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AN APPROACH TO MODELING ALTRUISTIC EQUILIBRIUM IN GAMES

Abstract

We present an approach to modeling equilibrium in non-cooperative non-zero sum games taking into account player altruism. The altruistic preferences concern the relations between changes of the given player's and the other players' pay-offs. The degree of altruism is represented by the altruistic coefficient for each pair of players. We prove that any Pareto optimal strategy profile can be an equilibrium if the level of player altruism is high enough.

Keywords

Game theory, altruistic equilibrium, altruistic trade-off.

Introduction

In recent years one can observe a tendency toward enhancing the game--theoretical apparatus integrated into the economic theory. The "classical" game theory models are based on the assumption that a player aims only at increasing his/her own pay-off. Such models are unable to explain why cooperation emerges in the wide variety of prisoner's dilemma-like economic affairs in real life. Let us refer to the journalistic article by Paul Krugman[*] [2009], where he criticizes the current state of economical science in the context of the world economy crisis. He points out, among other methodological defects, that the view of individual behavior of economic agents is primitively rational. The cognitive and behavioral approaches to economics, in contrast, are considered to be new directions of research for tackling the complex behavior of economic agents in the context of their personality. Thus, new models explaining player behavior are needed.

[*] The winner of the Nobel Memorial Prize in Economic Sciences in 2008.

Since the initial models of player behavior are based on the assumption of absolute egoism, the attempts to enhance them imply introducing non--egoistic features into the player behavior. The idea of modeling altruistic behavior can be traced back to Edgeworth [1881] (as described in Collard [1975]). Edgeworth proposed to increase an individual's utility by a value proportional to the utility of another person. The most popular models of altruistic player behavior in games have the form of utility functions which depend not only on the player's pay-off, but also on pay-offs of the other players. For example, the utility function by Fehr and Schmidt [1999] includes negative terms as penalties for distributional unfairness. The function by Bolton and Ockenfels [2000] depends on the relation between the player's own and the average pay-offs. Charness and Rabin [2002] built their function assuming that the player is interested in increasing the minimal and the average pay-offs of the other players.

Our approach differs from those mentioned above. We consider the situation where a player chooses his/her strategy while the other players' strategies are fixed. We formulate a condition when the player prefers not to change his/her strategy. Thus, the proposed preference model is bound directly to the notion of equilibrium, and the existence of a utility function characterizing the player preferences is not required.

1. The definition of altruistic equilibrium

Consider a p-person, $p > 1$, non-cooperative non-zero sum game $(\mathbf{S},\mathbf{a})$, where

$\mathbf{S}=S_1\times S_2\times \dots \times S_p$ is the set of *strategy profiles*, $S_k:= \{1,2, \dots , m_k\}$, $m_k > 1$, is the strategy set of k-th player, $k\in\mathrm{N}_p:= \{1,2, \dots , p\}$;

$\mathbf{a}=(a^1,a^2, \dots , a^p)$: $\mathbf{S} \to \mathbf{R}^p$ is the vector of pay-off functions, a^k: $\mathbf{S} \to \mathbf{R}$ is the *pay-off function* of k-th player yielding pay-off $a^k(I)$ for each strategy profile $I \in \mathbf{S}$.

For any strategy profile $I=(i_1,i_2, \dots , i_p)$ and any player k, define another strategy profile which differs from I only by strategy of player k:

$$I_{\langle k,j\rangle}=(i'_1, i'_2, \dots , i'_p), \text{ where } i'_l=i_l \text{ for any } l\neq k \text{ and } i'_k=j, j\in S_k, j\neq i_k.$$

Definition 1. *Strategy profile I is a* ***Nash equilibrium*** *in game* $(\mathbf{S},\mathbf{a})$*, if*

$$a^k(I) \geq a^k(I_{\langle k,j\rangle}) \text{ for any } k \in N_p \text{ and any } j \in S_k.$$

Definition 2. *Strategy profile I is **Pareto optimal** in game $(\mathbf{S},\boldsymbol{a})$, if there does not exist any other strategy profile I' such that*

$$\boldsymbol{a}(I') \geq \boldsymbol{a}(I), \ \ \boldsymbol{a}(I') \neq \boldsymbol{a}(I).$$

We propose the following assumption about the altruistic behavior of players:

each player evaluating one strategy versus another,
prefers not to gain in his/her pay-off, if this leads to
disproportionately large loss in pay-offs of other players.

