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SOLUTIONS OF SOME FUNCTIONAL EQUATIONS IN A CLASS OF GENERALIZED HÖLDER FUNCTIONS

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Abstract. The existence and uniqueness of solutions a nonlinear iterative equation in the class of r-times differentiable functions with the r-derivative satisfying a generalized Hölder condition is considered.

Keywords: *iterative functional equation, generalized Hölder condition*

1. Introduction

In [1, 2] the space $W_\gamma[a,b]$ ($W_\gamma^r[a,b]$) of r times differentiable functions with the r-the derivative satisfying generalized γ-Hölder condition was introduced and some of its properties proved. In the present paper we examine the existence and uniqueness of solutions of a nonlinear iterative functional equation in this class of functions. We apply some ideas from Kuczma [3], Matkowski [4, 5] (see also Kuczma, Choczewski, Ger [6]), where differentiable solutions, Lipschitzian solutions, bounded variation solutions of different type of itrerative functional equations were investigated.

2. Preliminaries

Consider non-linear functional equation

$$\varphi(x) = h(\varphi[f(x)]) + g(x) \tag{1}$$

where f, g, h are given and φ is a unknown function.

We accept the following notation: $I = [a, b], a, b \in R, d := b - a$, $W_\gamma(I)$ - is the Banach space of the r-time differentiable functions defined on the interval I with values in R, such that, for some $M \geq 0$; its r-th derivative satisfies the following γ-*Hölder condition*

$$\left|\varphi^{(r)}(x)-\varphi^{(r)}(\bar{x})\right| \leq M\gamma(|x-\bar{x}|), \qquad \bar{x}, x \in \mathrm{I}.$$

where a fixed function γ satisfies the following condition (see [1, 2]):

(Γ) $\gamma\colon [0,d] \to [0,\infty)$ is increasing and concave, $\gamma(0)=0$, $\lim_{t\to 0^+}\gamma(t)=\gamma(0)$, $\lim_{t\to d^-}\gamma(t)=\gamma(d)$, $\gamma'_+(0)=+\infty$

We assume that

(i) $f\colon I \to I,\ f \epsilon W_\gamma(I),\ \sup_I |\mathrm{f}'| \leq 1$

(ii) $g\colon I \to R,\ g \epsilon W_\gamma(I)$

(iii) $h\colon R \to R,\ h \in C^r,\ h^{(r)}$ fulfils the Lipschitz condition in R.

(iv) there exists $\xi \in \mathrm{I}$ such that $\lim_{n\to\infty} f^n(x)=\xi,\ x \in \mathrm{I}$, where f^n is the n-th iteration function f

(v) is analityc function at η_0, where η_0 is the solution of equation $\eta_0 = h(\eta_0)+g(\xi)$

We define functions $h_k\colon I \times R^{k+1} \to R, k=0,1,\dots,r-1$ by the formula

$$\begin{cases} h_0(x,y_0) := h(y_0)+g(x) \\ h_{k+1}(x,y_0,\dots,y_{k+1}) := \dfrac{\partial h_k}{\partial x} + f'(x)\left(\dfrac{\partial h_k}{\partial y_0}y_1+\cdots+\dfrac{\partial h_k}{\partial y_k}y_{k+1}\right). \end{cases} \tag{2}$$

Lemma 1. [4]

By assumptions (i)-(iii), h_k defined by (2) are of the form:

1. for $r=1$

$$h_1(x,y_0,y_1) = h'(y_0)y_1 f'(x) + g'(x); \tag{3}$$

2. for $r \geq 2,\ k=2,\dots,r$

$$h_k(x,y_0,\dots,y_k) = p_k(x,y_0,\dots,y_{k-1}) + h'(y_0)y_k\left(f'(x)\right)^k + \\ + h'(y_0)y_1 f^{(k)}(x) + g^{(k)}(x), \tag{4}$$

where

$$p_k(x,y_0,\dots,y_{k-1}) + h'(y_0)y_k\left(f'(x)\right)^k = \\ = \sum_{i=1}^{k} h^{(k-i+1)}(y_0) \sum_{\alpha_1+\cdots+\alpha_i=k-i+1} u_{\alpha_1\dots\alpha_i,k}(x) {y_1}^{\alpha_1} \dots {y_i}^{\alpha_i} \tag{5}$$

and $u_{\alpha_1\dots\alpha_i,k}(x)$ are of the class C^{r-k+1} in I, for all numbers $\alpha_1,\dots,\alpha_i \in N$ such that $\alpha_1+\cdots+\alpha_i = k-i+1,\ k=2,\dots,r,\ i=1,\dots,k$.

Remark 1.

