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ON A FACTORIZATION OF MAPPINGS WITH A PRESCRIBED BEHAVIOUR OF THE CAUCHY DIFFERENCE

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Abstract. We deal with functional congruences of the form

$$f(x+y)-f(x)-f(y) \in U+V$$

where U and V are given sets subjected to satisfy some "separability" conditions essentially weaker than that occurring in [4] which proved to be pretty useful especially while investigating various types of Hyers-Ulam stability problems. The goal is to factorize f into a sum of two functions whose Cauchy differences remain in U and V, respectively, or, at least to obtain an approximation of f by such a sum. An application of the newly established result in that spirit is given. Moreover, a stability result for the celebrated cocycle equation is presented and, finally, the behaviour of mappings whose Cauchy differences fall into a given Hamel basis is described.

1. Introduction. Stability problems in the sense of Hyers & Ulam may sometimes be reduced to functional congruences of the form

$$(*) \qquad f(x+y)-f(x)-f(y) \in U+V\,.$$

For instance, in a recent paper of R. Ger & P. Šemrl [4] results on representation of mappings satisfying $(*)$ were used to investigate various aspects of the stability of exponential functions. K. Baron, A. Simon & P. Volkmann [1] have found another application of those results. For convenience, below we quote explicitly the statement of a basic Theorem 2.1 from [4], using the

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following notational convention: for an arbitrary set $U \subset X$ of an Abelian group $(X, +)$ we put

$$U^+ := U + U \qquad \text{and} \qquad U^- := U - U,$$

getting, in particular, $(U^+)^- = U + U - U - U$.

Let $(S, +)$ be a cancellative Abelian semigroup and let $(X, +)$ be a torsion-free divisible Abelian group. Let further $U, V \subset X$ be such that

$$(**) \qquad (U^+)^- \cap (V^+)^- = \{0\}.$$

Then each function $f : S \longrightarrow X$ fulfilling condition $(*)$ admits a representation of the form $f = \alpha + \beta$ where $\alpha, \beta : S \longrightarrow X$ satisfy the relationships

$$\alpha(x + y) - \alpha(x) - \alpha(y) \in U, \quad x, y \in S,$$

and

$$\beta(x + y) - \beta(x) - \beta(y) \in V, \quad x, y \in S,$$

The functions α and β are determined uniquely up to an additive function.

Obviously, the "separation" condition $(**)$ plays here the crucial role. However, as we shall see later on, in some cases it happens to be too strong. The aim of the present paper is to obtain a version of the result just quoted with condition $(**)$ weakened considerably. For that purpose, a generalization of L. Szekélyhidi's result [8] on the stability of the celebrated cocycle equation will be presented, which might be of independent interest.

2. Stability of the cocycle equation. We shall be using the following notation: given a nonempty set S and a normed linear space $(X, \|\cdot\|)$, by $\mathcal{B}(S, X)$ we denote the real linear space of all bounded functions mapping S into X, endowed with the norm: $\| f \|_\infty := \sup\{\| f(s) \| : s \in S\}, f \in \mathcal{B}(S, X)$; by $B(x, \rho)$ (resp. $B_E(x, \rho)$) we mean the ball in X (resp. in a subspace E of X) centered at x and having radius $\rho > 0$, whereas $\overline{B}(x, \rho)$ will stand for the closure of $B(x, \rho)$.

Let us recall that a semigroup $(S, +)$ is termed *left* (resp. *right*) *amenable* provided that there exists a real linear functional M on $\mathcal{B}(S, \mathbb{R})$ such that

$$\inf f(S) \leq M(f) \leq \sup f(S), \qquad f \in \mathcal{B}(S, \mathbb{R}),$$

and M is *left* (resp. *right*) *invariant* in the sense that

$$(1) \qquad M({}_af) = M(f) \qquad (\text{resp.}\, M(f_a) = M(f))$$

for all $f \in \mathcal{B}(S,\mathbb{R})$ and all $a \in S$; here $({}_af)(x) := f(a+x)$ and $f_a(x) := f(x+a)$, $x, a \in S$. It is well-known that any commutative semigroup is amenable.