To quantify this assumption, for each pair of players we introduce the *altruistic coefficient*. Denote two players by k and l, $k \neq l$, and denote the altruistic coefficient of player k with respect to player l by α_{kl}, $\alpha_{kl} \geq 0$. This coefficient applies in the following situation. Let player k evaluate one of his/her strategies, say i, over another his/her strategy, say j, under the assumption that the strategies of the other players are known. Let strategy i give player k a greater pay-off in comparison to j, but if player k chooses i over j, then player l loses in his/her pay-off. In these terms, the above assumption is reformulated as follows:

*player **k** does not prefer strategy **i** to strategy **j**,*
*if the pay-off loss of player **l** multiplied by α_{kl}*
*is greater or equal to the pay-off gain of player **k**.*

We define the matrix of altruistic coefficients $\mathrm{A} = (\alpha_{kl})_{p\times p} \in \mathbf{R}_{>}^{p\times p}$, where $\alpha_{kk} := 1$, $k \in N_p$, and $\mathbf{R}_{>}^{p\times p}$ is the set of non-negative matrices with ones on the main diagonal.

Definition 3. *Strategy profile I is called **altruistic equilibrium** or **A-equilibrium**, if for any player $k \in N_p$ and any his/her strategy $j \in S_k$, $j \neq i_k$, the following implication holds:*

$$\text{if } \ a^k(I_{\langle k,j\rangle}) > a^k(I), \text{ then for some player } l \in N_p,\ l \neq k, \text{ it follows}$$

$$\alpha_{kl}(a^l(I) - a^l(I_{\langle k,j\rangle})) \geq a^k(I_{\langle k,j\rangle}) - a^k(I). \tag{1}$$

Literally, A-equilibrium is a strategy profile such that no player wants to change his/her strategy for the following reason: if the player can gain in pay--off by changing his/her strategy, then this leads to pay-off loss of another player such that the absolute value of the pay-off loss multiplied by the corresponding altruistic coefficient is greater than or equal to the pay-off gain of the first player.

A player's altruistic behavior restricts the domains in which players act exclusively in their own interests. The greater a player's altruistic coefficient is, the more severe this restriction. On the other hand, if $\alpha_{kl} = 0$ then player k does not feel any altruism with respect to player l and acts as in an "ordinary" game. Indeed, for $\alpha_{kl} = 0$ the implication in Definition 3 takes the form:

$$\text{if } a^k(I_{\langle k,j\rangle}) > a^k(I) \text{ then } a^k(I_{\langle k,j\rangle}) - a^k(I) \le 0$$

which holds true if and only if $a^k(I_{\langle k,j\rangle}) \le a^k(I)$. It follows that the definition of A-equilibrium is equivalent to the definition of Nash equilibrium, if all the altruistic coefficients are equal to zero.

Let us compare our concept of altruistic equilibrium to the concept implied by the Edgeworth's [1881] proposition on altruism (see Collard [1975]). Under the Edgeworth's assumption that the player's pay-off is increased by a value proportional to the pay-off of the other player, the equilibrium definition in a two player game takes the following form:

Strategy profile I is an Edgeworth A-equilibrium in game $(\boldsymbol{S},\boldsymbol{a})$
if and only if

$$a^1(I) - a^1(I_{\langle 1,j\rangle}) \ge \alpha^{12}(a^2(I_{\langle k,j\rangle}) - a^2(I)) \text{ and}$$
$$a^2(I) - a^2(I_{\langle 1,j\rangle}) \ge \alpha^{21}(a^1(I_{\langle k,j\rangle}) - a^1(I)) \text{ for any } j \in S_k.$$