If (i)-(iii) are fulfilled, then $h_r\colon I \times R^{k+1} \to R$, given by

$$h_r(x,y_0,\dots,y_r) = h'(y_0)y_1 f^{(r)}(x) + g^{(r)}(x) + \\ + \sum_{i=1}^{r} h^{(r-i+1)}(y_0) \sum_{\alpha_1+\cdots+\alpha_i=r-i+1} u_{\alpha_1\dots\alpha_i,r}(x) {y_1}^{\alpha_1} \dots {y_i}^{\alpha_i}$$

fulfill γ-Hölder condition for $x \in I$ and Lipschitz condition with respect to $y_i,\ i = 0, \dots, r$ in $Z \coloneqq [a_0, b_0] \times [a_1, b_1] \times \dots \times [a_r, b_r]$. It means, that there are positive constants $m, l_0, \dots, l_{r-1}$ and

$$l_r = \sup_{I \times [a_0, b_0]} |h'(f')^r|,$$

such that for $(x, y_1, \dots, y_r)$, $(\overline{x}, \overline{y_1}, \dots, \overline{y_r}) \in Z$ we have

$$|h_r(x, y_0, \dots, y_r) - h_r(\bar{x}, \overline{y_0}, \dots, \overline{y_r})| \leq m\gamma(|x - \bar{x}|) + l_0|y_0 - \overline{y_0}| + \dots + \\ + l_r|y_r - \overline{y_r}|.$$

Define the functions $w_{r,i}: I \times R^i \to R, i = 1,2, \dots, r$ by the following formulas:

$$w_{r,i}(x, y_1, \dots, y_i) \coloneqq \sum_{\alpha_1 + \dots + \alpha_i = r-i+1} u_{\alpha_1 \dots \alpha_i, r}(x) {y_1}^{\alpha_1} \dots {y_i}^{\alpha_i}. \tag{6}$$

Remark 2.
The functions $w_{r,i}$ defined by (6) fulfill γ-Hölder condition with respect to variable x in I and Lipschitz condition with respect to the variable $y_i, i = 1, \dots, r$ in each set $Z_i \coloneqq [a_1, b_1] \times \dots \times [a_i, b_i]$.

Remark 3.
If f, g, h satisfy the assumptions (i)-(iii) and $\varphi \epsilon W_\gamma(I)$ is a solution of equation (1) then the derivatives $\varphi^{(k)}, k = 0, \dots, r$ satisfy the system of equations

$$\varphi^{(k)}(x) = h_k\big(x, \varphi[f(x)], \dots, \varphi^{(k)}[f(x)]\big), \qquad x \in I.$$

If assumptions (i)-(iv) are fulfilled and $\varphi \epsilon W_\gamma(I)$ is a solution of equation (1) in I, then the numbers

$$\eta_k = \varphi^{(k)}(\xi),\ k = 0, \dots, r \tag{7}$$

satisfy the system of equations

$$\eta_k = h_k(\xi, \eta_0, \dots, \eta_k),\ k = 0, \dots, r, \tag{8}$$

where h_k are defined by (2).

Remark 4.
Let $\varphi \epsilon W_\gamma(I)$ be a solution of the equation (1). Present φ in the following form

$$\varphi(x) = P(x) + \psi(x - \xi), x \in I = [a, b] \tag{9}$$

where $\psi: [a - \xi, b - \xi] \to R$ and $P(x) = \sum_{i=0}^{r} \frac{\eta_i}{i!}(x - \xi)^i, x \in [a, b]$.

Define the functions

$$\bar{f}(x) := f(x+\xi) - \xi, \qquad x \in [a-\xi, b-\xi]$$

$$\bar{g}(x) := g(x+\xi), \qquad x \in [a-\xi, b-\xi]$$

and for $y \in R$, $x \in [a-\xi, b-\xi]$

$$\bar{h}(x) := h(P[f(x+\xi)] + y) - P(x+\xi).$$

It follows from above definitions and equation (9) that ψ satisfies the following equation

$$\psi(x) = \bar{h}(\psi[\bar{f}(x)]) + \bar{g}(x), x \in [a-\xi, b-\xi].$$

It is easy to prove, that if assumptions (i)-(iv) are fulfilled and $\eta_i, i = 0, \dots, r,$ are the solution of equations (8), then the function $\varphi \epsilon W_\gamma[a,b]$ satisfies the equation (1) in $[a,b]$ and the condition (7) if and only if the function ψ given by (9) belongs to $W_\gamma[a-\xi, b-\xi]$ and satisfies

$$\psi^{(k)}(0) = 0, k = 0, \dots, r.$$

Thus, we assume that $0 \in I$ and consider the equation (1) whose solution satisfies the condition

$$\varphi^{(k)}(0) = 0, k = 0, \dots, r.$$

Then system of equations (8) takes the following form

$$h_k(0, \dots, 0) = 0, k = 0, \dots, r.$$

3. Main result

Theorem 1.

If assumptions (i)-(iii) are fulfilled, f is a monotone function in the interval I, the conditions (iv) and (v) are fulfilled for $\xi = 0,\ \eta_0 = 0$ and

$$h_k(0, \dots, 0) = 0, \quad k = 1, \dots, r; \tag{10}$$

$$|h'(0)(f'(0))^r| < 1 \tag{11}$$

then equation (1) has exactly one solution $\varphi \epsilon W_\gamma(I)$ satisfying the condition

$$\varphi^{(k)}(0) = 0, k = 0, \dots, r. \tag{12}$$

Moreover, there exists a neighbourhood U of the point $\xi = 0$ and the number r_0 such that for a function $\varphi_0 \epsilon W_\gamma(\overline{U})$, satisfying the condition (12) and the inequality $\|\varphi_0\| \leq r_0$, a sequence of functions

$$\varphi_n(x) = h(\varphi_{n-1}[f(x)]) + g(x), \qquad x \in \overline{U},$$

converges to a solution of (1) according to the norm in the space $W_\gamma(\overline{U})$.

