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DELTA-CONVEXITY WITH GIVEN WEIGHTS

Roman Ger

Dedicated to Professor Zygfryd Kominek on the occasion of his 75th birthday

Abstract. Some differentiability results from the paper of D.Ş. Marinescu & M. Monea [7] on delta-convex mappings, obtained for real functions, are extended for mappings with values in a normed linear space. In this way, we are nearing the completion of studies established in papers [2], [5] and [7].

1. Motivation and main results

While solving Problem 11641 posed by a Romanian mathematician Nicolae Bourbăcuţ in [2] I was announcing in [5] (without proof) the following

Theorem 1.1. *Assume that we are given a differentiable function φ mapping an open real interval (a,b) into the real line $\mathbb{R}$. Then each convex solution $f\colon (a,b) \longrightarrow \mathbb{R}$ of the functional inequality*

$$(*)\qquad \frac{f(x)+f(y)}{2} - f\Big(\frac{x+y}{2}\Big) \le \frac{\varphi(x)+\varphi(y)}{2} - \varphi\Big(\frac{x+y}{2}\Big), \quad x,y\in(a,b),$$

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is differentiable and the inequality

$$|f'(x) - f'(y)| \le |\varphi'(x) - \varphi'(y)|$$

holds true for all $x, y \in (a, b)$.

The proof reads as follows.

Put $g := f - \varphi$. Then $(*)$ states nothing else but the Jensen concavity of g, i.e.

$$\frac{1}{2}g(x) + \frac{1}{2}g(y) \le g\Big(\frac{x+y}{2}\Big), \quad x, y \in (a, b).$$

It is widely known that a continuous Jensen concave function is concave in the usual sense. Since f itself is continuous (as a convex function on an open interval) and φ is differentiable then, obviously, our function g is continuous and hence concave. In particular, the one-sided dervatives of g do exist on (a, b) and we have

$$g'_+(x) \le g'_-(x) \quad \text{for all} \quad x \in (a, b).$$

Therefore

$$f'_+(x) = g'_+(x) + \varphi'(x) \le g'_-(x) + \varphi'(x) = f'_-(x) \le f'_+(x)$$

for all $x \in (a, b)$ because of the convexity of f, which proves the differentiability of f on (a, b).

To show that f satisfies the assertion inequality, observe that whenever $x, y \in (a, b)$ are such that $x \le y$, then

$$\begin{aligned} |f'(x) - f'(y)| &= f'(y) - f'(x) = g'(y) + \varphi'(y) - g'(x) - \varphi'(x) \\ &\le \varphi'(y) - \varphi'(x) = |\varphi'(x) - \varphi'(y)|, \end{aligned}$$

because the derivative of a differentiable convex (resp. concave) function is increasing (resp. decreasing). In the case where $y \le x$ it suffices to interchange the roles of the variables x and y in the latter inequality, which completes the proof.

Note that the convexity assumption imposed upon f in the above result renders $(*)$ to be equivalent to

$$\left|\frac{f(x)+f(y)}{2} - f\Big(\frac{x+y}{2}\Big)\right| \le \frac{\varphi(x)+\varphi(y)}{2} - \varphi\Big(\frac{x+y}{2}\Big), \quad x, y \in (a, b),$$

defining (in the class of continuous functions) the notion of delta convexity in the sense of L. Veselý and L. Zajíček (see [10]).

In that connection, D.Ş. Marinescu and M. Monea have proved, among others, the following result (see [7, Theorem 2.7]).

Theorem M-M. *Let* $\varphi\colon (a,b) \longrightarrow \mathbb{R}$ *be a differentiable function and let* $f\colon (a,b) \longrightarrow \mathbb{R}$ *be a convex function admitting some scalars* $s,t \in (0,1)$ *such that the inequality*

$$tf(x)+(1-t)f(y)-f(sx+(1-s)y) \le t\varphi(x)+(1-t)\varphi(y)-\varphi(sx+(1-s)y)$$

is satisfied for all $x,y \in (a,b)$*. Then the function* f *is differentiable and the inequality*

$$|f'(x)-f'(y)| \le |\varphi'(x)-\varphi'(y)|$$

holds true for all $x,y \in (a,b)$*.*

Without any convexity assumption we offer the following counterpart of Theorem M-M for vector valued mappings.