Our Definition 3 has the following form in the case of two players:

Strategy profile I is an A-equilibrium in game $(\boldsymbol{S},\boldsymbol{a})$
if and only if

$$a^1(I) - a^1(I_{\langle 1,j\rangle}) \ge \min\{0,\ \alpha^{12}(a^2(I_{\langle k,j\rangle}) - a^2(I))\} \text{ and}$$
$$a^2(I) - a^2(I_{\langle 1,j\rangle}) \ge \min\{0,\ \alpha^{21}(a^1(I_{\langle k,j\rangle}) - a^1(I))\} \text{ for any } j \in S_k.$$

The main difference is that the approach based on the Edgeworth's proposition may lead to a situation where a player sacrifices a small part of his/her pay-off to the benefit of another player. Our approach does not imply such a possibility.

Note that in the general case, the Edgeworth's A-equilibrium in game (**S**,**a**) is equivalent to the Nash equilibrium in game (**S**,**a**′) with the linearly transformed pay-off function: $\mathbf{a}'(I) = A\,\mathbf{a}(I)$ for any $I \in S$.

2. Altruism and cooperation

One important consequence of altruistic behavior is that it may lead to cooperation among players. We will prove that if the players are altruistic enough, then there exists an altruistic equilibrium which is efficient (Pareto optimal).

Definition 4. We call strategy profile I **locally efficient**, if there does not exist $k \in p$, $j \in S_k \setminus \{i_k\}$ such that

$$a^k(I_{\langle k,j\rangle}) > a^k(I) \text{ and } a^l(I_{\langle k,j\rangle}) \geq a^l(I) \text{ for any } l \in N_p \setminus \{k\}.$$

In other words, I is locally efficient if it is not "dominated" by any "neighbor" strategy profile $I_{\langle k,j\rangle}$. Here "domination" differs from the Pareto domination relation by the requirement $a^k(I_{\langle k,j\rangle}) > a^k(I)$, and "neighborhood" of strategy profiles is understood as difference in only one player's strategy.

Theorem 1. *Let $I \in \boldsymbol{S}$. There exists $A = (\alpha_{kl}) \in \mathbf{R}^{p\times p}_{>}$ such that I is an A-equilibrium if and only if I is locally efficient.*

Proof. Sufficiency. Suppose that I is locally efficient. Then for any player k such that:

$$a^k(I_{\langle k,j\rangle}) > a^k(I) \text{ for some strategy } j \in S_k, j \neq i_k,$$

there exists another player l such that

$$a^l(I_{\langle k,j\rangle}) < a^l(I).$$

If α_{lk} satisfies

$$\alpha_{lk} \geq \frac{a^k\left(I_{\langle k,j\rangle}\right) - a^k(I)}{a^l(I) - a^l\left(I_{\langle k,j\rangle}\right)},$$

then we have (1). It follows that I is an A-equilibrium, if the altruistic coefficients are large enough.

Necessity. If I is not locally efficient, then for some $k \in p$ and some $j \in S_k \setminus \{i_k\}$ we have

$$a^k(I_{\langle k,j \rangle}) > a^k(I) \text{ and } a^l(I_{\langle k,j \rangle}) \geq a^l(I) \text{ for any } l \in N_p \setminus \{k\}.$$

It follows that there does not exist $l \in N_p$, $l \neq k$, and positive α_{lk} satisfying (1). Therefore I is not an A-equilibrium for any altruistic coefficients. □

It is evident that any Pareto optimal strategy profile is locally efficient. Therefore Theorem 1 implies:

Corollary 1. *For any Pareto optimal strategy profile $I \in \mathbf{S}$ in game ($\mathbf{S}$, $\mathbf{a}$), there exists $A \in \mathbf{R}_{>}^{p \times p}$ such that I is an A-equilibrium.*

Actually, Corollary 1 is a stronger proposition than the existence of an altruistic equilibrium being Pareto optimal. We have proved that **any** Pareto optimal strategy profile can be an altruistic equilibrium, if the altruistic coefficients of players are large enough. Observe that the existence of a Nash equilibrium in the game is not required.