Proof.
From (v) we have $h(y) = \sum_{n=0}^{\infty} a_n y^n$ in some neighbourhood of the point 0. Denote by R_0 the radius of convergence of this series. From (11) and from the continuity of functions $(f')^r$ and h', from definition of the function γ there exists a neighbourhood V of the point $\xi = 0$ and $d < R_0, 0 < \theta < 1$ such that

$$sup_{\overline{V}\times[-d,d]}|h'(f')^r| \leq \theta, f(V) \subset V, \gamma(diam\overline{V}) \geq diam\overline{V} . \tag{13}$$

From Remark 1, definition of γ and from (13) there are positive constants $m, l_0, \dots, l_{r-1,}$ and $l_r = \theta$, that in $\overline{V} \times [-d, d]^{r+1}$ we have

$$|h_r(x, y_0, \dots, y_r) - h_r(\bar{x}, \overline{y_0}, \dots, \overline{y_r})| \leq m\gamma(|x - \bar{x}|) + l_0|y_0 - \overline{y_0}| + \dots + \\ + \theta|y_r - \overline{y_r}|. \tag{14}$$

From Remark 2, definition of γ there are in $Z_i = \overline{V} \times [-d, d]^i$ constants $B_{i,0}$, $B_{i,k}$, $i = 1, \dots, r,\ k = 1, \dots, i,$ such that

$$|w_{r,i}(x, y_1, \dots, y_i) - w_{r,i}(\bar{x}, \overline{y_1}, \dots, \overline{y_i})| \leq B_{i,0}\gamma(|x - \bar{x}|) + \sum_{k=1}^{i} B_{i,k} |y_k - \overline{y_k}| \tag{15}$$

We accept the following notation:

$$W_i \coloneqq \sup_{\overline{V}\times[-d,d]} |w_{r,i}|, \ i = 1,2, \dots, r; \tag{16}$$

$$H_i \coloneqq \sup_{\overline{V}\times[-d,d]} |h^{(i)}|, \ i = 1,2, \dots, r + 1; \tag{17}$$

$$F \coloneqq \sup_{\overline{V}} |f^{(r)}|; \ K \text{ is a } \gamma\text{-Hölder constant of } f^{(r)} \text{ in } \overline{V}; \tag{18}$$

$$C_{\alpha_1 \dots \alpha_i, r} \coloneqq \sup_{\overline{V}} |u_{\alpha_1 \dots \alpha_i, r}|, \ i = 1,2, \dots, r, \ \alpha_1 + \dots + \alpha_i = r - i + 1; \tag{19}$$

$$D_{\alpha_1 \dots \alpha_i, r} \coloneqq \sup_{\overline{V}} |u'_{\alpha_1 \dots \alpha_i, r}|, \ i = 1,2, \dots, r, \ \alpha_1 + \dots + \alpha_i = r - i + 1 . \tag{20}$$

By $\sum a_{\alpha_1 \dots \alpha_i, r}$ we denote the sum of $a_{\alpha_1 \dots \alpha_i, r}$ for all $\alpha_1, \dots, \alpha_i \in N$ such that $\alpha_1 + \dots + \alpha_i = r - i + 1,\ i = 1,2, \dots, r$.
In view of Lemma 1, we have

$$u_{0 \dots 01_i, r} = (f')^r$$

and, from (13), we get

$$\left| h'(y) u_{0 \dots 01_i, r}(x) \right| \leq \theta,\ x \epsilon \overline{V}, y \epsilon [-d, d] \ . \tag{21}$$

Let us take $c_1 \epsilon (0, b - a], c_1 \leq \gamma(c_1) \leq 1$ and

$$\gamma(c_1) \sum_{i=0}^{r-1} l_i \, c_1^{r-i-1} < 1 - \theta.$$

Put

$$r_0 := \frac{m}{1 - \theta - \gamma(c_1) \sum_{i=0}^{r-1} l_i \, c_1^{r-i-1}}. \tag{22}$$

Then let's take $c_2 \epsilon (0, b - a]$ such that $c_2 \leq \gamma(c_2) \leq \min\{1, \frac{d}{r_0}\}$ and

