in [1], we restrict the class of temperature fields to that in which $\theta(\cdot,t)$ is described by the formula

$$\theta(\mathrm{x},t)=\vartheta(\mathrm{x},t)+g(x_3)\Psi(\mathrm{x},t) \tag{3}$$

where $\vartheta(\cdot)$ and $\Psi(\cdot)$ are differentiable functions which are slowly varying in argument $x_3 \in \langle 0, L\rangle$, and $g(x_3)$ is assumed to be a saw-like shape function, [1]. Then, after substituting (3) to (1) and using *tolerance averaging technique* [1], we obtain finally

$$\begin{gathered}\langle c\rangle\dot{\vartheta}-\langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\vartheta-\langle K_{33}\rangle\partial_3\partial_3\vartheta-[K_{33}]\partial_3\Psi=0\\ l^2\langle c\rangle\dot{\Psi}-l^2\langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\Psi+\{K_{33}\}\Psi+[K_{33}]\partial_3\vartheta=0\end{gathered} \tag{4}$$

where $\langle c\rangle$, $\langle K_{\alpha\beta}\rangle$, $[K_{33}]$, $\{K_{33}\}$ are certain constant coefficients. The above equations with pertinent boundary and initial conditions represent what is called *a tolerance model* of a periodically laminated rigid conductor. The detailed discussion of equations (4) can be found in [1].

The homogenized model can be treated as a special case of the tolerance model by the formal neglecting terms $O(l^2)$ in (4). Hence Ψ can be eliminated from $(4)_1$ and after denotation

$$K^o=\frac{K'_{33}K''_{33}}{\nu' K''_{33}+\nu'' K'_{33}}$$

we obtain

$$\begin{gathered}\langle c\rangle\dot{\vartheta}-\langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\vartheta-K^o\partial_3\partial_3\vartheta=0\\ \Psi=-\frac{[K_{33}]}{\{K_{33}\}}\partial_3\vartheta\end{gathered} \tag{5}$$

The above equations with the boundary and initial conditions for ϑ represent the homogenized model for the heat transfer in a periodically laminated rigid conductor [3].

3. Simplified boundary model equations

Now we shall transform equations (4) to a new form by introducing the decomposition

$$\Psi=-\frac{[K_{33}]}{\{K_{33}\}}\partial_3\vartheta+\varphi \tag{6}$$

where φ is a new unknown function slowly varying in x_3. The above decomposition makes it possible to treat φ as a difference between temperature fluctuations in the tolerance averaged model and temperature fluctuations directly depended on averaged temperature like in the homogenized model. Substituting the right-hand side of (6) into (4) we obtain

$$\begin{gathered}\langle c\rangle\dot{\vartheta}-\langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\vartheta-K_{33}^{o}\partial_3\partial_3\vartheta=[K_{33}]\partial_3\varphi\\ l^2\langle c\rangle\dot{\varphi}-l^2\langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\varphi+\{K_{33}\}\varphi=l^2\frac{[K_{33}]}{\{K_{33}\}}\left(K_{33}^{o}\partial_3\partial_3\partial_3\vartheta+[K_{33}]\partial_3\vartheta\right)\end{gathered}\tag{7}$$

Notice that ϑ and φ are slowly varying function of argument $x_3\in\langle 0,H\rangle$ and by means of (6), values of φ and $\partial_3\vartheta$ are of the same order when related to a tolerance parameter. Thus, we conclude that the right-hand side of the second from the above equations is small enough when compared to the left-hand side. Similarly, the right-hand side of the first from equations (7) also can be omitted. So we obtain finally

$$\begin{gathered}\langle c\rangle\dot{\vartheta}-\langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\vartheta-K_{33}^{o}\partial_3\partial_3\vartheta=0\\ l^2\langle c\rangle\dot{\varphi}-l^2\langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\varphi+\{K_{33}\}\varphi=0\end{gathered}\tag{8}$$

It follows that the tolerance model equations (4) of a periodically laminated heat conductor can be also represented in the simplified form (7) with ϑ and φ as basic unknowns. Notice that the first from equations (8) coincides with that of the homogenized model.

3. Initial-boundary value problem

In order to verify the proposed simplified tolerance averaging model and compare this model with the homogenized model we shall restrict to the solution of a certain stationary benchmark problem. Let us assume following boundary conditions

$$\theta(x_1,0)=\theta(x_1,H)=0,$$

$$\theta(0,x_3)=\theta(L,x_3)=A\sin\frac{\pi}{H}x_3.$$

Solutions obtained both in the framework of the proposed simplified tolerance averaged model and the homogenized model are presented in Figure 2.

In order to verify these results we have to solve initial boundary value problem in the framework of the exact Fourier model. The above problem will be solved numerically using a finite difference method. The exact solutions is showed in Figure 2.

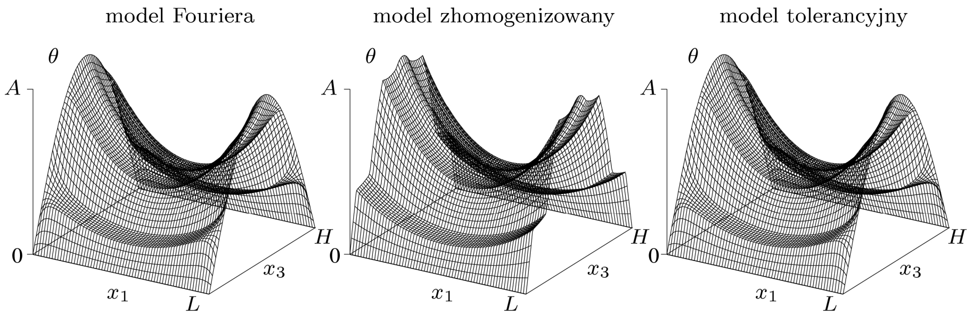

Fig. 2. Exact solution obtained numerically using Fourier model and solutions obtained in the framework of homogenized model, simplified tolerance averaging model

Concluding remarks

Let us summarize new result and information on the heat conduction in a laminated rigid conductor which have been obtained in this contribution:

1° It was shown that the temperature field in a laminated rigid conductor can suffer certain fluctuations near the boundaries of the conductor. These fluctuations are caused by the inhomogeneous periodic structure of the conductor.

2° A simplified mathematical model for the analysis of the aforementioned temperature fluctuations has been proposed. This model allows calculate separately the distribution of the temperature together with the related temperature fluctuations depending on the averaged temperature and fluctuations generated independently by boundary conditions.

3° The proposed model, in contrast to the known homogenized model of heat conduction, makes it possible to satisfy boundary conditions not only by the averaged temperature but also by the temperature fluctuations.

4° Solutions to the selected problem obtained in this contribution are compared with those derived from homogenization. Some comparisons between boundary layer type solutions and exact solutions to the heat conduction problems are presented. The main conclusion is that the proposed model (in contrast to the known homogenized model) describes the boundary layer effect on the heat conduction in a laminated rigid conductor. Moreover, differences between solutions obtained in the framework of the Fourier equation and those related to the proposed model are negligible in the boundary layer. Outside this layer solutions obtained by using the proposed model, those derived from the homogenized model, as well as results calculated in the framework of the Fourier theory nearly coincide.

References

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