We begin with the following

THEOREM 1. *Let $(S,+)$ be a left (right) amenable semigroup and let $(X, \|\cdot\|)$ be a real Banach space which is either*

(i) *reflexive*

or

(ii) *has the Hahn-Banach extension property*

or

(iii) *forms a boundedly complete Banach lattice with a strong unit e. Given a number $\varepsilon \geq 0$ and a mapping $F : S \times S \longrightarrow X$ such that*

$$\|F(x+y,z) + F(x,y) - F(x,y+z) - F(y,z)\| \leq \varepsilon, \quad x,y,z \in S,$$

there exists a function $f : S \times S \longrightarrow X$ such that

$$f(x+y,z) + f(x,y) = f(x,y+z) + f(y,z), \quad x,y,z \in S,$$

and

$$\|F(x,y) - f(x,y)\| \leq \varepsilon, \quad x,y \in S,$$

in cases (i) *and* (ii), *whereas*

$$\|F(x,y) - f(x,y)\| \leq c_o\varepsilon, \quad x,y \in S,$$

with $c_o := \inf\{c > 0 : \overline{B}(0,1) \subset c[-e,e]\}$ *in case* (iii).

PROOF. Without loss of generality we may assume that the semigroup $(S,+)$ is *left* amenable. An appeal to author's result [3, Theorem 1] gives the existence of a continuous linear operator $M : \mathcal{B}(S,X) \longrightarrow X$ satisfying the first part of (1) for all $f \in \mathcal{B}(S,X)$, $a \in S$, such that $M(c) = c$ for all $c \in X$ and

(2) $$\| M \| \leq 1$$

in cases (i) and (ii), whereas

(3) $$\| M \| \leq c_o$$

in case (iii). In what follows, to avoid ambiguities, we shall write $M_z\varphi(x,y,z)$ in the case where the operator M is assumed to act on a bounded function $\varphi(x,y,\cdot)$. Since, for arbitrarily fixed variables x, y from S the function

$$F(x+y,\cdot) - F(x,y+\cdot) - F(y,\cdot) : S \longrightarrow X$$

is bounded (by $\|F(x,y)\| + \varepsilon$), the formula

$$(4) \qquad f(x,y) := M_z\,(F(x,y+z) + F(y,z) - F(x+y,z))\,, \quad x,y \in S,$$

correctly defines a map $f : S \times S \longrightarrow X$. This map turns out to be a solution of the cocycle equation

$$f(x+y,s) + f(x,y) = f(x,y+s) + f(y,s), \quad x,y,s \in S.$$

Indeed, on account of the linearity of the operator M and its left invariance property one has

$$\begin{aligned} f(x,y+s)+&f(y,s) - f(x+y,s) \\ =&M_z\,(F(x,y+s+z) + F(y+s,z) - F(x+y+s,z)) \\ &+ M_z\,(F(y,s+z) + F(s,z) - F(y+s,z)) \\ &- M_z\,(F(x+y,s+z) + F(s,z) - F(x+y+s,z)) \\ =&M_z\,(F(x,y+s+z) + F(y,s+z) - F(x+y,s+z)) \\ =&M_z\,(F(x,y+z) + F(y,z) - F(x+y,z)) \\ =&f(x,y) \end{aligned}$$

for all $x,y,s \in S$.

To finish the proof, it remains to observe that by means of the equality $M(c) = c$ valid for all constant functions $\varphi(x) = c$, $x \in S$, $c \in S$, we have

$$\begin{aligned} \|\, F(x,y)-&f(x,y)\| \\ =&\|\, M_z\,(F(x,y)) - M_z\,(F(x,y+z) + F(y,z) - F(x+y,z))\,\| \\ =&\|\, M_z\,(F(x,y) - F(x,y+z) - F(y,z) + F(x+y,z))\,\| \\ \le&\|\, M_z\|\,\|\, F(x,y) - F(x,y+z) - F(y,z) + F(x+y,z)\| \\ \le&\|\, M_z\| \cdot \varepsilon, \end{aligned}$$

which gives the estimation desired because of (2) and (3).

