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COMPROMISE HYPERSPHERE FOR STOCHASTIC DOMINANCE MODEL

Abstract

The aim of the work is to present a method of ranking a finite set of discrete random variables. The proposed method is based on two approaches: the stochastic dominance model and the compromise hypersphere. Moreover, a numerical illustration of the method presented is given.

Keywords

Stochastic dominance, compromise programming, multiple criteria optimization.

Introduction

This paper presents a method of ranking a finite set of discrete random variables. The method is based on one of the multiple criteria methods: the compromise hypersphere, Gass and Roy [2003]. The source of the compromise hypersphere is the compromise programming, Charnes and Cooper [1957], Zeleny [1982]. Adaptations of the compromise hypersphere, in optimization with random variables, are based on stochastic dominance, Levy [1992]. The proposed method consists of the following steps:

Step 1. Establish feasible decisions and corresponding random variables.

Step 2. Compute nondominated random variables in the sense of stochastic dominance.

Step 3. Find the compromise hypersphere.

Step 4. Build a ranking of nondominated random variables using the compromise hypersphere.

Our paper consists of four sections: Section 1 presents a description and properties of the compromise hypersphere; in Section 2 a model of stochastic dominance is considered; Section 3 presents the four steps of the method in detail and the numerical illustration of the proposed algorithm is presented in section 4. The paper concludes with remarks and suggestions for further research.

1. Compromise hypersphere

The presented method originates in the work of Gass and Roy [2003]. The aim of this method is to rank the finite set of nondominated vectors $\mathbf{y}^1 \in R^n$, … , $\mathbf{y}^m \in R^n$. In detail, the method looks as follows:

1. Solve the program:

$$\min_{\mathbf{y}^0, r_0} \max_{i=1,\dots,m} \left| r_0 - d\left(\mathbf{y}^i, \mathbf{y}^0\right) \right|, \tag{1}$$

where

d: $R^n \times R^n \rightarrow R$ denotes the distance between two vectors.

We denote the optimal solution of (1) by $\overline{\mathbf{y}^0}$, $\overline{r^0}$ and the minimal value of the cost function as $\overline{\min(1)}$.

2. Find the ranking of the points $\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^m$ based on the distances:

$$| \overline{r^0} - d(\overline{\mathbf{y}^0}, \mathbf{y}^i)| \qquad i = 1, \dots, m. \tag{2}$$

In particular, we look for the point $\mathbf{y}^i$ closest to the hypersphere:

$$\min_{i=1,\dots,m} | \overline{r^0} - d(\overline{\mathbf{y}^0}, \mathbf{y}^i)| . \tag{3}$$

Remark 1

Problem (1) is to find a hypersphere with the centre $\mathbf{y}^0 \in R^n$ and the radius $r_0 \in R$ with a minimal distance from the set $\{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^m\}$.

Remark 2

In problem (1) one can use the well known family of metrics $l^p : R^n \times R^n \rightarrow R$ as the function d with the parameter $p \in [1,\infty]$. The function $l^p : R^n \times R^n \rightarrow R$ is described as follows:

$$l^p(\mathbf{y},\mathbf{z}) = \begin{cases} \left(\sum_{j=1}^{n} \left|y_j - z_j\right|^p\right)^{\frac{1}{p}}, & p \in [1,\infty) \\ \max_{j=1,\dots,n} \left|y_j - z_j\right|, & p = \infty \end{cases}$$

where $\mathbf{y} = (y_1, \dots, y_n) \in R^n$, $\mathbf{z} = (z_1, \dots, z_n) \in R^n$.

Remark 3

In general, problem (1) is a complicated optimization problem and we use genetic algorithms to solve it, Koza [1992, 1994].

Remark 4

Problem (3) is to find the point closest to the hypersphere found in step 1. Problems (2) and (3) are trivial; it is enough to compare n numbers, used in step 1.