Let us illustrate the altruistic equilibrium concept by the example of the prisoner's dilemma game. The classical interpretation of the game is that both players are suspected in a crime they committed together. They are separated from each other and interrogated simultaneously. Each of them have to decide either to betray the partner or to stay silent. The absolute values of pay-offs indicate how many years of imprisonment will a player get depending on both players' decisions.

Example 1. *Denote the players by Player A and Player B. The pay-off matrix is following:*

	Player A stays silent	***Player A betrays***
Player B stays silent	*(–0.5, –0.5)*	*(0, –10)*
Player B betrays	*(–10, 0)*	*(–2, –2)*

Here the two numbers in parentheses denote Player A's and Player B's pay-offs, respectively.

The paradox is that the cooperative solution (stay silent, stay silent) is not an equilibrium (in the sense of Nash), if the players behave rationally. On the contrary, the unique equilibrium is (betray, betray) which yields a non-efficient outcome.

Now suppose that both altruistic coefficients are equal to 0.1.

Consider the situation where Player A stays silent. If Player B had betrayed instead of staying silent, he/she would condemn Player A to additional 9.5 years of imprisonment while avoiding only 0.5 year imprisonment for him/herself. The pay-off loss of Player A multiplied by the altruistic coefficient is greater than the pay-off gain of Player B ($9.5 \cdot 0.1 > 0.5$). Then according to our assumption, Player B prefers to stay silent. Analogously, if Player B stays silent, then Player A prefers to stay silent too. Thus, (stay silent, stay silent) is an equilibrium in the sense that no player deviates from his/her strategy if the partner does not.

3. Characterization of strategy profiles in terms of altruistic equilibrium

According to Theorem 1, any (and only such) locally efficient strategy profile can be an altruistic equilibrium for sufficiently large altruistic coefficients. The following question arises: *for what values of altruistic coefficients a given strategy profile is an altruistic equilibrium?*. Answering this question means characterizing a locally efficient strategy profile I by a set of matrices $\Omega(I)$ such that I is an A-equilibrium if and only if $A \in \Omega(I)$. We can build such a characterization with the help of the trade-off concept.

Trade-off coefficients are widely used in multiple criteria decision making to characterize solutions in terms of partial preferences concerning relative importance of criteria (see Kaliszewski [2006]). We define the trade-off coefficient in a game as the ratio between the improvement of a player's pay-off and the worsening of another player's pay-off caused by the former player's strategy change.

Definition 5. *For any strategy profile I, any pair of players $k,\ l \in N_p$, $k \neq l$, and any k-th player's strategy $j \in S_k$ such that $a^k(I_{\langle k,j\rangle}) > a^k(I)$ and $a^l(I_{\langle k,j\rangle}) < a^l(I)$, the number*

$$T_{kl}(I,j) = \frac{a^k\left(I_{\langle k,j\rangle}\right) - a^k(I)}{a^l(I) - a^l\left(I_{\langle k,j\rangle}\right)}$$

*is called **altruistic trade-off coefficient** of player k with respect to player l for strategy profile I and strategy j.*

In the following obvious proposition, we reformulate the definition of A-equilibrium in terms of altruistic trade-off coefficients.