To get the "right" version, instead of (4) one has to put

$$f(y,z) := M_x\,(F(x+y,z) + F(x,y) - F(x,y+z))\,, \quad y,z \in S.$$

This completes the proof. □

REMARK 1. The approximating solution of the cocycle equation need not be unique even in the case of scalar valued mappings. However, with the aid of Theorem 2.1 from L. Székelyhidi's paper [8] one can easily show that the difference between any two scalar solutions of the cocycle equation that approximate the same (scalar) function F on a right amenable semigroup $(S,+)$ and fulfilling the inequality

$$|F(x+y,z)+F(x,y)-F(x,y+z)-F(y,z)|\le\varepsilon,$$

for every $x,y,z\in S$, has to be of the form

$$\varphi(x+y)-\varphi(x)-\varphi(y),\quad x,y\in S,$$

where φ, is a *bounded* scalar function on S.

REMARK 2. In the case where $(X,\|\cdot\|)=(\mathbb{C},|\cdot|)$ Theorem 1 has been proved by L. Székelyhidi [8].

3. Main result. Armed with the stability result just established we are able to prove the following

THEOREM 2. *Let $(S,+)$ be a cancellative Abelian semigroup and let $(X,\|\cdot\|)$ be a real Banach space. Assume that U and V are nonempty subsets of X such that the set*

$$B:=(U^+)^-\cap(V^+)^-$$

is bounded and put $c:=\sup\{\|b\|:\ b\in B\}$. Let $f:S\longrightarrow X$ be such that

$$f(x+y)-f(x)-f(y)\in U+V,\quad x,y\in S.$$

If the spaces $X_U:=\text{cl Lin}U$ and $X_V:=\text{cl Lin}V$ satisfy at least one of the conditions (i), (ii) *or* (iii) *spoken of in Theorem 1 (not necessarily the same), then there exist functions $\alpha,\beta:S\longrightarrow X$ satisfying the relations*

$$\alpha(x+y)-\alpha(x)-\alpha(y)\in U+\overline{B}_{X_U}(0,\tilde{c}),\qquad x,y\in S,$$

$$\beta(x+y)-\beta(x)-\beta(y)\in V+\overline{B}_{X_V}(0,\tilde{c}),\qquad x,y\in S,$$

and such that

$$\|f(x)-(\alpha+\beta)(x)\|\le 2\tilde{c}$$

for all $x\in S$; here $\tilde{c}$ stands for c or c_0c depending on whether condition (i), (ii) *or* (iii), *respectively, is assumed.*

PROOF. There exist functions $\varphi:S\times S\longrightarrow U\subset X_U$ and $\psi:S\times S\longrightarrow V\subset X_V$ such that

(5) $\quad d(x,y):=f(x+y)-f(x)-f(y)=\varphi(x,y)+\psi(x,y),\quad x,y\in S.$

Obviously, d is a symmetric solution to the cocycle equation. Moreover,

$$\begin{aligned}(U^+)^- \ni\ & \varphi(x+y,z)+\varphi(x,y)-\varphi(x,y+z)-\varphi(y,z)\\ &= d(x+y,z)+d(x,y)-d(x,y+z)-d(y,z)\\ &\quad -\psi(x+y,z)-\psi(x,y)+\psi(x,y+z)+\psi(y,z)\\ &= -\psi(x+y,z)-\psi(x,y)+\psi(x,y+z)+\psi(y,z)\in (V^+)^-,\end{aligned}$$

which states that

$$\varphi(x+y,z)+\varphi(x,y)-\varphi(x,y+z)-\varphi(y,z)\in B,$$

or, equivalently, that

$$\|\varphi(x+y,z)+\varphi(x,y)-\varphi(x,y+z)-\varphi(y,z)\|\le c,$$

for all $x,y,z\in S$. Plainly, X_U is a Banach space and, by assumption, it does have at least one of the properties (i), (ii) or (iii). So, by Theorem 1 there exists a solution $\Phi : S\times S \longrightarrow X_U$ of the cocycle equation such that

$$\|\varphi(x,y)-\Phi(x,y)\|\le \tilde{c},\quad x,y\in S. \tag{6}$$

Now, under the assumptions imposed upon the semigroup $(S,+)$, with the aid of M. Hosszú's theorem from [5], we infer that there exists a skew--symmetric and biadditive map $A : S\times S \longrightarrow X_U$ and a map $\alpha_o : S \longrightarrow X_U$ such that

$$\Phi(x,y)=A(x,y)+\alpha_o(x+y)-\alpha_o(x)-\alpha_o(y),\qquad x,y\in S. \tag{7}$$

