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FREQUENCY ANALYSIS OF COMPOSITE ANNULAR MEMBRANES

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Abstract. In this paper, the free vibration of an annular membrane consisting of three concentric segments is considered. The frequency equation and mode shapes are obtained by the use of the Green's function method. A numerical example to vibration problem of non-homogeneous annular membrane is presented.

Introduction

The problem of transverse vibrations of composite annular membranes has been considered by several authors [1-4]. In the papers [1, 2] the free vibration of composite membranes consisting of two annular segments is investigated. The exact solution of the problem and vibration analysis of membranes with discontinuously varying thickness is presented. In reference [2] the finite element method in the analysis was also used. The free vibration problem of annular membrane with many discontinuous variation of the density is the subject of paper [3]. Although the formulation of the problem deals with the vibration of membrane which consist of n segments (each of constant density), the numerical examples concern the antisymmetric modes of composite membrane consisting of two segments. In this case the frequency equation is obtained by setting the determinant of a 4x4 matrix of coefficients to a derived system of equation, which equals zero. In many papers various methods are applied to solution of the eigenvalue problems. The authors of paper [4] in solving the eigenproblem for annular membrane propose the method of fundamental solution. In this method the free space Green's function are applied.

In this paper the free vibration problem of a composite annular membrane consisting of three segments of constant densities is presented. The solution of the problem (frequency equation, mode shapes) is derived by using the properties of Green's functions corresponding to the Helmholtz operator in an annular domain. An example of numerical frequency analysis is given.

1. Formulation and solution of the problem

The free vibration of a membrane consisting of three annular segments (Fig. 1) is governed by the following differential equations:

$$\nabla^2 u_i - k_i^2 \frac{\partial^2 u_i}{\partial t^2} = -s_{i-1}(t)\frac{\delta(r-r_{i-1})}{r} + s_i(t)\frac{\delta(r-r_i)}{r}, \quad i = 1,2,3 \tag{1}$$

where u_i is the displacement of the i-th membrane segment, $k_i = \sqrt{\rho_i / \tilde{s}}$, $\tilde{s}$ is the tension per unit length, ρ_i is the density of i-th segment of the membrane, $\delta(\cdot)$ is the Dirac delta function, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ is the Laplace operator, r, θ are polar coordinates and t is time. The densities of the membrane change stepwise at circles with radii r_1 and r_2. These circles determine the uniform segments of the membrane.

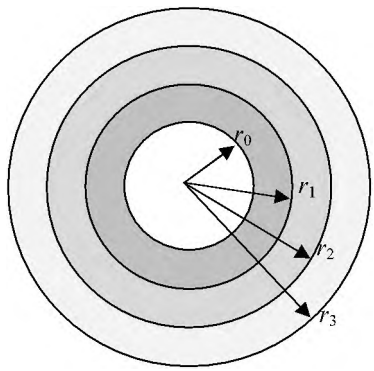

Fig. 1. A sketch of a membrane under study

The functions u_i (i = 1,2,3) satisfy the continuity conditions:

$$u_1(r_1,\theta,t) = u_2(r_1,\theta,t), \quad u_2(r_2,\theta,t) = u_3(r_2,\theta,t)$$
$$\left.\frac{\partial u_1(r,\theta,t)}{\partial r}\right|_{r=r_1} = \left.\frac{\partial u_2(r,\theta,t)}{\partial r}\right|_{r=r_1}, \quad \left.\frac{\partial u_2(r,\theta,t)}{\partial r}\right|_{r=r_2} = \left.\frac{\partial u_3(r,\theta,t)}{\partial r}\right|_{r=r_2} \tag{2}$$

and the zero boundary conditions

$$u_1(r_0,\theta,t) = 0, \quad u_3(r_3,\theta,t) = 0 \tag{3}$$

Considering the free vibration of the membrane we assume the functions $u_i(r,\theta,t)$ and $s_i(t)$ in the form:

$$u_i(r,\theta,t) = U_{in}(r)\cos n\theta \cos \omega t, \quad s_i(t) = S_i \cos \omega t \tag{4}$$

where ω is the natural frequency of the membrane. Taken into account equations (4) differential equation (1), continuity and boundary conditions (2)-(3), we obtain:

$$\mathbf{L}\left[U_{in}(r)\right] = -S_{i-1}\frac{\delta(r-r_{i-1})}{r} + S_i\frac{\delta(r-r_i)}{r} \quad i=1,2,3 \tag{5}$$

$$U_{1n}(r_1) = U_{2n}(r_1), \quad U_{2n}(r_2) = U_{3n}(r_2) \tag{6}$$

$$\left.\frac{dU_{1n}(r)}{dr}\right|_{r=r_1} = \left.\frac{dU_{2n}(r)}{dr}\right|_{r=r_1}, \quad \left.\frac{dU_{2n}(r)}{dr}\right|_{r=r_2} = \left.\frac{dU_{3n}(r)}{dr}\right|_{r=r_2} \tag{7}$$

$$U_{1n}(r_0) = 0, \quad U_{3n}(r_3) = 0 \tag{8}$$

where $\mathrm{L}_m = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \left(\lambda_i^2 - \frac{n^2}{r^2}\right)$.