Theorem 1.2. *Given an open interval* $(a,b) \subset \mathbb{R}$*, a normed linear space* $(E,\|\cdot\|)$*, and two real numbers* $s,t \in (0,1)$ *(weights) assume that a map* $F\colon (a,b) \longrightarrow E$ *is delta* (s,t)*-convex with a differentiable control function* $f\colon (a,b) \longrightarrow \mathbb{R}$*, i.e. that a functional inequality*

$$\|tF(x)+(1-t)F(y)-F(sx+(1-s)y)\| \le tf(x)+(1-t)f(y)-f(sx+(1-s)y)$$

is satisfied for all $x,y \in (a,b)$*. If the function*

$$(a,b) \ni x \longmapsto \|F(x)\| \in \mathbb{R}$$

is upper bounded on a set of positive Lebesgue measure, then F *is differentiable and the inequality*

$$\|F'(x)-F'(y)\| \le |f'(x)-f'(y)|$$

holds true for all $x,y \in (a,b)$*.*

Corollary. *Under the assumptions of Theorem* 1.2*, the vector valued map* F *is continuously differentiable.*

Proof. Fix arbitrarily an $x \in (a,b)$ and $h \in \mathbb{R}$ small enough to have $x+h \in (a,b)$ as well. Then

$$\|F'(x+h) - F'(x)\| \leq |f'(x+h) - f'(x)|$$

and the right-hand side difference tends to zero as $h \to 0$ because a differentiable convex function is of class C^1. □

The assumption that the function

$$(a,b) \ni x \longmapsto \|F(x)\| \in \mathbb{R}$$

is upper bounded on a set of positive Lebesgue measure, may be replaced by numerous alternative conditions forcing a scalar Jensen convex function on (a,b) to be continuous.

Theorem 1.3. *Given an open interval* $(a,b) \subset \mathbb{R}$*, a normed linear space* $(E, \|\cdot\|)$ *that is reflexive or constitutes a separable dual space, and two weights* $s,t \in (0,1)$*, assume that a map* $F\colon (a,b) \longrightarrow E$ *is delta* (s,t)*-convex with a* C^2*-control function* $f\colon (a,b) \longrightarrow \mathbb{R}$*. If the function*

$$(a,b) \ni x \longmapsto \|F(x)\| \in \mathbb{R}$$

is upper bounded on a set of positive Lebesgue measure, then F *is twice differentiable almost everywhere in* (a,b) *and the domination*

$$\|F''(x)\| \leq f''(x)$$

holds true for almost all $x \in (a,b)$*.*

The assumption that a normed linear space $(E, \|\cdot\|)$ spoken of in Theorem 1.3 is reflexive or constitutes a separable dual space may be replaced by a more general requirement that $(E, \|\cdot\|)$ has the Radon-Nikodym property (RNP), i.e. that every Lipschitz function from $\mathbb{R}$ into E is differentiable almost everywhere. This definition (of Rademacher type character) is not commonly used but is more relevant to the subject of the present paper. R.S. Phillips [9] showed that reflexive Banach spaces enjoy the RNP whereas N. Dunford and B.J. Pettis [3] proved that separable dual spaces have the RNP.

2. Proofs

To prove Theorem 1.2 we need the following

Lemma. *Given weights $s,t \in (0,1)$ assume that a map $F\colon (a,b) \longrightarrow E$ is delta (s,t)-convex with a control function $f\colon (a,b) \longrightarrow \mathbb{R}$. Then the inequality*

$$\|\lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y)\| \\ \leq \lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y)$$

is valid for all $x,y \in (a,b)$ and every rational $\lambda \in (0,1)$. In particular, F is delta Jensen convex with a control function f, i.e. the inequality

$$\left\|\frac{F(x)+F(y)}{2} - F\Big(\frac{x+y}{2}\Big)\right\| \leq \frac{f(x)+f(y)}{2} - f\Big(\frac{x+y}{2}\Big)$$

holds true for all $x,y \in (a,b)$.