However,

$$\begin{aligned}(U^+)^- \supset U-U \ni \varphi(x,y)-\varphi(y,x) &= d(x,y)-\psi(x,y)-d(y,x)+\psi(y,x)\\ &\in V-V\subset (V^+)^-,\end{aligned}$$

i.e. $\|\varphi(x,y)-\varphi(y,x)\|\le c$ for all $x,y\in S$, whence, by means of (6), we get

$$\begin{aligned}\|A(x,y)-&A(y,x)\|\\ &=\|\Phi(x,y)-\Phi(y,x)\|\\ &\le\|\Phi(x,y)-\varphi(x,y)\|+\|\varphi(x,y)-\varphi(y,x)\|+\|\varphi(y,x)-\Phi(y,x)\|\\ &\le\tilde{c}+c+\tilde{c}\end{aligned}$$

for all $x,y\in S$. Thus, the biadditive mapping

$$S\times S\ni(x,y)\longmapsto A(x,y)-A(y,x)\in X_U$$

remains bounded which is possible if and only if $A(x,y) = A(y,x)$ for every $x, y \in S$. Since A is skew-symmetric this implies that $A = 0$, and, consequently, in view of (7), we obtain the representation

$$\Phi(x,y) = \alpha_o(x+y) - \alpha_o(x) - \alpha_o(y), \qquad x, y \in S.$$

Setting

$$r_\varphi(x,y) := \Phi(x,y) - \varphi(x,y), \qquad x, y \in S,$$

by virtue of (6) we obtain

$$r_\varphi(x,y) \in \overline{B}_{X_U}(0,\tilde{c}), \quad x, y \in S,$$

whence

(8) $\alpha_o(x+y) - \alpha_o(x) - \alpha_o(y) = \Phi(x,y) = \varphi(x,y) + r_\varphi(x,y) \in U + \overline{B}_{X_U}(0,\tilde{c})$

for all $x, y \in S$.

Applying literally the same procedure with respect to the function $\psi : S \times S \longrightarrow V \subset X_V$ (see (5)) we deduce the existence of functions $\beta_o : S \longrightarrow X_V$ and $r_\psi : S \times S \longrightarrow \overline{B}_{X_V}(0,\tilde{c})$, such that

(9) $\beta_o(x+y) - \beta_o(x) - \beta_o(y) = \psi(x,y) + r_\psi(x,y) \in V + \overline{B}_{X_V}(0,\tilde{c})$

for all $x, y \in S$. From (8), (9) and (5) it follows that for any $x, y \in S$, one has

$$\begin{aligned} f(x+y) - f(x) - f(y) = {} & \alpha_o(x+y) - \alpha_o(x) - \alpha_o(y) - r_\varphi(x,y) \\ & + \beta_o(x+y) - \beta_o(x) - \beta_o(y) - r_\psi(x,y), \end{aligned}$$

i.e.

$$g(x+y) - g(x) - g(y) = -r_\varphi(x,y) - r_\psi(x,y) \in \overline{B}_{X_U}(0,\tilde{c}) + \overline{B}_{X_V}(0,\tilde{c})$$

for all $x, y \in S$, where $g := f - (\alpha_o + \beta_o)$. In other words, we have

$$\| g(x+y) - g(x) - g(y) \| \le 2\tilde{c}, \qquad x, y \in S,$$

whence, by J. Rätz's [7] generalization of the classical Hyers-Ulam stability result, there exists (a unique) additive mapping $a : S \longrightarrow X$ such that

(10) $$\| g(x) - a(x) \| \le 2\tilde{c}, \qquad x \in S.$$

Obviously, by virtue of the properties (8) and (9), respectively, functions

$$\alpha := \alpha_o + \frac{1}{2}a \qquad \text{and} \qquad \beta := \beta_o + \frac{1}{2}a$$

fulfil the conditions

$$\alpha(x+y) - \alpha(x) - \alpha(y) \in U + \overline{B}_{X_U}(0, \tilde{c}), \qquad x, y \in S,$$

and

$$\beta(x+y) - \beta(x) - \beta(y) \in V + \overline{B}_{X_V}(0, \tilde{c}), \qquad x, y \in S.$$