2. Stochastic dominance

In this section, we use the first order stochastic dominance, Shaked and Shanthikumar [1993], Ogryczak and Ruszczynski [1999].

The relation of the first order stochastic (FSD) dominance is defined as follows:

$$\xi_1 \leq_{\text{FSD}} \xi_2 \Leftrightarrow \forall_{x \in \mathrm{R}}\ F_{\xi_1}(x) \geq F_{\xi_2}(x),$$

where $F_\xi(x) = \mathbb{P}(\xi \leq x)$ is the right-continuous cumulative distribution function of the random variable ξ. We consider the family of discrete random variables $\{\xi_i : i = 1, 2, \dots, m\}$. Moreover, we assume that the following set:

$$\mathrm{X} = \{x \in R \colon \exists_{i \in \{1,2,\dots,m\}}\ \mathrm{P}(\xi_i = x) > 0\}$$

is finite. It means that we are able to enumerate the elements of the set X in the following way:

$$\mathrm{X} = \{x_1, x_2, \dots, x_n\},$$

Where $x_s < x_t$ for $s < t$.

We call ξ^* a nondominated random variable in set $\Omega = \{\xi_i \colon i = 1, 2, \dots, m\}$ in the sense of FSD if

$$\neg \exists_{\xi\in\Omega} \quad \xi^* \leq_{FSD} \xi \wedge F_\xi \neq F_{\xi^*} .$$

We build the vector $\mathbf{y}^i$ connected with discrete random variables ξ_i in the following way:

$$\mathbf{y}^i = [\, y_1^i,\ y_2^i\,, \dots ,\ y_n^i\,] = [\, F_{\xi_i}(x_1), F_{\xi_i}(x_2), \dots , F_{\xi_i}(x_n)\,].$$

In this case the FSD relation has the following form:

$$\xi_1 \leq_{FSD} \xi_2 \Leftrightarrow \mathbf{y}^1 \geq \mathbf{y}^2 \ \wedge\ \mathbf{y}^1 \neq \mathbf{y}^2.$$

Some additional aspects of FSD models one can find in papers by Ogryczak [2002] and Ogryczak and Romaszkiewicz [2001].

3. Method of ranking

The aim of the proposed procedure is to choose a decision from a finite set of decisions. The returns of decisions are described by means of random variables. The method is based on the stochastic order and the compromise hypersphere method. The procedure looks as follows:

Step 1. Establish feasible decisions with corresponding random variables $\{\xi_i : i = 1, 2, \dots , m\}$ and the right-continuous cumulative distribution function.

We obtain:

$$\mathbf{y}^1, \mathbf{y}^2, \dots , \mathbf{y}^m,$$

where $\mathbf{y}^i = [\, y_1^i,\ y_2^i, \dots ,\ y_n^i\,] = [\, F_{\xi_i}(x_1), F_{\xi_i}(x_2), \dots ,\ F_{\xi_i}(x_n)\,].$

Step 2. Compute nondominated vectors in the set $\{\mathbf{y}^1, \mathbf{y}^2, \dots , \mathbf{y}^m\}$ in the sense of minimalization, i.e. $\mathbf{y}^j$ is nondominated if

$$\neg \underset{i=1,\dots,m}{\exists} \mathbf{y}^j \leq \mathbf{y}^i \ \wedge\ \mathbf{y}^j \neq \mathbf{y}^i \quad .$$

We obtain

$$\mathbf{y}^{i_1}, \mathbf{y}^{i_2}, \dots , \mathbf{y}^{i_{kp}} \quad (p \leq m).$$

The above vectors are connected with the nondominated random variables in the sense of FSD.

Step 3. Solve problem (1) for $\mathbf{y}^{i_1}, \mathbf{y}^{i_2}, \dots , \mathbf{y}^{i_p}$.

Step 4. Use values (2) to obtain the ranking of $\mathbf{y}^{i_1}, \mathbf{y}^{i_2}, \dots , \mathbf{y}^{i_p}$ and corresponding nondominated random variables in the sense of FSD.

