ABOUT ONE METHOD OF INCOMES FORECASTING IN INSURANCE COMPANY

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Abstract. The stochastic model of processes of claims and rewards processing in insurance company was examined in this paper. The state of insurance company at a fixed moment of time was described by Markov process with continuous time and finite number of states. The theory of Markov processes with incomes was used for forecasting of expected incomes in insurance company.

Introduction

Let's examine the functioning of insurance company, which concludes uniform agreements of insurance. Total number of agreements, concluded by company up to moment of time t, $t \in [0, T]$, is described by the function of time $K(t)$, moreover $K(t) \le N$, where N - number of inhabitants of region in which company functions. Each of the company's clients can be found in one of the following states: C_0 - in the "waiting" stage (it does not necessary to contribute reward, the insurance case did not occur); C_1 - in the stage of estimation of the produced action; C_2 - in the stage of estimation of the contributed reward; C_3 - in the stage of payment on the action or payment to reward. A change of client's state occurs in accordance with the diagram, represented in Figure 1.

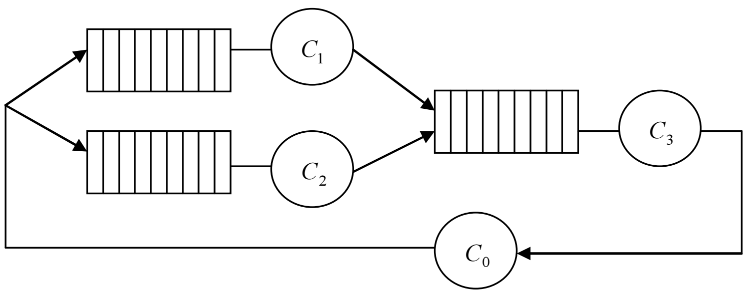

Fig. 1. Diagram of a change of clients states

Let $\mu_{01}(t)\Delta t + o(\Delta t)$, $\mu_{02}(t)\Delta t + o(\Delta t)$ - probability of the presentation of action and reward in the time interval $[t, t + \Delta t]$ respectively, i.e. probabilities of transition from state C_0 to C_i, i = 1,2. Transit time from state C_1 to C_3, from C_2, to C_3, from C_3 to C_0 are distributed according to the exponential law with the intensities μ_1, μ_2, μ_3 respectively. It means that servicing time of clients by each of m_1 claim's estimators is distributed according to the exponential law with the intensity μ_1, by each of m_2 rewards estimators - with the intensity μ_2. Servicing time of clients by each of m_3 cashiers we consider also distributed according to the exponential law with the intensity μ_3.

The vector

$$k(t) = (k,t) = (k_1(t),\ k_2(t),\ k_3(t)) \tag{1}$$

can describe the state of insurance company at the moment of time t, where $k_i(t)$ - number of clients, who are located in the state C_i at the moment of time t, $i = \overline{1,3}$; the number of clients in the state C_0 comprises $k_0(t) = K(t) - \sum_{i=1}^{3} k_i(t)$. It is obvious that $k(t)$ is Markov process with the continuous time and finite set of states.

1. Differential equation for incomes forecasting in insurance company

It is obvious that the income of insurance company is concerned with obtaining of rewards from the clients, and expenditure is caused by payment on the actions and expenditures for the clients care. Let us designate $V(k,t)$ - the complete expected income, which insurance company will obtain in the time t, if at initial moment of time company was in state (k,t). During the small time interval Δt insurance company can remain in the state $(k, t+\Delta t)$ or can transfer to one of the following states: $(k - I_3, t+\Delta t)$, $(k + I_3 - I_1, t+\Delta t)$, $(k + I_3 - I_2, t+\Delta t)$, $(k + I_1, t+\Delta t)$, $(k + I_2, t+\Delta t)$. Here I_i - the 3-dimensional vector, all components of which are equal to zero and i-th component equals to 1, $i = \overline{1,3}$.

