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FRACTIONAL HEAT CONDUCTION IN A RECTANGULAR PLATE WITH BENDING MOMENTS

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Abstract. In this research work, we consider a thin, simply supported rectangular plate defined as $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ and determine the thermal stresses by using a thermal bending moment with the help of a time dependent fractional derivative. The constant temperature is prescribed on the surface $y = 0$ and other surfaces are maintained at zero temperature. A powerful technique of integral transform is used to find the analytical solution of initial-boundary value problem of a thin rectangular plate. The numerical result of temperature distribution, thermal deflection and thermal stress component are computed and represented graphically for a copper plate.

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1. Introduction

Thermal stress analysis of a rectangular plate and its thermal stress intensity factor for a compressive stress field have been discussed by Tanigawa and Komatsubara [1]. Gogulwar and Deshmukh [2] studied thermal stresses in a rectangular plate due to partially distributed heat supply. Kulkarni and Deshmukh [3] deals with the realistic problem of the quasi-static thermal stresses in a rectangular plate subjected to constant heat supply on the extreme edges $(x = a, y = b)$, whereas the initial edges $(x = 0, y = 0)$ are thermally insulated. Deshmukh et al. [4] discussed thermal stresses in a simply supported plate with thermal bending moments with a heat source.

Fractional-order differential equations have been the forefront of research due to their applications in many real-life problems of fluid mechanics, viscoelasticity, biology, physics, and engineering. It is a well-known fact that the integer-order differential operator is a local operator but the fractional-order differential operator is non-local. Therefore, the next state of a system depends not only upon its current state, but also upon all of its historical states. This is much more realistic and due to this reason, the fractional derivative is also known as a memory dependent derivative. In recent times, the approaches of fractional order derivatives have become more popular amongst many researchers. The reason behind the introduction of the fractional theory is that it predicts a delayed response to physical stimuli, as found in nature, as opposed to the instantaneous response predicted by the generalized theory of thermoelasticity.

The fractional-order theory of thermoelasticity was developed by Sherief et al. [5]. The work on quasi-static fractional-order thermoelasticity can be found in the literature [6-11]. Raslan [12] studied the application of the fractional-order theory of thermoelasticity in a thick plate under axisymmetric temperature distribution. Warbhe et al. [13, 14] discussed the fractional heat conduction problem in a thin circular plate with constant temperature distribution and associated thermal stresses within the context of the quasi-static theory and studied the fractional order thermoelastic deflection in a thin hollow circular disk. Tripathi et al. [15] solved a fractional order thermoelastic deflection in a thin circular plate with constant temperature distribution within the context of the quasi-static theory. Tripathi et al. [16] solved a problem on fractional order generalized thermoelastic response in a half space due to a periodically varying heat source.

In this article, we investigate the thermal stress components due to the thermal bending moment in a thin rectangular plate subjected to constant temperature in the context of the fractional order theory by the quasi-static approach. Copper material is chosen for numerical purposes, and the numerical calculations has been carried out with the help of computational mathematical software Mathcad 2000. The results of temperature, thermal stresses and thermal deflection are illustrated graphically.

2. Mathematical formulation of the problem

Consider a thin rectangular plate with its dimension $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. The constant temperature Q_0 is applied to the boundary surface $y = 0$ and the some other boundaries are maintained at zero temperature. A hypothetical mathematical model is prepared considering a nonlocal Caputo type time fractional heat conduction equation of order α for a thin rectangular plate.

The definition of the Caputo type fractional derivative is given by [6]

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \quad n-1 < \alpha < n \tag{1}$$

For finding the Laplace transform, the Caputo derivative requires knowledge of the initial values of the function $f(t)$ and its integer derivatives of the order $k = 1, 2, ..., n-1$

$$L\left\{\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}}\right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) s^{\alpha-1-k}, \quad n-1 < \alpha < n \tag{2}$$

where the asterisk denotes the Laplace transform with respect to time, s is the Laplace transform parameter.

