Tadeusz Trzaskalik

Sebastian Sitarz

DYNAMIC STOCHASTIC PROBLEMS OF PROFIT MAXIMIZATION WITH PARTIALLY ORDERED CRITERIA SPACE

Abstract

Stochastic dynamic programming (DP) is a strong mathematical tool allowing modeling and solving many multiperiod decision processes. Multiple objective and dynamics characterize many sequential decision problems. In the paper we consider returns in partially ordered criteria set as a way of generalization of single criterion DP models to multiobjective case.

In the present paper, on the basis of theoretical findings, described in our previous papers we consider exemplary stochastic DP profit maximization processes. Because of the lack of space we omit the general, formal description of such a process and concentrate on explanation, how the theory of DP models in partially ordered criteria space works. Both in level-volume and velocity-volume process we will consider formulated problems step by step, first as single criterion problems and next as bi-criteria ones. Conclusions are presented in the last section.

Keywords

Stochastic dynamic programming, multiobjective dynamic optimization, profit maximization, partially ordered criteria space.

Introduction

Stochastic dynamic programming (DP) is a strong mathematical tool for modeling and solving many multiperiod decision processes. There are many stochastic DP applications in different fields. One of the most important of them is the profit maximization problem. Different aspects of this problem have been considered in literature. Recently Teunter [12] proposed a stochastic DP algorithm for determining the optimal disassembly and recovery strategy, the quality-dependent recovery options and associated profits for assemblies. Zhang and Piplani [19] utilized yield management technique and stochastic DP modeling to achieve maximization of expected profit in Make-To-Order manufacturing companies. Jonker et al. [13] present a joint optimization approach addressing the segmentation of customers into homogeneous groups and determining the optimal policy towards each segment. They propose a stochastic DP procedure based on the long-run maximization of expected average profit. Kamrad and Siddique [4] consider supply chain contracts as the producer's profit maximization problem with respect to the supplier's reaction and analyze risk reduction in a unique framework as a stochastic DP problem. Sboui et al. [11] present a profit maximization stochastic DP model for supply chain management.

Many sequential decision problems are characterized by multiple objective and dynamics. Research extending the principle of optimality formulated in Bellman [1] to multiobjective case made it possible to apply the vector principle of optimality to deterministic, stochastic and fuzzy problems. A review of multiobjective dynamic programming (MODP) models was done by Li and Haimes [6] and more recently by Trzaskalik [13].

Changeable hierarchy problems belong to the most challenging issues in MODP. Period criteria for separate periods and multiperiod criteria for the whole process can be distinguished. A period criterion is called important in a given period if it is considered in the evaluation of the process in that period. The following questions can be asked: how multiperiod criteria depend on period criteria and how to define preference structure in such a case? The notion of importance of criteria and the definition of preference structure was introduced and elaborated by Trzaskalik [16, 15, 14].

Another way of generalization of single criterion DP models is to consider returns in partially ordered criteria space. First attempts were done by Mitten [7], Sobel [9], Steinberg and Parks [10], Henig [2]. More recently, discrete DP problems with partially ordered criteria space were considered by Trzaskalik and Sitarz [18, 17]. It is worth noticing that MODP is based on the Pareto concept of optimality that determines a partial order in the criteria space, so each MODP model is also a DP model with returns in partially ordered criteria space. On the other hand, there exist single criterion DP problems with returns in partially ordered criteria space, which obviously are not MODP problems. Examples can be found below.

In the present paper we will consider stochastic problems of profit maximization as examples of DP problems in partially ordered criteria space. We will consider a situation in which important criteria for consecutive periods depend on the progress of the process until now, and, in particular, on the cumulated values of criteria from the beginning of the process.

The problem is stated as follows. We consider an investment multiperiod decision process. Decisions can be made at the beginning of consecutive periods. Probabilistic distributions of period returns are known. Two situations are of interest.

In the *level-volume* case we assume that the decision maker applies two criteria:

1a. The profit should be greater or equal to a given level (*level criterion*).

2. The profit should be as big as possible (*volume criterion*).

In the *velocity-volume* case it is assumed that the decision maker's criteria are as follows:

1b. The profit should be greater or equal to a given level as soon as possible (*velocity criterion*).

2. The profit should be as big as possible (*volume criterion*).

Such problems are examples of multiperiod, multiobjective processes, whose sets of important criteria depend on cumulated values of profit from the beginning of the process. We will define the set of important criteria as a function of these values. Such problems are close to real-life problems considered by decision makers in financial assessment of investments.