Theorem. *Density of the distribution of the income v(x,t), when it is differentiated on t and almost everywhere twice differentiated on x, satisfies the following partial differential equation:*

$$\frac{\partial v(x,t)}{\partial t} = -\sum_{i=1}^{3} A_i(x,t)\frac{\partial v(x,t)}{\partial x_i} - \frac{\varepsilon(t)}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} B_{ij}(x,t)\frac{\partial^2 v(x,t)}{\partial x_i \partial x_j} - \tag{2}$$
$$-3\varepsilon'(t)K(t)v(x,t) +$$
$$+\left[R - K(t)\sum_{i=1}^{3}\mu_i \min(l_i(t), x_i(t))R_i + K(t)\left(1 - \sum_{j=1}^{3} x_j(t)\right)\left(\mu_{02}(t)R_{02} - \mu_{01}(t)R_{01}\right)\right]$$

where:

$$A_i(x,t) = \sum_{j=1}^{3} \mu_j q_{ji} \min(l_j(t), x_j(t)) - \mu_{0i}(t)\left(1 - \sum_{j=1}^{3} x_j(t)\right),\ \mu_{03}(t) = 0 \quad (3)$$

$$B_{ii}(x,t) = \sum_{j=1}^{3} \mu_j q^*_{ji} \min(l_j(t), x_j(t)) - \mu_{0i}(t)\left(1 - \sum_{j=1}^{3} x_j(t)\right),\ i = \overline{1,3} \quad (4)$$

$$q_{ji} = \begin{cases} 1,\ j = i,\ i = \overline{1,3}, \\ -1,\ j \neq i,\ i = 3, \\ 0,\ \text{in other cases}; \end{cases} \qquad q_{ji} = \begin{cases} -1,\ j = i,\ i = \overline{1,3}, \\ -1,\ j \neq i,\ i = 3, \\ 0, \text{in other cases}; \end{cases}$$

$$B_{i3}(x,t) = \mu_i \min(l_i(t), x_i(t)) = B_{3i}(x,t),\ i = \overline{1,2}$$

Proof. We will consider that if in the time interval $[t, t+\Delta t]$ insurance company transfers from state (k,t) to state $(k - I_3, t+\Delta t)$ with probability $\mu_3 \min(m_3, k_3(t))\Delta t + o(\Delta t)$, then the income of insurance company will comprise - R_3 arbitrary units plus the expected income $V(k - I_3, t)$, which the company will obtain in the remained time *t*, if the initial state was $(k - I_3, t)$. While transferring from state (k,t) to state $(k + I_3 - I_i, t+\Delta t)$ in the time interval $[t, t+\Delta t]$, the income of the company will comprise - R_i arbitrary units plus $V(k + I_3 - I_i, t)$, the probability of such transition equals $\mu_i \min(m_i, k_i(t))\Delta t + o(\Delta t)$, i = 1,2. The passage from state (k,t) to state $(k + I_1, t+\Delta t)$, that is achieved in the time interval $[t, t+\Delta t]$ with probability $\mu_{01}(t)k_0(t)\Delta t + o(\Delta t)$, brings to the insurance company - R_{01} arbitrary units of income plus $V(k + I_1, t)$. The passage from the state (k,t) to state $(k + I_2, t+\Delta t)$ during the small time interval Δt is achieved with the probability $\mu_{02}(t)k_0(t)\Delta t + o(\Delta t)$ and brings to the insurance company R_{02} arbitrary units of income plus $V(k + I_2, t)$. In other words R_3, R_1 and R_2 - are losses of insurer, concerned with the care of clients in the payment stage, in the estimation stage of the produced action and in the estimation stage of the contributed reward accordingly; R_{01} - loss that concerned with the payment on the action, R_{02} - income from the entering of insurance reward. In addition to this, we will consider that the insurance company derives revenues in the size *R* arbitrary units for the unit of time during its stay in the state (k,t). During the small time interval Δt company remains in the state (k,t) with the probability $1 - \left[\sum_{i=1}^{3} \mu_i \min(m_i, k_i(t)) + \sum_{i=1}^{2} \mu_{0i}(t)k_0(t)\right]\Delta t$, in this case its income will comprise $R\Delta t + V(k,t)$.