Finally, in view of (10),

$$\|f(x) - (\alpha + \beta)(x)\| \leq 2\tilde{c} \tag{11}$$

for all $x, y \in S$, as claimed. Thus the proof has been completed. □

REMARK 3. Once we have $c = 0$, which is equivalent to the statement that $B = \{0\}$, relation (11) says that $f = \alpha + \beta$ along with the relationships

$$\alpha(x+y) - \alpha(x) - \alpha(y) \in U$$

and

$$\beta(x+y) - \beta(x) - \beta(y) \in V$$

valid for all $x, y \in S$. Therefore, in such a case, the result just proved reduces itself to Theorem 2.1 from [4], quoted explicitely in the Introduction; however, merely in the case where the target group $(X, +)$ yields a suitable Banach space.

REMARK 4. Unfortunately, among the three properties (i), (ii) and (iii) nothing but reflexivity is inherited by closed subspaces of a given Banach space. This forced us to assume that the subspaces X_U and X_V occurring in Theorem 2 enjoy at least one of these properties. Dealing with reflexive spaces we obtain more concise statement which reads as follows.

THEOREM 3. *Let $(S, +)$ be a cancellative Abelian semigroup and let $(X, \|\cdot\|)$ be a real reflexive normed linear space. Assume that U and V are nonempty subsets of X such that the set*

$$B := (U^+)^- \cap (V^+)^-$$

is bounded and put $c := \sup\{\|b\| : b \in B\}$. Let $X_U := \text{cl Lin}\, U$ and $X_V :=$ cl Lin V. If $f : S \longrightarrow X$ is such that

$$f(x+y) - f(x) - f(y) \in U + V, \qquad x, y \in S,$$

then there exist functions $\alpha, \beta : S \longrightarrow X$ *satisfying the relations*

$$\alpha(x+y) - \alpha(x) - \alpha(y) \in U + \overline{B}_{X_U}(0,c), \quad x, y \in S,$$

$$\beta(x+y) - \beta(x) - \beta(y) \in V + \overline{B}_{X_V}(0,c), \quad x, y \in S,$$

and such that

$$\|f(x) - (\alpha+\beta)(x)\| \le 2c$$

for all $x \in S$.

4. An application. The goal of Theorem 2 was to cover the situation where both of the sets U and V under consideration are simultaneously unbounded (otherwise, the requirement for the set $(U^+)^- \cap (V^+)^-$ to be bounded is automatically satisfied). If that is the case, Theorem 2 reduces the problem to a similar one with one of the new summands simply being a ball (in a suitable subspace) and hence bounded. To visualize the utility of such a method we are going to present the following

EXAMPLE. Let us consider any cancellative Abelian semigroup $(S,+)$ and assume a Banach space $(X, \|\cdot\|)$ to be the real plane $\mathbb{R}^2$ endowed with the usual Euclidean norm. Fix arbitrarily an ε from the interval $(0, \frac{1}{20})$ and take

$$U := (\mathbb{Z} + (-\varepsilon, \varepsilon)) \times \{0\}, \quad V := \{(x,y) \in \mathbb{R}^2 : \ x \in \mathbb{R}, \ x - \varepsilon < y < x + \varepsilon\},$$

where $\mathbb{Z}$ stands for the set of all integers. Obviously, both U and V are unbounded and the intersection

$$\begin{aligned} B :=& (U^+)^- \cap (V^+)^- \\ =& ((\mathbb{Z} + (-4\varepsilon, 4\varepsilon)) \times \{0\}) \cap \{(x,y) \in \mathbb{R}^2 : \ x \in \mathbb{R}, \ x - 4\varepsilon < y < x + 4\varepsilon\} \\ =& (-4\varepsilon, 4\varepsilon) \times \{0\} \end{aligned}$$