In the present paper, on the basis of theoretical findings described in our previous papers [17, 18] we will consider examples of stochastic DP profit maximization processes. Because of the lack of space we will omit the general, formal description of such processes and concentrate on explanation of how the theory of DP models in partially ordered criteria space works. Both in the level-volume and velocity-volume process we will consider the problems step by step, first as single criterion problems and next as bi--criteria ones.

The paper consists of six sections. In Section 1 we will describe the considered process and in particular, its dynamics and outcomes. In Section 2 we will consider the level-volume case and in Section 3, the velocity-volume case. In Section 4 we will review all the process realizations from the point of view of criteria considered and problems solved. Conclusions are presented in section 5.

1. Description of the process

Let us consider an example of the multiperiod decision process, presented in Figure 1. The nodes of the graph correspond to the states of the process and the arcs correspond to the feasible period decisions. Paths in the graph leading from the initial states of the process to its end correspond to process realizations. We assume that the transition functions of the process are deterministic. This means that if in a given state a decision is made, then the state of the process at the beginning of the next period is determined by means of the appropriate transition function. Outcomes (profits) of the process in subsequent periods are realizations of discrete random variables with given distributions. The values on arcs are intepreted as probabilities of profits equal to 0, 1, 2,... For instance, if at the beginning of period 4 the process is in the state 0 and we take decision 0, the probability of profits are realizations of period random variable $\xi_4(0,0)$. We have the following probabilities (see Figure 1):

$$P[\xi_4(0,0)=0] = 0.1 \quad P[\xi_4(0,0)=1] = 0.2 \quad P[\xi_4(0,0)=2] = 0.7$$

We denote this probability distribution as (0.1, 0.2, 0.7).

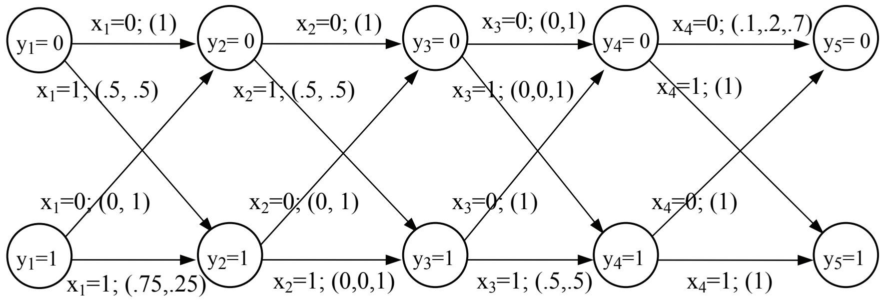

Figure 1. The graph of the process under consideration

Let us consider a process realization starting in the state y_1 and consisting of states and decisions: y_1, x_1, y_2, x_2, y_3, x_3, y_4, x_4, y_5. Since the transition functions are deterministic, it is sufficient to consider the states only and to omit the decisions, so we denote this realization as $d = (y_1, y_2, y_3, y_4, y_5)$. The total outcome for the realization d is a realization of random variable $\xi(d)$, which is the sum of realizations of random variables $\xi_t(y_t, x_t)$ for t=1,...,4, hence

$$\xi(d) = \xi_1(y_1, x_1) + \xi_2(y_2, x_2) + \xi_3(y_3, x_3) + \xi_4(y_4, x_4)$$

For instance, for the process realization $d^0 = (0, 0, 0, 0, 0)$ we have

$$\xi(d^0) = \xi_1(0,0) + \xi_2(0,0) + \xi_3(0,0) + \xi_4(0,0)$$

It is easy to find the probability distribution $p(d^0)$ for $\xi(d^0)$. We obtain:

$$P[\xi(d^0)=0]=0 \qquad P[\xi(d^0)=1]=0.1$$

$$P[\xi(d^0)=2]=0.2 \qquad P[\xi(d^0)=3]=0.7$$

We denote the set of all process realization as D and the set of all discrete probability distributions for all the realizations of the process as $\Xi(D)$.

Let us consider the second example of the realization of the process, for instance $d^4 = (0, 0, 1, 0, 0)$ (for the numbering of realizations see Table 1). The probability distribution for $\xi(d^4)$ is as follows:

$$P[\xi(d^4)=0] = 0.05 \quad P[\xi(d^4)=1] = 0.15$$
$$P[\xi(d^4)=2] = 0.45 \quad P[\xi(d^4)=3] = 0.35$$

We can compare process realizations according to the FSD (first stochastic dominance) rules (see [8]). Let us compare d^0 and d^4. For the realizations considered we obtain the following cumulated values:

1. Realization d^0:

$$c_0(d^0) = P[\xi(d^0)=0] = 0$$
$$c_1(d^0) = P[\xi(d^0)=0] + P[\xi(d^0)=1] = 0.1$$
$$c_2(d^0) = P[\xi(d^0)=0] + P[\xi(d^0)=1] + P[\xi(d^0)=2] = 0.3$$
$$c_3(d^0) = P[\xi(d^0)=0] + P[\xi(d^0)=1] + P[\xi(d^0)=2] + P[\xi(d^0)=3] = 1$$
$$c_k(d^0) = 1 \text{ for } k \geq 4$$

2. Realization d^4:

$$c_0(d^4) = P[\xi(d^4)=0] = 0.05$$
$$c_1(d^4) = P[\xi(d^4)=0] + P[\xi(d^4)=1] = 0.2$$
$$c_2(d^4) = P[\xi(d^4)=0] + P[\xi(d^4)=1] + P[\xi(d^4)=2] = 0.65$$
$$c_3(d^4) = P[\xi(d^4)=0] + P[\xi(d^4)=1] + P[\xi(d^4)=2] + P[\xi(d^4)=3] = 1$$
$$c_k(d^4) = 1 \text{ for } k \geq 4$$

Since for each k=0,1,2,... we have $c_k(d^0) \leq c_k(d^4)$ and $c_k(d^0) \neq c_k(d^4)$, it means that $\xi(d^0)$ dominates $\xi(d^4)$ according to the first degree stochastic dominance rule. We denote it as $p(d^0)$ FSD $P(d^4)$.

2. Level-volume case

2.1. Level criterion

The level criterion is important as long as the cumulated profit is less than 2. Because we analyze the process *ex ante*, before it has started, we are only interested in process realizations, whose cumulated probability distributions have the form $(r_0, r_1, r_2, \ldots,)$, and

$$r_0 = r_1 = 0 \tag{1}$$

Probability distributions for all these realizations dominate (according to FSD rule) the "weakness" probability distribution $\overline{p}$ = (0, 0, 1).

For any process realization the multiperiod level criterion function F^L takes one of the following values:
1 – assumed level of profit will be reached,
0 – assumed level of profit will not be reached.
It is possible to find the value $F^L(d)$ applying the formula

$$F^L(d) = \begin{cases} 1, & \text{if} \quad p(d)\ \text{FSD}\ \overline{p} \quad \text{or} \quad p(d) = \overline{p} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Depending on the value $F^L(d)$, each realization d is classified to one of two classes. Let $D(i)=\{d: F^L(d)=i\}$ $(i=0,1)$ be the set of all process realizations from the class i. We have $D(0) \cup D(1)=D$, $D(0) \cap D(1) = \varnothing$.

The preference structure can be described as follows: each process realization from D(1) dominates any realization from D(0). Realizations belonging to the same class are equally preferred. Let D^L denote the set of efficient realizations. We have $D^L = D(1)$. The set D^L can be obtained by means of the forward procedure for dynamic process with partially ordered criteria space (see [17]). Process realizations belonging to D^L are marked in Table 1.

2.2. Volume criterion

Volume criterion is important in all the periods considered. The multiperiod volume criterion function F^V has the form:

$$F^V(d) = p(d) \tag{3}$$

Let d^i, d^j, $d^i \neq d^j$ be feasible process realizations. d^i dominates d^j iff $p(d^i)$ FSD $p(d^j)$. A realization d^V is efficient if there doesn't exist any other realization d such that:

$$p(d)\ \text{FSD}\ p(d^V) \tag{4}$$

The set D^V of all efficient realizations can be obtained by means of forward dynamic procedure. Process realizations belonging to D^V are marked in Table 1 (column 12).

2.3. Bi-criteria case

The criteria space is defined as the product $\{0, 1\} \times \Xi(D)$. The vector multiperiod criterion function has the form $F^{LV} = [F^L, F^V]'$. For each process realization d we have $F^{LV}(d) = [i, p(d)]'$ ($i\in\{0,1\}$, $p(d)\in\Xi(D)$). The preference structure is defined as follows. A realization d^{LV} is efficient if

$$\neg \underset{d}{\exists} \quad i \geq i_{LV} \ \wedge \ p(d) \text{ FSD } p(d^{LV}) \wedge (i \neq i_{LV} \vee p \neq p^{LV}) \qquad (5)$$

The set D^{LV} of all the efficient realizations can be obtained by means of the forward dynamic procedure. Process realizations belonging to D^{LV} are marked in Table 1 (column 13).

Let us notice that some maximal elements from the criteria space are generated by more than one process realization.