Then complete expected income $V(k,t+\Delta t)$ at the time moment $t+\Delta t$ satisfies the following set of difference equations:

$$V(k,t+\Delta t)=\left\{1-\left[\sum_{i=1}^{3}\mu_i\min(m_i,k_i(t))+\sum_{i=1}^{2}\mu_{0i}(t)k_0(t)\right]\Delta t\right\}(R\Delta t+V(k,t))+$$
$$+\mu_3\min(m_3,k_3(t))\Delta t(-R_3+V(k-I_3,t))+$$
$$+\sum_{i=1}^{2}\mu_i\min(m_i,k_i(t))\Delta t(-R_i+V(k+I_3-I_i,t))+$$
$$+\mu_{01}(t)k_0(t)\Delta t(-R_{01}+V(k+I_1,t))+\mu_{02}(t)k_0(t)\Delta t(R_{02}+V(k+I_2,t))+o(\Delta t)$$

The given set can be represented in the form of the set of the difference-differential equations:

$$\frac{\partial V(k,t)}{\partial t}=R-\sum_{i=1}^{3}\mu_i\min(m_i,k_i(t))R_i-\mu_{01}(t)k_0(t)R_{01}+\mu_{02}(t)k_0(t)R_{02}+$$
$$+\mu_3\min(m_3,k_3(t))(V(k-I_3,t)-V(k,t))+$$
$$+\sum_{i=1}^{2}\mu_i\min(m_i,k_i(t))(V(k+I_3-I_i,t)-V(k,t))+ \tag{5}$$
$$+\sum_{i=1}^{2}\mu_{0i}(t)k_0(t)(V(k+I_i,t)-V(k,t))$$

We will further examine the case of the large number of agreements in the insurance company; let us assume that the function *K*(*t*) is taken the sufficiently great values $1<<K(t)\le N$. Let us switch over to the vector of the relative variables $\xi(t)=$ $=(\xi,t)=\left(\frac{k_1(t)}{K(t)},\frac{k_2(t)}{K(t)},\frac{k_3(t)}{K(t)}\right)$. Possible values of the vector $\xi(t)$ belong to the bounded closed set $G(t)=\left\{x(t)=(x,t)=(x_1(t),\ x_2(t),\ x_3(t)):\ x_i(t)\ge 0,\ \sum_{i=1}^{3}x_i(t)\le 1\right\}$, in which they are located in the points of three-dimensional lattice at a distance $\varepsilon(t)=\frac{1}{K(t)}$ from each other. With an increase of values of the function *K*(*t*) the "density of filling" of the set *G*(*t*) with the possible values of the vector *ξ*(*t*) increases and it becomes possible to consider that the vector has continuous distribution in the region *G*(*t*). We can consider with these assumptions that the complete expected income of insurance company continuously changes in the dependence on the initial state (*x*,*t*). Therefore we can put into consideration the function of the expected income's density distribution (concentration) in the region *G*(*t*). By analogy, for example, with the density of mass distribution

$$\rho = \frac{m}{V} = \lim_{\varepsilon \to 0} \frac{m(x_1 \le \xi_1 < x_1 + \varepsilon, x_2 \le \xi_2 < x_2 + \varepsilon, x_3 \le \xi_3 < x_3 + \varepsilon)}{\varepsilon^3}$$

the density of income's distribution is defined as the following limit

$$v(x,t) = \lim_{\varepsilon(t) \to 0} \frac{V(x_1(t) \le \xi_1 < x_1(t) + \varepsilon(t), x_2(t) \le \xi_2 < x_2(t) + \varepsilon(t), x_3(t) \le \xi_3 < x_3(t) + \varepsilon(t))}{\varepsilon^3(t)} \tag{6}$$

It is obvious that the density of the distribution of the income $v(x,t)$ will have the following property:

$$\frac{1}{V_{sum}} \iiint\limits_{G(t)} v(x,t)dx = 1$$

And income $V(\xi,t)$ has property

$$V(\xi,t)\Big|_{\xi=(x_1,x_2,x_3)} = 0, \; V(\xi \in D,t) = \iiint\limits_{D(t)} v(x,t)dx$$

It follows of (6) that for $V(k,t)$ is valid the following approximation with $K(t) \to \infty$

$$V(k,t) = V(x \cdot K,t) = \varepsilon^3(t)v(x,t) \text{ or } v(x,t) = K^3(t)V(x \cdot K,t)$$