is greater than $\{0\}$. This prevents us from applying the "old" result recalled in the Introduction. However, B is bounded (we have here $c := \sup\{\|b\| : b \in B\} = 4\varepsilon$) and we can make use of Theorem 3 getting that any function $f : S \longrightarrow \mathbb{R}^2$ whose Cauchy difference $f(x+y) - f(x) - f(y)$ stays in $U + V$ for all $x, y \in S$ admits an approximation of the form

$$\|f(x) - (\alpha+\beta)(x)\| \le 8\varepsilon,$$

where $\alpha, \beta : S \longrightarrow \mathbb{R}^2$ satisfy the relations

(12) $$\alpha(x+y) - \alpha(x) - \alpha(y) \in U + ([-4\varepsilon, 4\varepsilon] \times \{0\}), \qquad x, y \in S,$$

and

$$(13) \qquad \beta(x+y)-\beta(x)-\beta(y) \in V+\overline{B}((0,0),4\varepsilon), \qquad x,y \in S.$$

Relation (12) says that the first real component α_1 of the function α has to satisfy the congruence

$$\alpha_1(x+y)-\alpha_1(x)-\alpha_1(y) \in \mathbb{Z}+(-5\varepsilon,5\varepsilon), \qquad x,y \in S,$$

whereas the other real component α_2 is simply additive. Since $5\varepsilon \in (0,\frac{1}{4})$ an appeal to Corollary 2.4 from [4] gives the existence of a function $p: S \longrightarrow \mathbb{R}$ such that

$$(14) \qquad p(x+y)-p(x)-p(y) \in \mathbb{Z}, \qquad x,y \in S,$$

and

$$(15) \qquad |\alpha_1(x)-p(x)| \leq 5\varepsilon, \qquad x \in S.$$

An easy calculation shows that relation (13) forces the real components β_1 and β_2 of the function β to satisfy the functional inequality

$$|(\beta_2-\beta_1)(x+y)-(\beta_2-\beta_1)(x)-(\beta_2-\beta_1)(y)| < 5\varepsilon, \qquad x,y \in S,$$

whence, by J. Rätz's [7] generalization of the classical Hyers-Ulam stability result, there exists (a unique) additive mapping $\delta_o : S \longrightarrow \mathbb{R}$ such that

$$\|(\beta_2-\beta_1)(x)-\delta_o(x)\| \leq 5\varepsilon, \qquad x \in S.$$

Thus $\beta_2 = \beta_1 + \delta_o + \varrho$ with $|\varrho(x)| \leq 5\varepsilon$, which implies that

$$\beta(x) = d(x) + (0, \delta_o(x)+\varrho(x)), \qquad x \in S,$$

where d is a function from S into the main diagonal of $\mathbb{R}^2$ (we have put $d(x) := (\beta_1(x), \beta_1(x))$, $x \in S$). Setting $\delta := \alpha_2 + \delta_o$ we arrive at

$$(\alpha+\beta)(x) = d(x) + (\alpha_1(x), \delta(x)+\varrho(x)), \qquad x \in S.$$

Summarizing, Theorem 3 implies that function f in question differs by 8ε in absolute value from a function of the form

$$S \ni x \longmapsto d(x) + (\alpha_1(x), \delta(x)+\varrho(x)), \qquad x \in S,$$

where d maps S into the main diagonal of $\mathbb{R}^2$, function $\alpha_1 : S \longrightarrow \mathbb{R}$ satisfies estimation (15) with $p : S \longrightarrow \mathbb{R}$ fulfilling the congruence (14), whereas $\delta : S \longrightarrow \mathbb{R}$ is additive and $|\varrho(x)| \leq 5\varepsilon$, for all $x \in S$.

5. Supplementary results. Even in the (relatively) simplest case where the separation condition

$$(**) \qquad (U^+)^- \cap (V^+)^- = \{0\}$$

holds true, the algebraic sum $U + V$ may happen to be surprisingly large. Obviously, the larger is that sum the more interesting is a result upon a functional congruence of the form

$$(*) \qquad f(x+y) - f(x) - f(y) \in U + V.$$

In the present section we shall discuss some special congruences to that effect.