Proof. Fix arbitrarily a continuous linear functional x^* from the unit ball in the dual space E^*. Then the delta (s,t)-convexity of F implies that for all $x,y \in (a,b)$ one has

$$t(x^* \circ F)(x) + (1-t)(x^* \circ F)(y) - (x^* \circ F)(sx + (1-s)y) \\ \leq tf(x) + (1-t)f(y) - f(sx + (1-s)y)$$

or, equivalently,

$$(f - x^* \circ F)(sx + (1-s)y) \leq t(f - x^* \circ F)(x) + (1-t)(f - x^* \circ F)(y).$$

By means of Theorem 3 from N. Kuhn's paper [6] we deduce that the function $g := f - x^* \circ F$ enjoys the convexity type property

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y), \quad x,y \in (a,b),\ \lambda \in (0,1) \cap \mathbb{Q},$$

where $\mathbb{Q}$ stands for the field of all rationals. Consequently, for all $x,y \in (a,b)$ and $\lambda \in (0,1) \cap \mathbb{Q}$, we get the inequality

$$\lambda(x^* \circ F)(x) + (1-\lambda)(x^* \circ F)(y) - (x^* \circ F)(\lambda x + (1-\lambda)y) \\ \leq \lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y).$$

Replacing here the functional x^* by $-x^*$ we infer that *a fortiori*

$$|x^*(\lambda F(x)+(1-\lambda)F(y)-F(\lambda x+(1-\lambda)y))| \\ \leq \lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y),$$

which due to the unrestricted choice of x^* gives the assertion desired. □

REMARK 2.1. Using another method, A. Olbryś ([8, Lemma 1]) with the aid of the celebrated Daróczy and Páles identity

$$\frac{x+y}{2}=s\Big[s\frac{x+y}{2}+(1-s)y\Big]+(1-s)\Big[sx+(1-s)\frac{x+y}{2}\Big],$$

has proved that any delta (s,t)-convex map on a convex subset of a real Banach space is necessarily delta Jensen convex.

PROOF OF THEOREM 1.2. In view of the Lemma, F is delta Jensen convex with a control function f. Due to the differentiability of f and the regularity assumption upon F the map

$$(a,b)\ni x\longmapsto f(x)+\|F(x)\|\in\mathbb{R}$$

is upper bounded on a set of positive Lebesgue measure. Thus, with the aid of author's result from [4], we obtain the local Lipschitz property of F and, in particular, the fact that F is a delta convex map controlled by f in the sense of L. Veselý & L. Zajíček (see [10]). Therefore, for any member x^* from the unit ball in the dual space E^* the function $g_* := f + x^*\circ F$ is convex. Moreover, on account of Proposition 3.9 (i) in [10, p. 22] (see also Remark 2.2, below), F yields a differentiable map. Hence, g_* is differentiable as well and the derivative g_*' is increasing. Consequently, for any two fixed elements $x,y\in(a,b), x\leq y$, we obtain the inequality

$$(x^*\circ F)'(x)-(x^*\circ F)'(y)=g_*'(x)-f'(x)-g_*'(y)+f'(y) \\ \leq -f'(x)+f'(y)\leq |f'(x)-f'(y)|.$$

Replacing here the functional x^* by $-x^*$ we arrive at

$$|x^*(F'(x)-F'(y))|=|(x^*\circ F)'(x)-(x^*\circ F)'(y)|\leq|f'(x)-f'(y)|,$$

which, due to the unrestricted choice of x^* from the unit ball in E^*, implies that

$$\|F'(x)-F'(y)\|\leq|f'(x)-f'(y)|.$$

In the case where $y \le x$ it suffices to interchange the roles of x an y in the latter inequality. Thus the proof has been completed. $\square$

REMARK 2.2. Actually, Proposition 3.9 (i) in [10, p. 22] states that F is even *strongly differentiable* at each point $x \in (a,b)$, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ and an element $c(x) \in E$ such that for all points $u, v \in (x-\delta, x+\delta) \subset (a,b), u \ne v$, one has

$$\left\| \frac{F(v)-F(u)}{v-u} - c(x) \right\| \le \varepsilon.$$

Obviously, every strongly differentiable map is differentiable (in general, in the sense of Fréchet).