Proposition 1. *A locally efficient strategy profile I is an A-equilibrium if and only if for any player* $k \in N_p$ *and any strategy from his/her strategy set* $j \in S_k$, $j \neq i_k$, *the following implication holds:*

$$\text{if } a^k(I_{\langle k,j\rangle}) > a^k(I), \text{ then there exists}$$
$$\text{another player } l \in N_p \text{ such that } a^l(I_{\langle k,j\rangle}) < a^l(I) \text{ and } T_{kl}(I,j) \leq \alpha_{kl}.$$

Let us apply this proposition to characterize the cooperative solution of the Prisoner's Dilemma game

Example 2. *Consider the Prisoner's Dilemma game described in Section 3, where the players and the strategies are numbered in the following way: Player A = 1, Player B = 2, "stay silent" = 1 and "betray" = 2.*

Consider the strategy profile $I := (1,1)$. *It is locally efficient. Let us calculate the altruistic trade-off coefficients for I:*

$$T_{12}(I,2) = T_{21}(I,2) = 0.5/9.5 = 1/19.$$

According to Proposition 1, strategy profile I is an A-equilibrium if and only if $\alpha_{12} \geq 1/19$ *and* $\alpha_{21} \geq 1/19$. *So it suffices that each player considers the other player's interests 19 times less important than his/her own interests, to make the cooperation possible.*

It is easy to characterize a strategy profile in a game with two players with the help of the following evident corollary from Proposition 1.

Corollary 2. *Let* $p = 2$. *A locally efficient strategy profile* $I = (i_1,i_2)$ *is an A-equilibrium if and only if* $\alpha_{12} \geq \tau_{12}$ *and* $\alpha_{21} \geq \tau_{21}$, *where*

$$\tau_{kl} = \max\left\{T_{kl}(I,j) : j \in S_k,\ j \neq i_k,\ a^k(I_{\langle k,j\rangle}) > a^k(I),\ a^l(I_{\langle k,j\rangle}) < a^l(I)\},\right.$$
$$\left.(k,l) \in \{(1,2),(2,1)\}\right\}$$

and the maximum over the empty set is assumed to be zero.

Unfortunately, in a game with more than two players it is impossible to characterize a strategy profile by lower bounds of altruistic coefficients. In other words, it is impossible to represent the characterization in the following form: *the strategy profile is A-equilibrium if and only if* $\alpha_{kl} \geq \tau_{kl}$ *for any* $k,l \in N_p$, $k \neq l$, *where* τ_{kl} *is the lower bound for altruistic coefficient*. This difficulty is illustrated by the following example.

Example 3. *Consider the game with 3 players each having 2 strategies and following pay-off functions:*

Player 1 pay-off function

	$i_2=1$		$i_2=2$	
	$i_3=1$	$i_3=2$	$i_3=1$	$i_3=2$
$i_1=1$	**4**	**3**	**0**	**1**
$i_1=2$	**5**	**1**	**1**	**3**

Player 2 pay-off function

	$i_1=1$		$i_1=2$	
	$i_3=1$	$i_3=2$	$i_3=1$	$i_3=2$
$i_2=1$	**3**	**3**	**5**	**1**
$i_2=2$	**4**	**1**	**1**	**4**

Player 3 pay-off function

	$i_1=1$		$i_1=2$	
	$i_2=1$	$i_2=2$	$i_2=1$	$i_2=2$
$i_3=1$	**10**	**2**	**0**	**1**
$i_3=2$	**8**	**1**	**1**	**6**

It is easy to check that the strategy profile $I := (1,1,1)$ is locally efficient. Let us characterize it with the help of altruistic trade-off coefficients.

Altruistic trade-off coefficients of Player 1: $T_{13}(I,2) = 0.1$; $T_{12}(I,2)$ *is undefined because when Player 1 changes his/her strategy from 1 to 2, Player's 2 pay-off is not decreased.*

Altruistic trade-off coefficients of Player 2: $T_{21}(I,2) = 0.5$; $T_{23}(I,2) = 0.25$.

Altruistic trade-off coefficients of Player 3 *are undefined because $a^3(1,1,2) < a^3(1,1,1)$, which means that Player 3 is not interested in changing his/her strategy. So the degree of this player's altruism does not influence the equilibrium.*

Applying Proposition 1, we obtain that I is an A-equilibrium if and only if

$$\alpha_{13} \geq 0.1 \text{ and } (\alpha_{21} \geq 0.5 \text{ or } \alpha_{23} \geq 0.25).$$

Thus, instead of a set of constraints on the altruistic coefficients, we have a logical expression which does not necessarily include all of them.