3. Velocity-volume case

3.1 Velocity criterion

The velocity criterion is important as long as the cumulated profit is less than 2. The moment of achievement of the required level of profit is important – the sooner the better. Similarly as before, we are interested only in those process realizations for which the condition (1) is fulfilled.

For any process realization, the multiperiod velocity (speed) criterion function F^S takes one of the following values:

4 – assumed level of profit will be reached at the end of the period 1,
3 – assumed level of profit will be reached at the end of the period 2,
2 – assumed level of profit will be reached at the end of the period 3,
1 – assumed level of profit will be reached at the end of the period 4,
0 – assumed level of profit will not reached.

Depending on the value $F^S(d)$, each realization d is classified to one of five classes. Let $D(i)=\{d: F^S(d)=i\}$ ($i=0,\dots,4$) be the set of all process realizations from the class i. We have $\cup_{i=0\dots4}D(i)=D$, $\cap_{i=0,\dots,4}D(i)=\varnothing$.

The preference structure is described as follows: for i>j each process realization from D(i) dominates any process realization from D(j). Realizations belonging to the same class are equally preferred. Efficient realizations D^S belong to the highest non-empty class. The set D^S can be obtained by means of dynamic forward procedure. Process realizations belonging to D^S are marked in Table 1 (column 14).

3.2. Volume criterion

The set of efficient realizations with respect to the volume criterion is obtained in the same way as in Section 2.2.

3.3. Bi-criteria case

The criteria space is defined as the product $\{0, 1, 2, 3, 4\} \times \Xi(D)$. The vector multiperiod criterion function has the form $F^{SV} = [F^S, F^V]'$. For each realization $d \in D$ we have $F(d) = [i, p(d)]'$. A realization d^{SV} is efficient if condition (4) is fulfilled. The set D^{SV} of all the efficient realizations can be obtained by means of the forward dynamic procedure. Process realizations belonging to D^{SV} are marked in Table 1 (column 15).

4. Review of process realizations

The set of process realizations and the results obtained for the considered process are shown in Table 1. Its structure is as follows:
column 1 – number of realization,
column 2 – trajectory (sequence of states),
column 3 – probability of profit at the level 0,
column 4 – probability of profit at the level 1,
column 5 – probability of profit at the level 2,
column 6 – probability of profit at the level 3,
column 7 – probability of profit at the level 4,
column 8 – probability of profit at the level 5,
column 9 – value of the level criterion,
column 10 – value of the velocity criterion,
column 11 – efficient realizations for the level criterion,
column 12 – efficient realizations for the volume criterion,
column 13 – efficient realizations for the level-volume case,
column 14 – efficient realizations for the velocity criterion,
column 15 – efficient realizations for the velocity-volume case.

Table 1

Process realizations and results

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	00000	0	0		0,1	0,2	0,7							
1	00001	0	0		1									
2	00010	1	2			1				x				
3	00011	1	2			1				x				
4	00100	0	0	0,05	0,15	0,45	0,35							
5	00101	0	0	0,5	0,5									
6	00110	0	0	0,25	0,5	0,25								
7	00111	0	0	0,25	0,5	0,25								
8	01000	1	2			0,05	0,15	0,45	0,35	x	x			
9	01001	1	2			0,5	0,5			x				
1	01010	1	2				0,5	0,5		x	x	x		x
11	01011	1	2				0,5	0,5		x	x	x		x
12	01100	1	3	0	0	0,05	0,15	0,45	0,35	x	x	x	x	x
13	01101	1	3	0	0	0,5	0,5			x			x	
14	01110	1	3	0	0	0,25	0,5	0,25		x			x	
15	01111	1	3	0	0	0,25	0,5	0,25		x			x	
16	10000	1	2	0	0	0,1	0,2	0,7		x				
17	10001	1	2	0	0	1				x				
18	10010	1	2	0	0	0	1			x				
19	10011	1	2	0	0	0	1			x				
20	10100	0	0	0	0,05	0,15	0,45	0,35						
21	10101	0	0	0	0,5	0,5								
22	10110	0	0	0	0,25	0,5	0,25							
23	10111	0	0	0	0,25	0,5	0,25							
24	11000	1	2	0	0	0,075	0,175	0,575	0,175	x				
25	11001	1	2	0	0	0,75	0,25			x				
26	11010	1	2	0	0	0	0,75	0,25		x				
27	11011	1	2	0	0	0	0,75	0,25		x				
28	11100	1	3	0	0	0,075	0,175	0,575	0,175	x			x	
29	11101	1	3	0	0	0,75	0,25			x			x	
30	11110	1	3	0	0	0,375	0,5	0,125		x			x	
31	11111	1	3	0	0	0,375	0,5	0,125		x			x	

Conclusions

It is worth comparing the number of efficient realizations for single-criterion processes with the number of efficient realizations in bi-criteria cases (see Table 2).