$$\begin{aligned} l_0 := {} & H_1 F \big(\gamma(c_2)\big)^{r-1} + H_2 F \big(\gamma(c_2)\big)^{2r} + H_1 K \big(\gamma(c_2)\big)^r + H_2 F r_0 (\gamma(c_2))^{2r} + \\ & + H_2 K r_0 (\gamma(c_2))^{2r+1} + F r_0 \big(\gamma(c_2)\big)^{2r} \sum_{n=2}^{\infty} n(n-1)^2 \, |a_n| r_0^{n-2} \big(\gamma(c_2)\big)^{(n-2)(r+1)} \\ & + \big(\gamma(c_2)\big)^r \sum_{i=1}^{r} W_i \sum_{n=r-i+2}^{\infty} |a_n| \, n(n-1)(n-r+i-2)^2 r_0^{n-r+i-2} \\ & \cdot \big(\gamma(c_2)\big)^{(n-r+i-2)(r+1)} + \\ & + \big(\gamma(c_2)\big)^{r+1} \left(\sum_{i=1}^{r} H_{r-i+2} \left(B_{i,0} + 2 r_0 \sum_{i=1}^{i} B_{i,k} \big(\gamma(c_2)\big)^{r-k} \right) \right) + \\ & + \sum_{i=1}^{r-1} H_{r-i+1} \sum C_{\alpha_1 \dots \alpha_i, r} \, r_0^{r-i} \big(\gamma(c_2)\big)^{r\alpha_1 + \dots + (r-i+1)\alpha_i - 1} (r-i+1)^2 + \\ & + H_1 \sum C_{\alpha_1 \dots \alpha_{r-1} 0, r} \big(\gamma(c_2)\big)^{r\alpha_1 + (r-1)\alpha_2 + \dots + 2\alpha_{r-1} - 1} + \\ & + \sum_{i=1}^{r} H_{r-i+1} r_0^{r-i} \, (r-i+1) \sum D_{\alpha_1 \dots \alpha_i, r} \big(\gamma(c_2)\big)^{r\alpha_1 + \dots + (r-i+1)\alpha_i} < 1 - \theta. \end{aligned} \tag{23}$$

Choose $c \leq min\{c_1, c_2\}$. Of course $c \leq \gamma(c) \leq \frac{d}{r_0}$. We will select a neighborhood of zero $U \subset V$ such that $f(U) \subset U$ and $diam\overline{U} \leq c$.

Consider the Banach space $W_\gamma(\overline{U})$ with the norm:

$$\|\varphi\| := \sum_{k=0}^{r}\left|\varphi^{(k)}(0)\right| + sup\left\{\frac{\left|\varphi^{(r)}(x)-\varphi^{(r)}(\bar{x})\right|}{\gamma(|x-\bar{x}|)};\ x,\bar{x}\epsilon\bar{U}, x\neq\bar{x}\right\}.$$

Let us define the set

$$A_{r_0} := \{\ \varphi\epsilon W_\gamma(\bar{U}), \varphi^{(k)}(0)=0, k=0,\dots,r, \|\varphi\|\le r_0\ \}.$$

Note that A_{r_0} is a closed subset of Banach space $W_\gamma(\bar{U})$ and for $\varphi\in A_{r_0}$ the norm is expressed by the formula

$$\|\varphi\| := sup\left\{\frac{\left|\varphi^{(r)}(x)-\varphi^{(r)}(\bar{x})\right|}{\gamma(|x-\bar{x}|)};\ x,\bar{x}\epsilon\bar{U}, x\neq\bar{x}\right\} \qquad (24)$$

Thus, the set A_{r_0} with the metric $\varrho(\varphi_1,\varphi_2) := \|\varphi_1-\varphi_2\|$ is a complete metric space.

By the mean value theorem and by definition of the number of c we have for $\varphi\epsilon A_{r_0}$

$$sup\left|\varphi^{(k)}\right| \le c^{r-k}\gamma(c)r_0 \le \gamma(c)r_0 \le d,\ k=0,\dots,r \qquad (25)$$

and so $\varphi^{(k)}\epsilon[-d,d],\ k=0,\dots,r.$

For $\varphi\epsilon A_{r_0}$ define the transformation T by the formula

$$(T\varphi)(x) := h(\varphi[f(x)]) + g(x), \qquad x\in\bar{U}.$$

We will show that $T\left(A_{r_0}\right)\subset A_{r_0}$.
Based on Remarks 1 and 3 the function $\psi := T\varphi$ belongs to $W_\gamma(\bar{U})$, from (iv) and (10), (12) appears that $\psi^{(k)}(0)=0,\ k=0,\dots,r$. Then using the formulas (12), (13), (22), (25) and the assumption (i) we obtain

$$\left|\psi^{(r)}(x)-\psi^{(r)}(\bar{x})\right| \le m\gamma(|x-\bar{x}|) + l_0|\varphi[f(x)]-\varphi[f(\bar{x})]| + \dots +$$

$$+ l_{r-1}\left|\varphi^{(r-1)}[f(x)]-\varphi^{(r-1)}[f(\bar{x})]\right| + \theta\left|\varphi^{(r)}[f(x)]-\varphi^{(r)}[f(\bar{x})]\right| \le$$

$$(m + l_0c^{r-1}\gamma(c)r_0 + \dots + l_{r-1}\gamma(c)r_0 + \theta r_0)\gamma(|x-\bar{x}|) \le r_0\gamma(|x-\bar{x}|).$$

Which means from (24) that $\|T\varphi\|\le r_0$. Thus $T\left(A_{r_0}\right)\subset A_{r_0}$.