Fix arbitrarily a nonmeasurable Hamel basis H of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ of all rationals and assume that $1 \in H$ (see e.g. M. Kuczma [6]). The set $H + \mathbb{Q}$ is "large" indeed; for, the one-dimensional inner Lebesgue measure $(\ell_1)_*$ of its *complement* vanishes:

$$(\ell_1)_* (\mathbb{R} \setminus (H + \mathbb{Q})) = 0.$$

This results immediately from Smital's lemma (cf. M. Kuczma [6, Chapter III]) because the one-dimensional outer Lebesgue measure of nonmeasurable Hamel basis H is necessarily positive and the field $\mathbb{Q}$ is dense in $\mathbb{R}$.

Nevertheless, we are able to obtain a remarkably precise description of the solutions of the functional congruence

$$(16) \qquad f(x+y) - f(x) - f(y) \in H + \mathbb{Q},$$

for functions f mapping a cancellative Abelian semigroup $(S, +)$ into $\mathbb{R}$. To this aim, we shall first prove the following

PROPOSITION. *Under the hypotheses assumed above a function $f : S \longrightarrow \mathbb{R}$ yields a solution to the functional congruence* (16) *if and only if*

$$f(x) = a(x) + q(x) + h(x), \qquad x \in S,$$

where $a : S \longrightarrow \mathbb{R}$ is additive, q is a function on S with rational values only and $h : S \longrightarrow \mathbb{R}$ fulfils the condition

$$(17) \qquad h(x+y) - h(x) - h(y) \in H,$$

for all $x, y \in S$.

PROOF. The "if" part is trivial. Assume that $f : S \longrightarrow \mathbb{R}$ yields a solution to (16). Clearly,

$$(H^+)^- \cap (\mathbb{Q}^+)^- = (H + H - H - H) \cap \mathbb{Q} = \{0\},$$

because H is a basis and $1 \in H$. By means of Theorem 2.1 from [4] (see the Introduction), f admits a factorization of the form $f = r + h$, where $r, h : S \longrightarrow \mathbb{R}$ are such that

$$r(x+y) - r(x) - r(y) \in \mathbb{Q}, \qquad x, y \in S, \tag{18}$$

and the congruence (17) is fulfilled.

Let $\mathbb{Q}^c$ denote the space complementary to $\mathbb{Q}$ in the $\mathbb{Q}$−vector space $\mathbb{R}$. Then r itself factorizes into the sum $q + a$ where q is a function on S with rational values only and a maps S into $\mathbb{Q}^c$. Substituting that representation into (18) immediately shows that a has to be additive which finishes the proof. □

It remains to solve the congruence (17). We are able to do that provided the domain of the unknown function forms an Abelian group.

THEOREM 4. *Let $(G, +)$ be an Abelian group and let H be a Hamel basis of a (real or complex) Banach space $(X, \|\cdot\|)$, understood as vector space over the field $\mathbb{Q}$ of all rationals ($\mathbb{Q}$−vector space). Assume that a function $h : G \longrightarrow X$ satisfies the functional congruence*

$$h(x+y) - h(x) - h(y) \in H, \qquad x, y \in G.$$

Then there exist two different members h_o and h_1 of the Hamel basis H, a scalar function $\lambda : G \longrightarrow [-1, 0]$ and an additive mapping $A : G \longrightarrow X$ such that

$$h(x) = A(x) - h_o + \lambda(x)(h_1 - h_o), \qquad x \in G, \tag{19}$$

and

$$\lambda(x+y) - \lambda(x) - \lambda(y) \in \{0, 1\}, \qquad x, y \in G. \tag{20}$$

In particular,

$$h(x+y) - h(x) - h(y) \in \{h_o, h_1\}, \qquad x, y \in G,$$

and the function h differs from an additive mapping by a function whose values are totally contained in the segment $[-h_o, -h_1] \subset X$.

Conversely, for any two members h_o and h_1 of the Hamel basis H, any scalar function $\lambda : G \longrightarrow [-1,0]$ fulfilling the congruence (20) and any additive mapping $A : G \longrightarrow X$, function $h : G \longrightarrow X$ given by formula (19) yields a solution of the congruence (17).