Table 2

The number of efficient realizations

Process	Criterion 1	Criterion 2	Bi-criteria case
level-volume	22	4	4
velocity-volume	8	4	3

The number of efficient realizations in the bi-criteria case (both in level--volume and in velocity-volume problems) is less or equal to the number of efficient realizations in the single-criterion case. Such situations occur infrequently in multiobjective programming. We can explain this by recalling that our single criteria problems have outcomes in partially ordered criteria spaces which usually contain more than one maximal element (contrary to single-objective mathematical programming problems, whose criterion space is an ordered set and usually there exists a unique optimal solution). Additional explanation for the case under consideration can be found in the construction of criteria and their dependences. If the required level of profit is reached faster, cumulated profits can be bigger than profits cumulated later.

The presented solution can be extended to any multiperiod process with finite number of periods, states and decisions.

References

[1] Bellman R.E.: *Dynamic programming*. Princeton University Press, Princeton 1957.

[2] Henig M.I.: *The Principle of Optimality in Dynamic Programming with Returns in Partially Ordered Sets*. “Mathematics of Operations Research” 1985, 10, 3, pp. 462-470.

[3] Jonker J.-J., Piersma N., Van Den Poel D.: *Joint Optimization of Customer Segmentation and Marketing Policy to Maximize Long-Term Profitability*. “Expert Systems with Applications” 2004, 27, 2, pp. 159-168.

[4] Kamrad B., Siddique A.: *Supply Contracts, Profit Sharing, Switching, and Reaction Options*. "Management Science" 2004, 50, 1, pp. 64-82.

[5] Kao E.P.: *A Preference Order Dynamic Program for Stochastic Traveling Salesman Problem*. "Operations Research" 1978, 26, 6, pp. 1033-1045.

[6] Li D., Haimes Y.Y.: *Multiobjective Dynamic Programming: The State of the Art*. "Control-Theory and Advanced Technology" 1989, 5, 4, pp. 471-483.

[7] Mitten L.G.: *Preference Order Dynamic Programming*. "Management Science" 1974, 21, 1, pp. 43-46.

[8] Rolski T.: *Order Relations in the Set of Probability Distributions and Their Applications in the Queuing Theory*. "Dissertation Mathematicae" 132, PAN, Warszawa 1976.

[9] Sobel M.M.: *Ordinal Dynamic Programming*. "Management Science" 1975, 21, 9, pp. 967-975.

[10] Steinberg E., Parks M.S.: *A Preference Order Dynamic Program for a Knapsack Problem with Stochastic Reward*. "Operational Research Society Journal" 1979, 30, 2, pp. 141-147.

[11] Sboui S., Rabenasolo B., Jolly-Desodt A.-M., De Waele N.: *A Profit-Maximization Dynamic Model for Supply Chain Planning.* "Proceedings of the IEEE International Conference on Systems, Man and Cybernetics" 2002, 5, pp. 667--672.

[12] Teunter R.H.: *Determining Optimal Disassembly and Recovery Strategies*. "Omega" 2006, 34, 6, pp. 533-537.

[13] Trzaskalik T.: *Multiobjective Analysis in Dynamic Environment*. The Karol Adamiecki University of Economics, Katowice 1998.

[14] Trzaskalik T.: *Hierarchy Depending on State in Multiple Objective Dynamic Programming.* "Operations Research and Decisions" 1997, 2, pp. 65-73.

[15] Trzaskalik T.: *Hierarchy Depending on Value in Multiple Criteria Dynamic Programming.* "Foundations of Computing and Decision Sciences" 1995, 20, 2, pp. 139-148.

[16] Trzaskalik T.: *Multicriteria Discrete Dynamic Programming. Theory and Economic Applications*. The Karol Adamiecki University of Economics, Katowice 1990.

[17] Trzaskalik T., Sitarz S.: *Discrete Dynamic Programming with Outcomes in Random Variable Structures*. "European Journal of Operational Research" 2007, 177, pp. 1535-1548.

[18] Trzaskalik T, Sitarz S.: *Dynamic Discrete Programming with Partially Ordered Criteria Set*. In: *Multiple Objective and Goal Programming. Recent Developments*. T. Trzaskalik, J. Michnik (eds). Phisica-Verlag, Heidelberg-New York 2002, pp. 186-195.

[19] Zhang W.F., Piplani R.: *Application of Yield Management to Capacity Rationing.* "EEE International Engineering Management Conference" 2004, 3, pp. 1229-1233.