PROOF OF THEOREM 1.3. In view of Theorem 1.2, F is differentiable and the inequality

$$\|F'(x) - F'(y)\| \le |f'(x) - f'(y)|$$

holds true for all $x, y \in (a,b)$. Let a closed interval $[\alpha, \beta]$ be contained in (a,b). Since, a continuously differentiable function, $f'|_{[\alpha,\beta]}$ yields an absolutely continuous function, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every finite collection of pairwise disjoint subintervals $(a_1,b_1),(a_2,b_2),\dots,(a_k,b_k)$ of $[\alpha,\beta]$ with $\sum_{i=1}^{k}(b_i-a_i) < \delta$, one has $\sum_{i=1}^{k}|f'(b_i)-f'(a_i)| < \varepsilon$, whence

$$\sum_{i=1}^{k} \|F'(b_i) - F'(a_i)\| \le \sum_{i=1}^{k} |f'(b_i) - f'(a_i)| < \varepsilon.$$

This proves that the map $F'|_{[\alpha,\beta]}$ is absolutely continuous as well. Since the space $(E, \|\cdot\|)$ enjoys the Radon-Nikodym property, in virtue of Theorem 5.21 from the monograph [1] by Y. Benyamini and J. Lindenstrauss, the map $F'|_{[\alpha,\beta]}$ is differentiable almost everywhere in $[\alpha,\beta]$, i.e. off some nullset $T \subset [\alpha,\beta]$ the second derivative $F''(x)$ of F at x does exist for all $x \in [\alpha,\beta] \setminus T$.

Now, fix arbitrarily a strictly decreasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ and a strictly increasing sequence $(\beta_n)_{n\in\mathbb{N}}$ such that $a < \alpha_n < \beta_n < b$, $n \in \mathbb{N}$, convergent to a and b, respectively. Then, for every $n \in \mathbb{N}$ one may find a nullset $T_n \subset [\alpha_n, \beta_n]$ such that the second derivative $F''(x)$ of F at x does exist for all $x \in [\alpha,\beta] \setminus T_n$. Putting $T := \bigcup_{n\in\mathbb{N}} T_n$ we obtain a set of Lebesgue measure zero, contained in (a,b), such that the second derivative $F''(x)$ does exist for all points $x \in (a,b) \setminus T$. Fix arbitrarily a point $x \in (a,b) \setminus T$. Then for any point $y \in (a,b) \setminus \{x\}$ we have

$$\left\| \frac{F'(y)-F'(x)}{y-x} \right\| \le \left| \frac{f'(y)-f'(x)}{y-x} \right|$$

and passing to the limit as $y \to x$ we conclude that

$$\|F''(x)\| \leq |f''(x)| = f''(x),$$

because of the convexity of f, which completes the proof. □

Remark 2.3. Theorem 5.21 from [1] states, among others, that any absolutely continuous map from the unit interval $[0, 1]$ into a normed linear space E with the Radon-Nikodym property is differentiable almost everywhere. It is an easy task to check (an affine change of variables) that any absolutely continuous map on a compact interval $[\alpha, \beta] \subset \mathbb{R}$ with values in E is almost everywhere differentiable as well.

References

[1] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, American Mathematical Society, Colloquium Publications, **48**, American Mathematical Society, Providence, 2000.
[2] N. Bourbăcuţ, *Problem 11641*, Amer. Math. Monthly **119** (2012), no. 4, p. 345.
[3] N. Dunford and B.J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323–392.
[4] R. Ger, *Stability aspects of delta-convexity*, in: Th.M. Rasssias, J. Tabor (eds.), *Stability of Mappings of Hyers-Ulam Type*, Hadronic Press, Palm Harbor, 1994, pp. 99–109.
[5] R. Ger, *A Functional Inequality, Solution of Problem 11641*, Amer. Math. Monthly **121** (2014), no. 2, 174–175.
[6] N. Kuhn, *On the structure of (s, t)-convex functions*, in: W. Walter (ed.), *General Inequalities. 5.* Proceedings of the Fifth International Conference held in Oberwolfach, May 4-10, 1986, International Series of Numerical Mathematics, **80**, Birkhäuser Verlag, Basel, 1987, pp. 161–174.
[7] D.Ş. Marinescu and M. Monea, *An extension of a Ger's result*, Ann. Math. Sil. **32** (2018), 263–274.
[8] A. Olbryś, *A support theorem for delta (s, t)-convex mappings*, Aequationes Math. **89** (2015), no. 3, 937–948.
[9] R.S. Phillips, *On weakly compact subsets of a Banach space*, Amer. J. Math. **65** (1943), 108–136.
[10] L. Veselý and L. Zajíček, *Delta-convex Mappings Between Banach Spaces and Applications,* Dissertationes Math. (Rozprawy Mat.) **289**, Polish Scientific Publishers, Warszawa, 1989.

Institute of Mathematics
University of Silesia in Katowice
Bankowa 14
40-007 Katowice
Poland
e-mail: roman.ger@us.edu.pl