Now we prove that T is a contraction map. Let us put $\psi_1 := T\varphi_1,\ \psi_2 := T\varphi_2$. Basing on formulas (4)-(5) of Lemma 1 and from (24) we have

$$\left|\psi_1^{(r)}(x)-\psi_1^{(r)}(\bar{x})-\psi_2^{(r)}(x)+\psi_2^{(r)}(\bar{x})\right|=$$

$$=\left|h'\left(\varphi_1\left[f(x)\right]\right)\varphi_1'\left[f(x)\right]f^{(r)}(x)-h'\left(\varphi_1\left[f(\bar{x})\right]\right)\varphi_1'\left[f(\bar{x})\right]f^{(r)}(\bar{x})+\right.$$

$$-h'\left(\varphi_2\left[f(x)\right]\right)\varphi_2'\left[f(x)\right]f^{(r)}(x)+h'\left(\varphi_2\left[f(\bar{x})\right]\right)\varphi_2'\left[f(\bar{x})\right]f^{(r)}(\bar{x})+$$

$$+\sum_{i=1}^{r}\Big(h^{(r-i+1)}(\varphi_1[f(x)])w_{r,i}(x,\ \varphi_1'\left[f(x)\right],\ldots,\varphi_1^{(i)}\left[f(x)\right])+$$

$$-h^{(r-i+1)}(\varphi_1[f(\bar{x})])w_{r,i}(\bar{x},\varphi_1'\left[f(\bar{x})\right],\ldots,\varphi_1^{(i)}\left[f(\bar{x})\right])+$$

$$-h^{(r-i+1)}(\varphi_2[f(x)])w_{r,i}(x,\varphi_2'\left[f(x)\right],\ldots,\varphi_2^{(i)}\left[f(x)\right])+$$

$$\left.+h^{(r-i+1)}(\varphi_2[f(\bar{x})])w_{r,i}(\bar{x},\varphi_2'\left[f(\bar{x})\right],\ldots,\varphi_2^{(i)}\left[f(\bar{x})\right])\Big)\right|\le$$

$$\left|h'\left(\varphi_1\left[f(x)\right]\right)\right|\left|f^{(r)}(x)\right|\left|\varphi_1'\left[f(x)\right]-\varphi_1'\left[f(\bar{x})\right]-\varphi_2'\left[f(x)\right]+\varphi_2'\left[f(\bar{x})\right]\right|+$$

$$+\left|f^{(r)}(x)\right|\left|\varphi_1'\left[f(\bar{x})\right]-\varphi_2'\left[f(\bar{x})\right]\right|\left|h'\left(\varphi_1\left[f(x)\right]\right)-h'\left(\varphi_1\left[f(\bar{x})\right]\right)\right|+$$

$$+\left|h'\left(\varphi_1\left[f(\bar{x})\right]\right)\right|\left|\varphi_1'\left[f(\bar{x})\right]-\varphi_2'\left[f(\bar{x})\right]\right|\left|f^{(r)}(x)-f^{(r)}(\bar{x})\right|+$$

$$+\left|f^{(r)}(x)\right|\left|\varphi_2'\left[f(x)\right]-\varphi_2'\left[f(\bar{x})\right]\right|\left|h'\left(\varphi_1\left[f(x)\right]\right)-h'\left(\varphi_2\left[f(x)\right]\right)\right|+$$

$$+\left|\varphi_2\left[f(\bar{x})\right]\right|\left|f^{(r)}(x)-f^{(r)}(\bar{x})\right|\left|h'\left(\varphi_1\left[f(x)\right]\right)-h'\left(\varphi_2\left[f(x)\right]\right)\right|+$$

$$+\left|\varphi_2\left[f(\bar{x})\right]\right|\left|f^{(r)}(\bar{x})\right|\left|h'\left(\varphi_1\left[f(x)\right]\right)-h'\left(\varphi_2\left[f(x)\right]\right)-h'\left(\varphi_1\left[f(\bar{x})\right]\right)+h'\left(\varphi_2\left[f(\bar{x})\right]\right)\right|+$$

$$+\sum_{i=1}^{r}\Big(\left|w_{r,i}(x,\varphi_1'\left[f(x)\right],\ldots,\varphi_1^{(i)}\left[f(x)\right])\right|\left|h^{(r-i+1)}(\varphi_1\left[f(x)\right])-h^{(r-i+1)}(\varphi_1\left[f(\bar{x})\right])+\right.$$

$$\left.-h^{(r-i+1)}(\varphi_2\left[f(x)\right])+h^{(r-i+1)}(\varphi_2\left[f(\bar{x})\right])\right|+$$

$$+\left|h^{(r-i+1)}(\varphi_1\left[f(\bar{x})\right])-h^{(r-i+1)}(\varphi_2\left[f(\bar{x})\right])\right|\cdot$$

$$\cdot\left|w_{r,i}(x,\varphi_1'\left[f(x)\right],\ldots,\varphi_1^{(i)}\left[f(x)\right])-w_{r,i}(\bar{x},\varphi_1'\left[f(\bar{x})\right],\ldots,\varphi_1^{(i)}\left[f(\bar{x})\right])\right|+$$

$$+\left|h^{(r-i+1)}(\varphi_2\left[f(x)\right])-h^{(r-i+1)}(\varphi_2\left[f(\bar{x})\right])\right|\cdot$$