The boundary value problem of heat conduction of a homogeneous isotropic solid is given as,

$$h\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \frac{\partial^{\alpha} T}{\partial t^{\alpha}}, \; 0 \le x \le a, 0 \le y \le b, 0 \le z \le c, t > 0 \tag{3}$$

with the conditions

$$T = 0 \quad \text{at } x = 0 \tag{4}$$

$$T = 0 \quad \text{at } x = a \tag{5}$$

$$T = Q_0 \quad \text{at } y = 0 \tag{6}$$

$$T = 0 \quad \text{at } y = b \tag{7}$$

$$T = 0 \quad \text{at } z = 0 \tag{8}$$

$$T = 0 \quad \text{at } z = c \tag{9}$$

and initially

$$T = 0 \quad \text{at } t = 0, \; 0 < \alpha < 2 \tag{10}$$

$$\frac{\partial T}{\partial t} = 0 \quad \text{at } t = 0, 1 < \alpha < 2 \tag{11}$$

where $T = T(x, y, z, t)$; h is thermal diffusivity of the material of the plate.

Here we consider a simply supported rectangular plate with its dimension $a \times b$ subjected to thermal load. The fundamental equation and the associated boundary conditions in the Cartesian coordinate system are given as [4]

$$\nabla^2 \nabla^2 w = \frac{-1}{(1-v)D} \nabla^2 M_T, \tag{12}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

with $$w = 0, \; \frac{\partial^2 w}{\partial x^2} = \frac{-1}{(1-v)D} M_T, \text{ on } x = 0, x = a, \tag{13}$$

and $$w=0,\ \frac{\partial^2 w}{\partial y^2}=\frac{-1}{(1-v)D}M_T,\ \text{on}\ \ y=b, \tag{14}$$

where w is the thermal deflection, M_T is the thermally induced resultant moment and D is the bending rigidity of the plate.

Considering the equilibrium state in the in-plate directions of x and y, the inplate resultant forces are

$$N_x=N_y=N_{xy}=0. \tag{15}$$

The resultant moments M_x, M_y, M_{xy} per unit length of the plate are defined as:

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}\right)-\frac{1}{(1-v)}M_T, \tag{16}$$

$$M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+v\frac{\partial^2 w}{\partial x^2}\right)-\frac{1}{(1-v)}M_T, \tag{17}$$

and $$M_{xy}=(1-v)D\frac{\partial^2 w}{\partial x \partial y}. \tag{18}$$

The equilibrium equations of moments about the x and y axes are:

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}-Q_x=0, \tag{19}$$

$$\frac{\partial M_y}{\partial y}-\frac{\partial M_{xy}}{\partial x}-Q_y=0, \tag{20}$$

where Q_x, Q_y are the shearing forces.

The bending rigidity D and thermally induced resultant moment M_T of the plate are defined as

$$D=\frac{Eh^3}{12(1-v^2)}, \tag{21}$$

$$M_T=a_t E\int_0^c T\,z\,dz \tag{22}$$

a_t, E and v are the coefficients of the linear thermal expansion, the Young's modulus, and Poisson's ratio of the plate material, respectively.

The thermal stress components in terms of the resultant forces and resultant moments are given as [17]

$$\sigma_{xx}=\frac{1}{c}N_x+\frac{12z}{c^3}M_x+\frac{1}{1-v}\left(\frac{1}{c}N_T+\frac{12z}{c^3}M_T-\alpha ET\right), \tag{23}$$

$$\sigma_{yy}=\frac{1}{c}N_y+\frac{12z}{c^3}M_y+\frac{1}{1-v}\left(\frac{1}{c}N_T+\frac{12z}{c^3}M_T-\alpha ET\right), \tag{24}$$

$$\sigma_{xy}=\frac{1}{c}N_{xy}-\frac{12z}{c^3}M_{xy}, \tag{25}$$

where the resultant force is

$$N_T=a_t E\int_0^c T\,dz. \tag{26}$$

The deflection with

$$w=0 \text{ at } x=a,\ y=b, \tag{27}$$

the moments

$$M_x=M_y=0 \text{ at } x=a,\ y=b, \tag{28}$$

the shearing forces

$$Q_x=Q_y=0, \tag{29}$$

and the thermal stresses

$$\sigma_{xx}=\sigma_{yy}=0 \text{ at } x=a, y=b \tag{30}$$

Equations (1) to (30) constitute the mathematical formulation of the problem.