Proof. Given a solution $h : G \longrightarrow X$ of the functional congruence (17) put

$$C(x,y) := h(x+y) - h(x) - h(y), \qquad x, y \in G.$$

Plainly, the symmetric function $C : G \times G \longrightarrow H$ given by that formula satisfies the cocycle equation

$$C(x+y,z) + C(x,y) = C(x,y+z) + C(y,z), \qquad x,y,z \in G. \tag{21}$$

Setting here $y = z = 0$ we infer that $C(x,0) = C(0,0) =: h_o \in H$ for all $x \in G$. Defining a function $c : G \longrightarrow H$ by the formula $c(x) := C(x,-x), x \in G$, and putting $y = -x$ in (21) we obtain that

$$h_o + c(x) = C(x, z-x) + C(z,-x), \qquad x,z \in G. \tag{22}$$

Take $z = x + y$ in (22) getting

$$C(x,y) = h_o + c(x) - C(x+y,-x), \qquad x,y \in G. \tag{23}$$

By virtue of the symmetry of C we have the equality

$$h_o + c(x) - C(x+y,-x) = h_o + c(y) - C(x+y,-y), \qquad x,y \in G,$$

i.e.

$$c(x) - c(y) = C(x+y,-x) - C(x+y,-y), \qquad x,y \in G.$$

Setting here $x - y$ in place of x gives

$$c(x-y) - c(y) = C(x,y-x) - C(x,-y), \qquad x,y \in G,$$

whence, by means of (22),

$$c(x-y) - c(y) = h_o + c(x) - C(y,-x) - C(x,-y), \qquad x,y \in G,$$

and, consequently, replacing here y by $-y$, and making use of the fact that c is an even function

$$c(x+y) + C(-x,-y) + C(x,y) = h_o + c(x) + c(y), \qquad x,y \in G.$$

Bearing in mind that all the summands occurring here are members of the Hamel basis we infer that necessarily

$$(24) \qquad c(x+y) \in \{h_o, c(x), c(y)\}, \qquad x, y \in G.$$

We are going to show that there exists an element $h_1 \in H$ such that

$$(25) \qquad c(x) \in \{h_o, h_1\}, \qquad x, y \in G.$$

To this aim, let us first observe that relation (24) and the evenness of c imply

$$c(x) \in \{h_o, c(x-y), c(y)\}, \qquad x, y \in G,$$

as well as

$$c(y) \in \{h_o, c(x-y), c(x)\}, \qquad x, y \in G.$$

Thus, for every $x, y \in G$ one has $c(x) = c(x-y) = c(y)$, provided that each two elements of the set $\{c(x), c(y), h_o\}$ are different. Obviously, this would lead to a contradiction unless

$$c(x) = c(y) \quad \text{or} \quad c(x) = h_o \quad \text{or} \quad c(y) = h_o \qquad \text{for all} \quad x, y \in G,$$

but this means nothing else but (25).

Without loss of generality, in the sequel we may assume that $h_o \neq h_1$. From (23) and (25) we deduce that

$$C(x,y) \in \{h_o, h_1\} \qquad \text{for all} \quad x, y \in G.$$

This means that for a function $k : G \longrightarrow X$ given by the formula

$$k(x) := h(x) + h_o, \qquad x \in G,$$

the functional congruence

$$k(x+y) - k(x) - k(y) \in \{0, h_1 - h_o\}, \qquad x, y \in G,$$

is valid. Now, an appeal to G. L. Forti's paper [2] guarantees the existence of a scalar function $\lambda : G \longrightarrow [-1, 0]$ and an additive mapping $A : G \longrightarrow X$ such that

$$h(x) + h_o = k(x) = A(x) + \lambda(x)(h_1 - h_o), \qquad x \in G,$$

which jointly with the former congruence forces the Cauchy difference of the function λ to satisfy relationship (20). Since the converse part of the theorem is trivial, the proof has been completed. □

REMARK 5. Without any changes in the proof, instead of a Hamel basis one might consider an arbitrary subset B of the Banach space in question enjoying the following two properties:

- for each elements $a, b, c, d \in B$ such that $a+b = c+d$ one has $a \in \{c, d\}$;
- for each elements $a, b, c, p, q, r \in B$ such that $a+b+c = p+q+r$ one has $a \in \{p, q, r\}$.

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