In general, the strategy profile characterization implied by Proposition 1 can be formulated as follows:

Locally efficient strategy profile I is A-equilibrium if and only if

$$\bigwedge_{k \in \hat{N}(I)} \bigwedge_{j \in \hat{S}_k(I)} \bigvee_{l \in \tilde{N}(I,k,j)} \left(\alpha_{kl} \geq T_{kl}(I,j)\right) \tag{2}$$

where

$\hat{N}(I) = \left\{k \in N_p : \hat{S}_k(I) \neq \emptyset\right\}$ is the subset of players who can improve their pay-offs by changing their strategies,

$\hat{S}_k(I) = \left\{j \in S_k : j \neq i_k, a^k\left(I_{\langle k,j \rangle}\right) > a^k(I)\right\}$ is the subset of player k's strategies, for which his/her pay-off is greater than the initial pay-off,

$\tilde{N}(I,k,j) = \left\{l \in N_p : a^l\left(I_{\langle k,j \rangle}\right) < a^l(I)\right\}$ is the set of players who suffer from player k changing his/her strategy to j.

Thus, the set of matrices characterizing a strategy profile may have a rather complicated structure in the case of a large number of players.

Conclusion

We presented an approach to modeling equilibrium in multi-player non--zero sum games taking into account the relative preferences of players, namely the relative importance of their own gains and other players' losses. We suppose that in addition to striving for their own profit, players are concerned with not harming the interests of other players disproportionately, which can be referred to as altruism. Such a deviation from the egoistic behavior in real life may be conditioned by moral and ethic concerns, fear, reputation concern, and many other motivation factors.

The intensity of mutual altruism of players is quantified by a non--negative altruistic coefficient. When the altruistic coefficients are zero, the altruistic equilibrium is reduced to the Nash equilibrium, so the concept proposed may be considered as a generalization of the Nash equilibrium concept.

Our approach does not require to characterize player preferences in terms of a utility function (in contrast to other approaches, see Fehr and Schmidt [1999], Bolton and Ockenfels [2000], Charness and Rabin [2002]). On the contrary, the proposed equilibrium condition is based on direct comparison of pay-off differences. It is worth to note that the proposed model is not the only possible formalization of the player altruism concept in terms of relative importance preferences. The model based on the Edgeworth's proposition (see the end of Section 2) describes a slightly different variant of the altruistic behavior.

The clear interpretation of the model proposed makes it useful for analyzing equilibrium situations in terms of relative preferences. For example, after estimating the degree of player altruism, one can describe a range of possible equilibria. And vice versa, analyzing the information about equilibria achieved and unachieved among locally efficient solutions, one can estimate the degree of player altruism in terms of bounds on altruistic coefficients. This can be done by characterizing strategy profiles in terms of altruistic trade-off coefficients (see Section 4).

Another possible application of the model proposed is in the field of repeatedly played games. This research area attempts to explain cooperative behavior through natural selection mechanisms. For example, Axelrod [1980, 1984] conducted game tournaments with two players and found out a long-term incentive for cooperation in their behavior. In the framework of evolutionary approach, Robson [1990] proposed a model where a prisoner's dilemma-like game is repeatedly played in a population of players and there are "mutants" who cooperate by playing with other "mutants" and betray by playing with the rest of individuals. An invasion of "mutants" displays the advantage of the cooperation strategy. Chlebuś et al. [2009] built a computer simulation of a society, where economic activity is modeled dynamically by repetitive playing of random prisoner's dilemma-like games. By varying parameters of players' behavior, one can analyze how the propensity to cooperate influences social welfare. Other examples of the evolutionary approach applied to the game theory can be found in Nowak and Sigmund [2004], Szabó and Fáth [2007]. Our model can be used to quantify the degree of player altruism in evolutionary simulations. It would be interesting to trace the dependence between the inclination to altruism and the survivability or welfare of the player population.

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