$$\cdot\left|w_{r,i}(x,\varphi_1'\left[f(x)\right],\ldots,\varphi_1^{(i)}\left[f(x)\right])-w_{r,i}(x,\varphi_2'\left[f(x)\right],\ldots,\varphi_2^{(i)}\left[f(x)\right])\right|+$$

$$+\left|h^{(r-i+1)}(\varphi_2\left[f(\bar{x})\right])\right|\left|w_{r,i}(x,\varphi_1'\left[f(x)\right],\ldots,\varphi_1^{(i)}\left[f(x)\right])-w_{r,i}(\bar{x},\varphi_1'\left[f(\bar{x})\right],\ldots,\varphi_1^{(i)}\left[f(\bar{x})\right])+\right.$$

$$\left.-w_{r,i}(x,\varphi_2'\left[f(x)\right],\ldots,\varphi_2^{(i)}\left[f(x)\right])+w_{r,i}(x,\varphi_2'\left[f(\bar{x})\right],\ldots,\varphi_2^{(i)}\left[f(\bar{x})\right])\right|.$$

Note, that if $\varphi_1,\varphi_2\in A_{r_0}$, then in view of the mean value theorem, from the definition of the number c and from (i) we have the following inequalities

$$\sup_{\bar{U}}\left|\varphi_i^{(k)}\right|\le r_0c^{r-k}\gamma(c)\le r_0\left(\gamma(c)\right)^{r-k+1},\ \ k=0,\ldots,r,\quad i=1,2; \tag{26}$$

$$\left|\varphi_1^{(k)}[f(x)]-\varphi_1^{(k)}[f(\bar{x})]\right| \le r_0\left(\gamma(c)\right)^{r-k}\gamma\left(\left|x-\bar{x}\right|\right),\ k=0,...,r,\quad x,\bar{x}\in\bar{U};\quad (27)$$

$$\left|\varphi_1^{(k)}[f(x)]-\varphi_2^{(k)}[f(x)]\right| \le \left\|\varphi_1-\varphi_2\right\|\left(\gamma(c)\right)^{r-k+1},\ k=0,...,r,\quad x\in\bar{U};\qquad (28)$$

$$\left|\varphi_1^{(k)}[f(x)]-\varphi_1^{(k)}[f(\bar{x})]-\varphi_2^{(k)}[f(x)]+\varphi_2^{(k)}[f(\bar{x})]\right| \le \left\|\varphi_1-\varphi_2\right\|\left(\gamma(c)\right)^{r-k}\gamma\left(\left|x-\bar{x}\right|\right),\quad (29)$$
$$k=0,...,r,\quad x,\bar{x}\in\bar{U}.$$

By induction on $l\in N$ we also obtain:

$$\left|\left(\varphi_1^{(k)}[f(x)]\right)^l-\left(\varphi_1^{(k)}[f(\bar{x})]\right)^l-\left(\varphi_2^{(k)}[f(x)]\right)^l+\left(\varphi_2^{(k)}[f(\bar{x})]\right)^l\right|\le \quad (30)$$
$$l^2 r_0^{l-1}\left(\gamma(c)\right)^{l(r-k)+l-1}\left\|\varphi_1-\varphi_2\right\|\gamma\left(\left|x-\bar{x}\right|\right),\ k=0,...,r,\quad x,\bar{x}\in\bar{U},\quad l=1,2,....$$

From (v) and by selection of d we have uniform and absolute convergence of the series

$$h'(y)=\sum_{n=1}^{\infty} n a_n y^{n-1}\quad \text{for}\quad y\in[-d,d].$$

Let's consider the expression:

$$\left|h'\left(\varphi_1[f(x)]\right)-h'\left(\varphi_1[f(\bar{x})]\right)-h'\left(\varphi_2[f(x)]\right)+h'\left(\varphi_2[f(\bar{x})]\right)\right|=$$
$$=\left|\sum_{n=2}^{\infty} n a_n\left(\left(\varphi_1[f(x)]\right)^{n-1}-\left(\varphi_1[f(\bar{x})]\right)^{n-1}-\left(\varphi_2[f(x)]\right)^{n-1}+\left(\varphi_2[f(\bar{x})]\right)^{n-1}\right)\right|.$$

From (30) we obtain

$$\left|\left(\varphi_1[f(x)]\right)^{n-1}-\left(\varphi_1[f(\bar{x})]\right)^{n-1}-\left(\varphi_2[f(x)]\right)^{n-1}+\left(\varphi_2[f(\bar{x})]\right)^{n-1}\right|\le$$
$$\le(n-1)^2 r_0^{n-2}\left(\gamma(c)\right)^{(n-1)r+n-2}\left\|\varphi_1-\varphi_2\right\|\gamma\left(\left|x-\bar{x}\right|\right),\quad x,\bar{x}\in\bar{U},\quad n=2,3,....$$

Note that a series

$$\sum_{n=2}^{\infty} A_n \text{ where } A_n := n\left|a_n\right|(n-1)^2 r_0^{n-2}\left(\gamma(c)\right)^{(n-1)r+n-2}$$

converges, because the numbers c, d have been selected in such a way that

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|=\frac{r_0}{R_0}\left(\gamma(c)\right)^{r+1}\le\frac{r_0\gamma(c)}{R_0}\le\frac{d}{R_0}<1.$$