Then $\frac{\partial V(k,t)}{\partial t} = \frac{\partial(\varepsilon^3(t)v(x,t))}{\partial t} = 3\varepsilon^2(t)\varepsilon'(t)v(x,t) + \varepsilon^3(t)\frac{\partial v(x,t)}{\partial t}$ and equation (5) for the density of the income's distribution can be represented in the form:

$$\begin{aligned}
\frac{\partial v(x,t)}{\partial t} = {} & -3\varepsilon'(t)K(t)v(x,t) + R - K(t)\sum_{i=1}^{3} \mu_i \min(l_i(t), x_i(t))R_i - \\
& - K(t)\mu_{01}(t)x_0(t)R_{01} + K(t)\mu_{02}(t)x_0(t)R_{02} + \\
& + K(t)\mu_3 \min(l_3(t), x_3(t))(v(x - e_3, t) - v(x,t)) + \\
& + \sum_{i=1}^{2} K(t)\mu_i \min(l_i(t), x_i(t))(v(x + e_3 - e_i, t) - v(x,t)) + \\
& + \sum_{i=1}^{2} K(t)\mu_{0i}(t)x_0(t)(v(x + e_i, t) - v(x,t))
\end{aligned} \tag{7}$$

where: $l_i(t) = \frac{m_i}{K(t)}$, $e_i = \varepsilon(t)I_i$, $i = \overline{1,3}$.

Assuming that $v(x,t)$ is differentiated on t and is almost everywhere twice differentiated on x_i, $i=\overline{1,3}$, the function $v(x+e_3-e_i,t)$, $v(x\pm e_i,t)$ let's decompose in the Taylor series in the environment of the point (x,t). Equation (7) can be written down in the form:

$$\frac{\partial v(x,t)}{\partial t}=-3\varepsilon'(t)K(t)v(x,t)+R-K(t)\sum_{i=1}^{3}\mu_i\min(l_i(t),x_i(t))R_i-$$
$$-K(t)\mu_{01}(t)x_0(t)R_{01}+K(t)\mu_{02}(t)x_0(t)R_{02}+$$
$$+\mu_3\min(l_3(t),x_3(t))\left[-\frac{\partial v(x,t)}{\partial x_3}+\frac{\varepsilon(t)}{2}\frac{\partial^2 v(x,t)}{\partial {x_3}^2}\right]+$$
$$+\sum_{i=1}^{2}\mu_i\min(l_i(t),x_i(t))\left[\left(\frac{\partial v(x,t)}{\partial x_3}-\frac{\partial v(x,t)}{\partial x_i}\right)+\right.$$
$$\left.+\frac{\varepsilon(t)}{2}\left(\frac{\partial^2 v(x,t)}{\partial {x_3}^2}-2\frac{\partial^2 v(x,t)}{\partial x_3\partial x_i}+\frac{\partial^2 v(x,t)}{\partial {x_i}^2}\right)\right]+$$
$$+\sum_{i=1}^{2}\mu_{0i}(t)x_0(t)\left[\frac{\partial v(x,t)}{\partial x_i}+\frac{\varepsilon(t)}{2}\frac{\partial^2 v(x,t)}{\partial {x_i}^2}\right]+O(\varepsilon^2(t))$$

Last equation with an accuracy down to the terms of the order $O(\varepsilon^2(t))$ can be write down in more compact form (2), using designations (3), (4). Theorem is proven.

Taking into account (2), expression $\frac{\varepsilon(t)}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}B_{ij}(x,t)\frac{\partial^2 v(x,t)}{\partial x_i\partial x_j}$ can be attributed to $O(\varepsilon^2(t))$. Therefore we will examine the following equation:

$$\frac{\partial v(x,t)}{\partial t}=-\sum_{i=1}^{3}A_i(x,t)\frac{\partial v(x,t)}{\partial x_i(t)}-3\varepsilon'(t)K(t)v(x,t)+$$
$$+\left[R-K(t)\sum_{i=1}^{3}\mu_i\min(l_i(t),x_i(t))R_i+K(t)\left(1-\sum_{j=1}^{3}x_j(t)\right)\left(\mu_{02}(t)R_{02}-\mu_{01}(t)R_{01}\right)\right]$$