4. Example

Step 1. Let us consider a set of seven discrete random variables:

$$\xi_i,\quad i \in \{1, 2, \dots, 10\}.$$

The probabilities characterizing these random variables are presented in table 1.

Table 1

Description of random variables

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}
$P(\xi_i=0)$	0	0.3	0.4	0.1	0	0.2	0.2	0.1	0.4	0.1
$P(\xi_i=1)$	0	0.1	0	0.4	0.5	0.3	0.1	0.6	0	0.4
$P(\xi_i=2)$	1	0.1	0	0.3	0.4	0.1	0.4	0.1	0.4	0.4
$P(\xi_i=3)$	0	0.5	0.6	0.2	0.1	0.4	0.3	0.2	0.2	0.1

Vectors $\mathbf{y}^i$ built for the random variables considered are presented in table 2.

Table 2

Vectors $\mathbf{y}^i$ for considered random variables

$\mathbf{y}^1$	$\mathbf{y}^2$	$\mathbf{y}^3$	$\mathbf{y}^4$	$\mathbf{y}^5$	$\mathbf{y}^6$	$\mathbf{y}^7$	$\mathbf{y}^8$	$\mathbf{y}^9$	$\mathbf{y}^{10}$
0	0.3	0.4	0.1	0	0.2	0.2	0.1	0.4	0.1
0	0.4	0.4	0.5	0.5	0.5	0.3	0.7	0.4	0.5
1	0.5	0.4	0.8	0.9	0.6	0.7	0.8	0.8	0.9
1	1	1	1	1	1	1	1	1	1

Step 2. Compute the nondominated vectors in the set $\{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^{10}\}$. The nondominated vectors are as follows:

$$\{\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^4, \mathbf{y}^5, \mathbf{y}^6, \mathbf{y}^7\}.$$

We denote the set of indices of the nondominated vectors by N, i.e.: $N = \{1, 2, 3, 4, 5, 6, 7\}$.

Step 3. By solving problem (1) with the set $\{\mathbf{y}^i\colon\ i \in N\}$ and $d = l^2$:

$$\min_{\mathbf{y}^0, r_0} \max_{i \in N} \left| r_0 - \sqrt{\sum_{j=1}^{4} \left(y_j^0 - y_j^i \right)^2} \right|,$$

we obtain the following optimal solution:

$$\overline{\mathbf{y}^0} = (-0.73808;\ 0.05522; -0.02151; -2.39575), \quad \overline{r^0} = 3.61347$$

and the minimal value of the cost function:

$$\overline{\min(1)} = 0.00908.$$

Step 4. By solving problem (3)

$$\min_{i \in N} \left| \overline{r^0} - \sqrt{\sum_{j=1}^{4} \left(\overline{y_j^0} - y_j^i \right)^2} \right|,$$

we obtain values (as distances between points and the hypersphere) shown in Table 3. Moreover, Table 3 presents the ranking based on these values.

Table 3

Ranking for $d = l^2$

	$\mathbf{y}^1$	$\mathbf{y}^2$	$\mathbf{y}^3$	$\mathbf{y}^4$	$\mathbf{y}^5$	$\mathbf{y}^6$	$\mathbf{y}^7$
$\left\| \overline{r^0} - \sqrt{\sum_{j=1}^{4} \left(\overline{y_j^0} - y_j^i \right)^2} \right\|$	0.00902	0.00798	0.00908	0.00678	0.00908	0.00858	0.00908
Ranking	4	2	5	1	5	3	5

Conclusions and further research

In this paper we have proposed a method of ranking discrete random variables. We have used two approaches: the stochastic dominance and the compromise hypersphere. In future, the following aspects of the presented method are worth studying: comparing with other methods of random variables ranking, the case of continuous random variables, an interactive version of the method, analysis of the method for different metrics *d*, applications to real life problems.

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