3. Solution

By applying Fourier, and Laplace transforms and successfully inverted by the inversions defined as [18, 19] to the boundary value problem of heat conduction, one obtains the temperature distribution function as

$$T(x,y,z,t)=\frac{Q_0 2\sqrt{2}}{\sqrt{abc}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}K(\beta_i,x)K(v_j,y)K(\eta_k,z)$$
$$\times\frac{v_j}{\beta_i\eta_k}\left[1-\cos(\beta_j a)\right]\left[1-\cos(\eta_k c)\right]\left\{1-E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\} \tag{31}$$

where, $K(\beta_i,x)=\sqrt{2/a}\sin(\beta_i x),\ \beta_i=\dfrac{i\pi}{a},\ i=1,2,3......$

$K(v_j,y)=\sqrt{2/b}\sin(v_j y),\ v_j=\dfrac{j\pi}{b},\ j=1,2,3......$

$K(\eta_k,z)=\sqrt{2/c}\sin(\eta_k z),\ \eta_k=\dfrac{k\pi}{c},\ k=1,2,3,......$

Here, $E_\alpha(.)$ represents the Mittag-Leffler function.

Now substitute (31) in (26), and one obtains the resultant force

$$N_T=\frac{Q_0 2\sqrt{2}\,a_t E}{\sqrt{abc}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}K(\beta_i,x)K(v_j,y)\frac{v_j}{\beta_i\eta_k^2}\left[(-1)^{k+1}+1\right]$$
$$\times\left[1-\cos(\beta_i a)\right]\left[1-\cos(\eta_k c)\right]\left\{1-E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\} \tag{32}$$

and substitute (31) in (22), then one gets the thermal moment as

$$M_T=\frac{Q_0 2\sqrt{2}\,a_t E}{\sqrt{abc}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}K(\beta_i,x)K(v_j,y)(-1)^{k+1}\frac{c\,v_j}{\beta_i\eta_k^2}$$
$$\times\left[1-\cos(\beta_i a)\right]\left[1-\cos(\eta_k c)\right]\left\{1-E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\} \tag{33}$$

Using equation (33) in (12), (13), (14), thermal deflection is obtained as

$$w=\frac{a_t E}{D(1-v)}\frac{Q_0 2\sqrt{2}}{\sqrt{abc}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}K(\beta_i,x)K(v_j,y)(-1)^{k+1}\frac{c\,v_j}{\beta_i\eta_k^2}\frac{1}{\beta_i^2+v_j^2}$$
$$\times\left[1-\cos(\beta_i a)\right]\left[1-\cos(\eta_k c)\right]\left\{1-E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\}. \tag{34}$$

Using (33) and (34) in (16), (17), (18), one obtains the resultant moment as:

$$M_x=-\,a_t E\frac{Q_0 2\sqrt{2}}{\sqrt{abc}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}K(\beta_i,x)K(v_j,y)(-1)^{k+1}\frac{c\,v_j}{\beta_i\eta_k^2}$$
$$\times\left[1-\cos(\beta_i a)\right]\left[1-\cos(\eta_k c)\right]\frac{v_j^2}{\beta_i^2+v_j^2}\left\{1-E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\} \tag{35}$$

$$M_y = -a_t E \frac{Q_0 2\sqrt{2}}{\sqrt{abc}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} K(\beta_i, x) K(v_j, y)(-1)^{k+1} \frac{c v_j}{\beta_i \eta_k^2}$$
$$\times \left[1-\cos(\beta_i a)\right]\left[1-\cos(\eta_k c)\right] \frac{\beta_i^2}{\beta_i^2+v_j^2} \left\{1 - E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\} \quad (36)$$

$$M_{xy} = a_t E \frac{Q_0 2\sqrt{2}}{\sqrt{abc}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (-1)^{k+1} \frac{c v_j^2}{(\beta_i^2+v_j^2)\eta_k^2} \cos(\beta_i x)\cos(v_j y)$$
$$\times \frac{2}{\sqrt{ab}}\left[1-\cos(\beta_i a)\right]\left[1-\cos(\eta_k c)\right]\left\{1 - E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\} \quad (37)$$