After dividing both parts of this equation to the volume of the region $G(t)$, let's integrate both parts of the equation for $x=(x_1,x_2,x_3)$ in the region $G(t)$:

$$\frac{1}{m(G(t))}\iiint\limits_{G(t)}\frac{\partial v(x,t)}{\partial t}dx=$$

$$=-\frac{1}{m(G(t))}\sum_{i=1}^{3}\iiint\limits_{G(t)}A_i(x,t)\frac{\partial v(x,t)}{\partial x_i(t)}dx-\frac{3}{m(G(t))}\iiint\limits_{G(t)}\varepsilon'(t)K(t)v(x,t)dx+\frac{1}{m(G(t))}\times \quad (8)$$

$$\times\iiint\limits_{G(t)}\left[R-K(t)\sum_{i=1}^{3}\mu_i\min(l_i(t),x_i(t))R_i+K(t)\left(1-\sum_{j=1}^{3}x_j(t)\right)\left(\mu_{02}(t)R_{02}-\mu_{01}(t)R_{01}\right)\right]dx$$

Considering that on the left side of this equality change in the order of integration and differentiation is permitted, we will obtain

$$\frac{1}{m(G(t))}\iiint\limits_{G(t)}\frac{\partial v(x,t)}{\partial t}dx=\frac{1}{m(G(t))}\frac{\partial}{\partial t}\iiint\limits_{G(t)}v(x,t)dx=\frac{d}{dt}\overline{v_G}(t)$$

where $\overline{v_G}(t)$ - average on x value of income with the condition of changing the initial state (x,t) in the region $G(t)$.

Let's examine integrals in right side of (8)

$$\frac{3}{m(G(t))}\iiint\limits_{G(t)}\varepsilon'(t)K(t)v(x,t)dx=3\varepsilon'(t)K(t)\overline{v_G}(t)$$

With the calculation of the remained integrals we will use integration in parts. Let us assume that following boundary conditions are satisfied [1]:

$$A_3(x,t)v(x,t)\Big|_{x_3=0}^{x_3=1-x_1-x_2}=0,\ \ A_2(x,t)v(x,t)\Big|_{x_2=0}^{x_2=1-x_1-x_3}=0,\ \ A_3(x,t)v(x,t)\Big|_{x_1=0}^{x_1=1-x_2-x_3}=0$$

which mean that the flow of income through the boundary of the region $G(t)$ is not allowed, or that at the frontier points of the region $G(t)$ are fixed the reflecting barriers. Then, taking into account that $\frac{\partial A_i(x,t)}{\partial x_i}=\text{const}$ we will obtain

$$\frac{1}{m(G(t))}\iiint\limits_{G(t)}A_i(x,t)\frac{\partial v(x,t)}{\partial x_i}dx=-\frac{\partial A_i(x,t)}{\partial x_i}\overline{v_G}(t),\ \ i=\overline{1,3}$$

In other words, we come to the following differential equation

$$\frac{d}{dt}\overline{v_G}(t)=\overline{v_G}(t)\left(\sum_{i=1}^{3}\frac{\partial A_i(x,t)}{\partial x_i}-3\varepsilon'(t)K(t)\right)+\frac{1}{m(G(t))}\times$$

$$\times\iiint\limits_{G(t)}\left[R-K(t)\sum_{i=1}^{3}\mu_i\min(l_i(t),x_i(t))R_i+K(t)\left(1-\sum_{j=1}^{3}x_j(t)\right)\left(\mu_{02}(t)R_{02}-\mu_{01}(t)R_{01}\right)\right]dx \quad (9)$$

We see of (6) that the coefficients $A_i(x,t)$ are piecewise-linear functions, i.e. (9) - differential equation with the piecewise-constant right side. Let us designate the set of the indices of the vector's $x(t) = (x_1(t), x_2(t), x_3(t))$ components like $\Omega(t) = \{1,2,3\}$. Let's divide $\Omega(t)$ into two disjoint sets $\Omega_0(\tau,t)$, $\Omega_1(\tau,t)$ such, that $\Omega_0(\tau,t) = \{j : l_j(t) < x_j(t) \le 1\}$, $\Omega_1(\tau,t) = \{j : 0 \le x_j(t) \le l_j(t)\}$. With fixed t the number of partitions of such type is equal $2^3 = 8$, $\tau = \overline{1,8}$. Each partition will assign the nonintersecting regions $G_\tau(t)$ in the set $G(t)$ such, that

$$G_\tau(t) = \left\{ x(t) : l_i(t) < x_i(t) \le 1, i \in \Omega_0(\tau,t); 0 \le x_j(t) \le l_j(t), j \in \Omega_1(\tau,t); \sum_{c=1}^{3} x_c(t) \le 1 \right\}$$

$$\tau = 1, 2, ..., 8, \bigcup_{\tau=1}^{8} G_\tau(t) = G(t)$$

Now in each of the regions of the phase space's partition we can write down explicit form (9), and with determined initial conditions we can find the average expected income for each of the regions $G_\tau(t)$.