Therefore

$$\left|h'\left(\varphi_1\left[f(x)\right]\right)-h'\left(\varphi_1\left[f(\bar{x})\right]\right)-h'\left(\varphi_2\left[f(x)\right]\right)+h'\left(\varphi_2\left[f(\bar{x})\right]\right)\right|\leq$$
$$\leq\sum_{n=2}^{\infty}n\left(n-1\right)^2\left|a_n\right|r_0^{n-2}\left(\gamma(c)\right)^{(n-1)r+n-2}\left\|\varphi_1-\varphi_2\right\|\gamma\left(\left|x-\bar{x}\right|\right),\quad x,\bar{x}\in\bar{U}. \tag{31}$$

Similarly for $x,\bar{x}\in\bar{U},\ i=1,\dots,r$ we get

$$\left|h^{(r-i+1)}\left(\varphi_1\left[f(x)\right]\right)-h^{(r-i+1)}\left(\varphi_1\left[f(\bar{x})\right]\right)-h^{(r-i+1)}\left(\varphi_2\left[f(x)\right]\right)+h^{(r-i+1)}\left(\varphi_2\left[f(\bar{x})\right]\right)\right|\leq$$
$$\sum_{n=r-i+2}^{\infty}\left|a_n\right|n\dots(n-r+i)(n-r+i-1)^2r_0^{n-r+i-2}\left(\gamma(c)\right)^{(n-r+i-1)r+n-r+i-2}\left\|\varphi_1-\varphi_2\right\|\gamma\left(\left|x-\bar{x}\right|\right). \tag{32}$$

By induction and from (26)-(29) we have

$$\left|\left(\varphi_1'\left[f(x)\right]\right)^{\alpha_1}\dots\left(\varphi_1^{(i)}\left[f(\bar{x})\right]\right)^{\alpha_i}-\left(\varphi_2'\left[f(x)\right]\right)^{\alpha_1}\dots\left(\varphi_2^{(i)}\left[f(\bar{x})\right]\right)^{\alpha_i}\right|\leq$$
$$\leq\left(\alpha_1+\dots+\alpha_i\right)r_0^{\alpha_1+\dots+\alpha_i-1}\left(\gamma(c)\right)^{r\alpha_1+\dots+(r-i+1)\alpha_i}\left\|\varphi_1-\varphi_2\right\|, \tag{33}$$
$$\alpha_1,\dots,\alpha_i\in N,\ i=1,\dots,r,\quad x,\bar{x}\in\bar{U},\varphi_1,\varphi_2\in A_{r_0}$$

$$\left|\left(\varphi_2'\left[f(x)\right]\right)^{\alpha_1}\dots\left(\varphi_2^{(i)}\left[f(x)\right]\right)^{\alpha_i}-\left(\varphi_2'\left[f(\bar{x})\right]\right)^{\alpha_1}\dots\left(\varphi_2^{(i)}\left[f(\bar{x})\right]\right)^{\alpha_i}\right|\leq$$
$$\leq\left(\alpha_1+\dots+\alpha_i\right)r_0^{\alpha_1+\dots+\alpha_i-1}\left(\gamma(c)\right)^{r\alpha_1+\dots+(r-i+1)\alpha_i-1}\gamma\left(\left|x-\bar{x}\right|\right), \tag{34}$$
$$i=1,\dots,r,\quad x,\bar{x}\in\bar{U},\quad \varphi_2\in A_{r_0}.$$

Now from (33) and (34) we get

$$\left|\left(\varphi_1'\left[f(x)\right]\right)^{\alpha_1}\dots\left(\varphi_1^{(i)}\left[f(x)\right]\right)^{\alpha_i}-\left(\varphi_1'\left[f(\bar{x})\right]\right)^{\alpha_1}\dots\left(\varphi_1^{(i)}\left[f(\bar{x})\right]\right)^{\alpha_i}+\right.$$
$$\left.-\left(\varphi_2'\left[f(x)\right]\right)^{\alpha_1}\dots\left(\varphi_2^{(i)}\left[f(x)\right]\right)^{\alpha_i}+\left(\varphi_2'\left[f(\bar{x})\right]\right)^{\alpha_1}\dots\left(\varphi_2^{(i)}\left[f(\bar{x})\right]\right)^{\alpha_i}\right|\leq$$
$$\leq\left(\alpha_1+\dots+\alpha_i\right)^2r_0^{\alpha_1+\dots+\alpha_i-1}\left(\gamma(c)\right)^{r\alpha_1+\dots+(r-i+1)\alpha_i-1}\left\|\varphi_1-\varphi_2\right\|\gamma\left(\left|x-\bar{x}\right|\right), \tag{35}$$
$$i=1,\dots,r,\quad x,\bar{x}\in\bar{U},\quad \varphi_1,\varphi_2\in A_{r_0}.$$