Using equations (15), (31), (32), (33), (35), (36), (37) in (23) (24) and (25), one obtains expressions for the thermal stresses as:

$$\sigma_{xx} = a_t E \frac{Q_0 2\sqrt{2}}{\sqrt{abc}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} K(\beta_i, x) K(v_j, y) \frac{v_j}{\beta_i \eta_k}\left[1-\cos(\beta_i a)\right]\left[1-\cos(\eta_k c)\right]$$
$$\times \left\{1 - E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\}\left[-\frac{12z}{c^2}\frac{v_j^2}{\eta_k\left(\beta_i^2+v_j^2\right)} + \frac{1}{c(1-v)}\frac{[(-1)^{k+1}+1]}{\eta_k}\right.$$
$$\left. + \frac{12z}{c^2}\frac{(-1)^{k+1}}{\eta_k} - K(\eta_k, z)\right] \quad (38)$$

$$\sigma_{yy} = a_t E \frac{Q_0 2\sqrt{2}}{\sqrt{abc}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} K(\beta_i, x) K(v_j, y) \frac{v_j}{\beta_i \eta_k}\left[1-\cos(\beta_i a)\right]\left[1-\cos(\eta_k c)\right]$$
$$\times \left\{1 - E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\}$$
$$\left[-\frac{12z}{c^2}\frac{(-1)^{k+1}}{\eta_k}\frac{\beta_i^2}{\left(\beta_i^2+v_j^2\right)} + \frac{1}{c(1-v)}\frac{[(-1)^{k+1}+1]}{\eta_k} + \frac{12z}{c^2}\frac{(-1)^{k+1}}{\eta_k} - K(\eta_k, z)\right] \quad (39)$$

$$\sigma_{xy} = -a_t E \frac{12z}{c^2}\frac{Q_0 2\sqrt{2}}{\sqrt{abc}}\sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}\sqrt{\frac{2}{c}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \cos(\beta_i x)\cos(v_j y)(-1)^{k+1}\frac{v_j^2}{\eta_k^2\left(\beta_i^2+v_j^2\right)}$$
$$\times \left[1-\cos(\beta_i a)\right]\left[1-\cos(\eta_k c)\right]\left\{1 - E_\alpha\left[-h(\beta_i^2+v_j^2+\eta_k^2)t^\alpha\right]\right\} \quad (40)$$

4. Numerical calculation

To prepare the hypothetical mathematical model for different values of parameters and functions for a copper material, the chosen dimensions are

4.1. Dimension

Length of the rectangular plate $a = 4$ m,
Breadth of the rectangular plate $b = 2.1$ m,
Height of the rectangular plate $c = 0.4$ m.

4.2. The material Properties as

To analyze this problem numerically, we use the following data for the copper (pure) thin rectangular plate with the material properties:
Thermal diffusivity $h = 112.34\times10^{-6}\,(\mathrm{m^2 s^{-1}})$, Density $\rho = 8954\,(\mathrm{kg/m^3})$,
Specific heat $c_p = 383\,(\mathrm{J/kgK})$, Poisson ratio $\nu = 0.35$,
Coefficient of linear thermal expansion $a_t = 16.5\times10^{-6}$ 1/K,
Lamé constant $\mu = 26.67$ GPa, $Q_0 = 500$.

The graphs are plotted for fractional order parameter $\alpha = 0, 0.5, 1, 1.5, 2$ depicting weak, normal and strong conductivity at time $t = 5$ s.