For example, let's assign the following partition $\Omega_0(1,t) = \{1,2,3\}$, $\Omega_1(1,t) = \{\varnothing\}$, τ = 1, it corresponds to the presence of turns in the stages of maintenance and corresponds to the real situation of the clients care in the insurance companies. Then solving (9) with the initial condition $\overline{v_{G_1}}(0) = S$ we can determine the average expected income with a change of the initial state according to the region $G_1(t)$.

2. Examples

Example 1. Let's examine the functioning of insurance company that works with the uniform agreements of insurance. Moreover, let's assume that the number of concluded agreements invariably and comprises K = 25 000. The remaining parameters of the functioning of insurance company are the following: $\mu_{01}(t) = 0.0002$, $\mu_{02}(t) = 0.003$, $\mu_1 = 0.008$, $\mu_2 = 0.05$, $\mu_3 = 0.1$, $m_1 = 3$, $m_2 = 2$, $m_3 = 1$, $R = 10$, $R_{01} = 500$, $R_{02} = 33$, $R_1 = 3$, $R_2 = 1$, $R_3 = 0.3$, $\overline{v_{G_1}}(0) = 5000$. Then the dependence from the time of the average expected income with the condition of changing the initial state on the region $G_1(t)$ is depicted in Figure 2.

Example 2. Let's examine the functioning of insurance company, which works with the uniform agreements of the insurances, whose number up to moment of the time t is determined from the formula $K(t) = 20000 + \dfrac{40000}{2\sin(2\pi t / 364 + 5)}$.

The parameters of the functioning of insurance company are the following: $\mu_{01}(t) = 0.00007$, $\mu_{02}(t) = 0.001$, $\mu_1 = 0.008$, $\mu_2 = 0.03$, $\mu_3 = 0.1$, $m_1 = 3$, $m_2 = 1$, $m_3 = 1$, $R = 10$, $R_{01} = 400$, $R_{02} = 30$, $R_1 = 5$, $R_2 = 1$, $R_3 = 0.3$, $\overline{v_{G_1}}(0) = 5000$. Then the dependence from the time of the average expected income with the condition of changing the initial state on the region $G_1(t)$ is depicted in Figure 3.

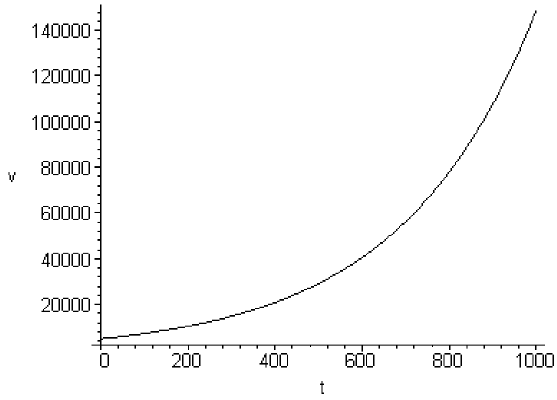

Fig. 2. Forecast of the income $\overline{v_{G_1}}(t)$ for example 1

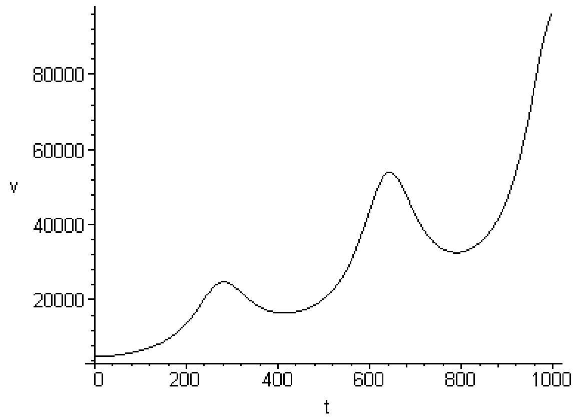

Fig. 3. Forecast of the income $\overline{v_{G_1}}(t)$ for example 2

Reference

[1] V.I. Tikhonov, Mironov M.A., Markov processes, Sov. radio, Moscow 1977 (in Russian).