From (6), by the mean value theorem and from (33) and (34) we get

$$\Big|w_{r,i}(x,\varphi_1'[f(x)],...,\varphi_1^{(i)}[f(x)])-w_{r,i}(\bar{x},\varphi_1'[f(\bar{x})],...,\varphi_1^{(i)}[f(\bar{x})])+$$
$$-w_{r,i}(x,\varphi_2'[f(x)],...,\varphi_2^{(i)}[f(x)])+w_{r,i}(\bar{x},\varphi_2'[f(\bar{x})],...,\varphi_2^{(i)}[f(\bar{x})]\Big|\le$$
$$\le\sum\left|u_{\alpha_1...\alpha_i,r}(x)\right|(r-i+1)^2 r_0^{r-i}\left(\gamma(c)\right)^{r\alpha_1+...+(r-i+1)\alpha_i-1}\|\varphi_1-\varphi_2\|\gamma\left(|x-\bar{x}|\right)+ \qquad (36)$$
$$+\sum\left|u_{\alpha_1...\alpha_i,r}(z)\right|(r-i+1)r_0^{r-i}\left(\gamma(c)\right)^{r\alpha_1+...+(r-i+1)\alpha_i}\|\varphi_1-\varphi_2\|\gamma\left(|x-\bar{x}|\right)$$
$$i=1,2,...,r,x,\bar{x}\in\bar{U},z \text{ is between } x \text{ and } \bar{x}.$$

Now, from (15)-(22), (27)-(32) and (36) we get

$$\left|\psi_1^{(r)}(x)-\psi_1^{(r)}(\bar{x})-\psi_2^{(r)}(x)+\psi_2^{(r)}(\bar{x})\right|\le$$
$$\le\left(H_1F\left(\gamma(c)\right)\right)^{r-1}+H_2Fr_0\left(\gamma(c)\right)^{2r}+H_1K\left(\gamma(c)\right)^r+$$
$$+H_2Fr_0\left(\gamma(c)\right)^{2r}+H_2Kr_0\left(\gamma(c)\right)^{2r+1}+$$
$$+Fr_0\left(\gamma(c)\right)^{2r}\sum_{n=2}^{\infty}|a_n|n(n-1)^2r_0^{n-2}\left(\gamma(c)\right)^{(n-2)(r+1)}+$$
$$+\sum_{i=1}^{r}W_i\left(\gamma(c)\right)^r\sum_{n=r-i+2}^{\infty}|a_n|n(n-1)...(n-r+i)(n-r+i-1)^2r_0^{n-r+i-2}\cdot$$
$$\cdot\left(\gamma(c)\right)^{(n-r+i-2)(r-1)r}+\left(\gamma(c)\right)^{r+1}\left(\sum_{i=1}^{r}H_{r-i+2}\left(B_{i,0}+2r_0\sum_{k=1}^{i}B_{i,k}\left(\gamma(c)\right)^{r-k}\right)\right)+$$
$$+\sum_{i=1}^{r-1}H_{r-i+1}\sum C_{\alpha_1...\alpha_i,r}(r-i+1)^2r_0^{r-i}\left(\gamma(c)\right)^{r\alpha_1+...+(r-i+1)\alpha_i-1}+$$
$$+H_1\sum_{\alpha_1+...+\alpha_{r-1}=1,\alpha_r=0}C_{\alpha_1...\alpha_{r-1}0,r}\left(\gamma(c)\right)^{r\alpha_1+...+2\alpha_{r-1}-1}+$$
$$+\sum_{i=1}^{r-1}H_{r-i+1}\sum D_{\alpha_1...\alpha_i,r}(r-i+1)r_0^{r-i}\left(\gamma(c)\right)^{r\alpha_1+...+(r-i+1)\alpha_i}+$$
$$+\sup_{\bar{U}}\left|h'u_{0...01,r}\right|)\|\varphi_1-\varphi_2\|\gamma\left(|x-\bar{x}|\right)\le\left(l_0+\theta\right)\|\varphi_1-\varphi_2\|\gamma\left(|x-\bar{x}|\right).$$

Putting $L=l_0+\theta$ and making use of definition (24) of the norm in $W_\gamma(\bar{U})$ we have

$$\|\psi_1-\psi_2\|\le L\|\varphi_1-\varphi_2\|,$$

which means that $\rho(\psi_1,\psi_2)\le L\ \rho(\varphi_1,\varphi_2)$, where $L<1$ in view on (23).

By the Banach fixed point theorem, there is exactly one solution $\overline{\varphi} \in W_\gamma(\overline{U})$ of (1) satisfying the condition (12). This solution is given as the limit of series of successive approximations.

$$\varphi_n(x) = h\left(\varphi_{n-1}[f(x)]\right) + g(x), \quad n \in N, \quad x \in \overline{U}$$

where $\varphi_0 \in A_{r_0}$. This sequence converges in the sense of the norm of $W_\gamma(\overline{U})$. By Lemma 4 in [7], there exists the unique extension φ of $\overline{\varphi}$ to the whole interval I such that $\varphi = \overline{\varphi}$ for $x \in \overline{U}$ and φ satisfies the equation (1) in I. This completes the proof.

Conclusions

In this paper, applying the Banach contraction principle, a theorem on the existence and uniqueness of W_γ-solutions of nonlinear iterative functional equation (1) has been proved. The suitable unique solution is determined as a limit of sequence of successive approximations.

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