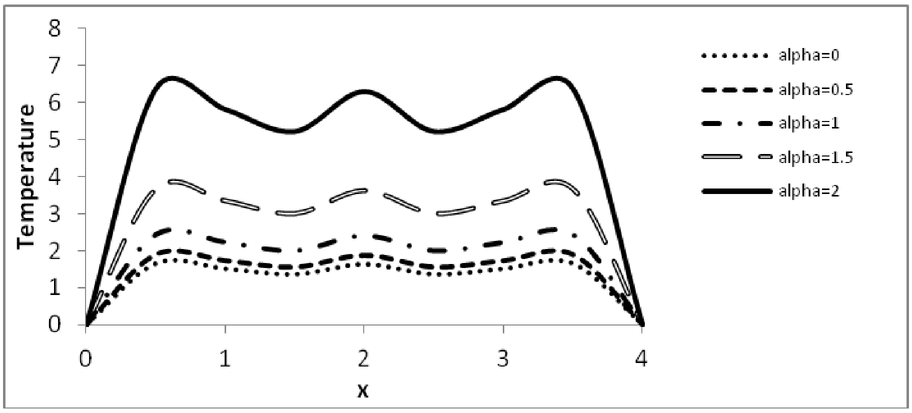

Fig. 1. Temperature distribution for different values of α in X-direction

From Figure 1 and 2, it can be observed that for different values of the fractional order parameter, the temperature distribution function and thermal deflection start to increase from the initial edge, after that it is fluctuating in the regions $x = 0.5$ to $x = 3.5$ and then decrease towards the extreme edge in the X direction.

From Figure 3 it is observed that the stress function σ_{xx} is tensile and it forms non-uniform pattern along the X-axis. Additionally, the stress function is directly proportional to the different values of fractional order parameter α in the X direction.

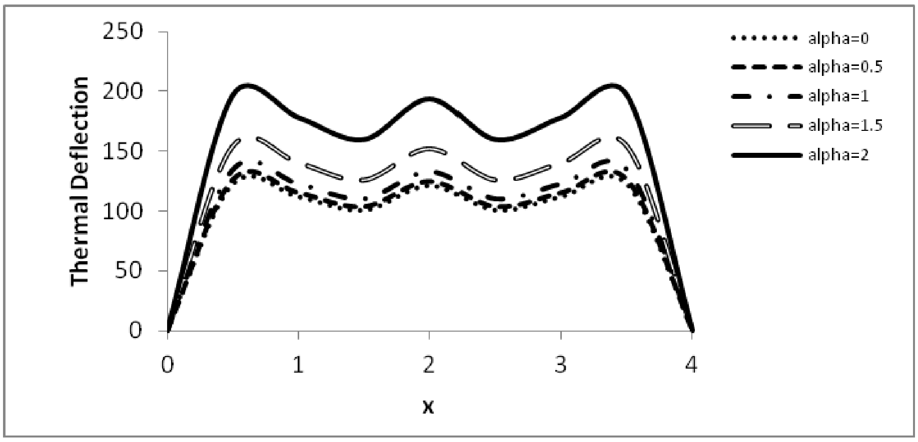

Fig. 2. Thermal deflection for different values of α in X-direction

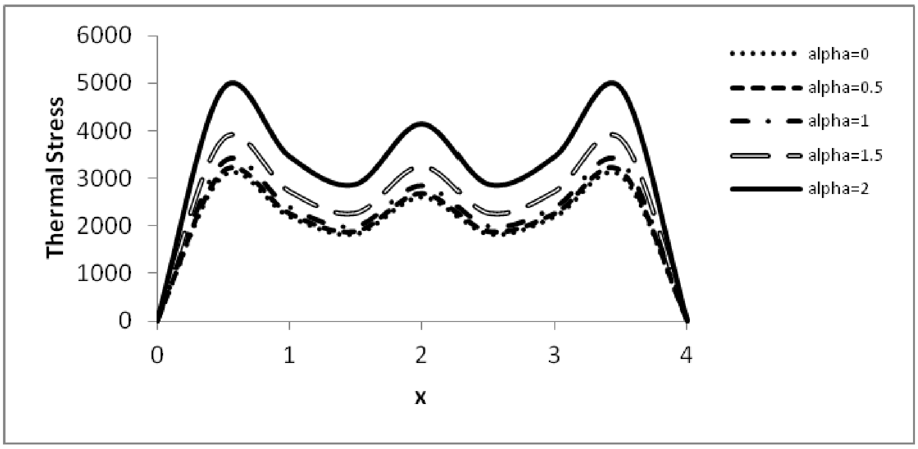

Fig. 3. Distribution of stress σ_{xx} for different values of α in X-direction

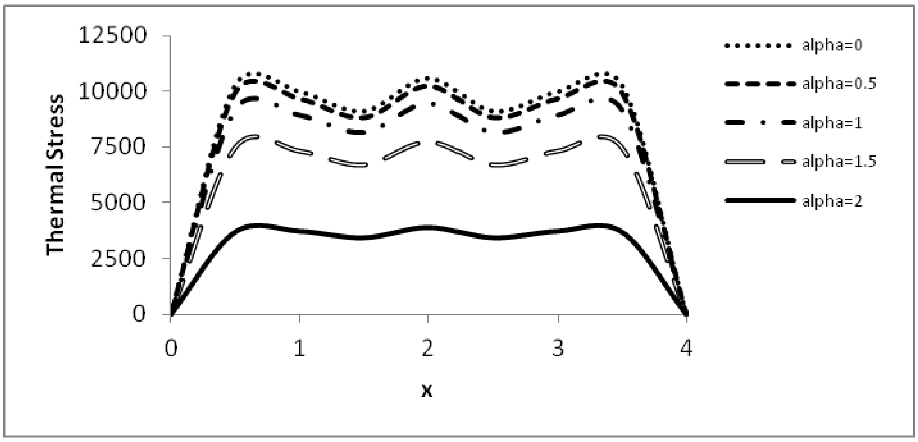

Fig. 4. Distribution of stress σ_{yy} for different values of α in X-direction

From Figure 4 it is observed that the stress function σ_{yy} is tensile and it forms non-uniform pattern along the X-axis. Additionally, the stress is inversely proportional to the different values of fractional order parameter α in the X direction.

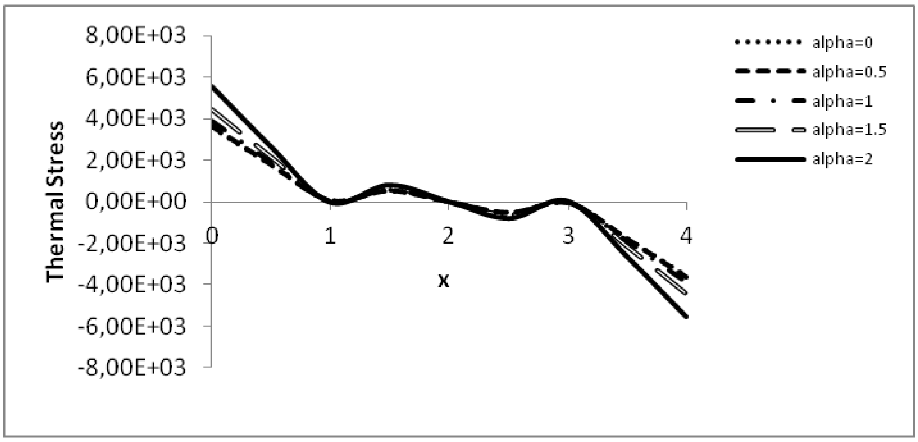

Fig. 5. Distribution of stress σ_{xy} for different values of α in X-direction

From Figure 5 it can be observed that for different values of fractional order parameter α, the stress function σ_{xy} is tensile in the range $0 \le x \le 2$ and it is compressive in the range $2 \le x \le 4$ in the X direction.

5. Conclusion

In this article, we study the time fractional heat conduction equation under zero initial conditions. The integral transform is used to solve the problem. Figures 1-5 depict the behavior of temperature, thermal stresses and bending moments along X direction for different values of the fractional order parameter $\alpha = 0, 0.5, 1, 1.5, 2$ and show the variation between classical thermoelasticity and fractional order thermoelasticity. It is also observed form the figures that when $\alpha = 0$, all the graphical representation satisfied the Helmholtz equation, whereas when $\alpha = 1$, it predicts the diffusion equation and $\alpha = 2$ gives the representation of wave equations. It is a general observation that while using time dependent fractional order which predicts the memory effects and interpolates the classical uncoupled theory of thermoelastic problems.

Furthermore, it is observed that for different values of the parameter α in the X directions the effects of temperature, thermal stresses and thermal deflection predicted weak conductivity, moderate conductivity and super conductivity for a fixed time. The reason behind the consideration of the fractional theory is that it predicts delayed response to physical stimuli, as is found in nature, as opposed to instantaneous response predicted by the generalized theory of thermoelasticity. This type of problem are particularly applicable to the researchers working in material science, the design of new materials and those working to further develop the theory of thermoelasticity by the classical approach using the Caputo fractional order derivative.

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