The solution of the problem (5)-(8) can be obtained by the use of the Green's function method. The Green's functions corresponding to the operator L_m satisfy non-homogeneous differential equation

$$\mathbf{L}_m\left[G_i(r,\zeta)\right] = \frac{\delta(r-\zeta)}{r}, \quad i=1,2,3 \tag{9}$$

and the homogeneous boundary conditions:

$$\begin{gathered} G_1(r,\zeta)\big|_{r=r_0} = 0, \quad \frac{\partial}{\partial r}G_1(r,\zeta)\bigg|_{r=r_1} = 0 \\ \frac{\partial}{\partial r}G_2(r,\zeta)\bigg|_{r=r_1} = \frac{\partial}{\partial r}G_2(r,\zeta)\bigg|_{r=r_2} = 0 \\ \frac{\partial}{\partial r}G_3(r,\zeta)\bigg|_{r=r_3} = 0, \quad G_3(r,\zeta)\big|_{r=r_3} = 0 \end{gathered} \tag{10}$$

Multiplying the equation (5) by $rG_i(r,\zeta)$ and integrating in interval $\langle r_{i-1}, r_i\rangle$ for $i = 1,2,3$, we obtain

$$\int_{r_{i-1}}^{r_i} r\mathrm{L}_m\left[U_m(\zeta)\right]G_i(r,\zeta)\,d\zeta = \int_{r_{i-1}}^{r_i}\left(-S_{i-1}\delta(\zeta-r_{i-1}) + S_i\delta(\zeta-r_i)\right)G_i(r,\zeta)\,d\zeta \tag{11}$$

Next, we use the self-adjointness of the operator L_m, i.e.:

$$\int_a^b r\left[vL_m(u)-uL_m(v)\right]dr=0 \text{ for } a\le r\le b \tag{12}$$

for all functions u, v which satisfy the same zero conditions (10) as the functions G_i. Taking into account (12) in equation (11) we can write

$$U_m(r)=-S_{i-1}G_i(r,r_{i-1})+S_iG_i(r,r_i), \quad r_{i-1}\le r\le r_i \tag{13}$$

where we have used the property of Dirac delta function

$$\int_a^b f(\zeta)\delta(\zeta-r)d\zeta=f(r) \text{ for } a\le r\le b \tag{14}$$

Using (13) in conditions (6) we obtain a system of equations:

$$\begin{cases} S_{1n}\left[G_{1n}(r_1,r_1)+G_{2n}(r_1,r_1)\right]-S_{2n}G_{2n}(r_1,r_2)=0 \\ -S_{1n}G_{2n}(r_2,r_1)+S_{2n}\left[G_{2n}(r_2,r_2)+G_{3n}(r_2,r_2)\right] \quad 0 \end{cases} \tag{15}$$

These equations have a non-trivial solution for the values of parameter ω which are roots of the frequency equation

$$\left[G_{1n}(r_1,r_1)+G_{2n}(r_1,r_1)\right]\left[G_{2n}(r_2,r_2)+G_{3n}(r_2,r_2)\right]-G_{2n}(r_1,r_2)G_{2n}(r_2,r_1)=0 \tag{16}$$

Equation (16) is then solved numerically with respect to the non-dimensional frequency $\Omega_n=\omega_n r_3\sqrt{\frac{\rho_3}{s}}$.

The mode shapes of vibration corresponding to the determined frequencies we obtain from equation (13)

$$\begin{aligned} U_{1n}(r)&=S_{1n}G_{1n}(r,r_1), && r_0\le r\le r_1 \\ U_{2n}(r)&=-S_{1n}G_{2n}(r,r_1)+S_{2n}G_{2n}(r,r_2), && r_1<r\le r_2 \\ U_{3n}(r)&=-S_{2n}G_{3n}(r,r_2), && r_2<r\le r_3 \end{aligned} \tag{17}$$

where S_{2n} is received from (15a)

$$S_{2n}=S_{1n}\frac{G_{1n}(r_1,r_1)+G_{2n}(r_1,r_1)}{G_{2n}(r_1,r_2)} \tag{18}$$

2. Numerical examples

Consider a non-homogeneous annular membrane whose density changes stepwise in a radial direction. The membrane is characterized by radii: $r_0 = 0.2$, $r_1 = 0.4$, $r_2 = 0.6$, $r_3 = 0.8$, and by ratios of densities of the segments: $\sigma_1 = \frac{\rho_1}{\rho_3}$ and $\sigma_2 = \frac{\rho_2}{\rho_3}$. The eigenfrequencies of the membrane we calculate numerically by using the frequency equation (16). The bisection method was applied. The non-dimensional frequencies corresponding to symmetric modes of vibrations for various values of ratios σ_1 and σ_2 are presented in Table 1. The comparison of the results presented in the table leads to the conclusion that increase of the densities ratios causes decrease of the frequencies of the membrane.

Table 1

First five frequency values $\Omega_{0i} = \omega_{0i} r_3 \sqrt{\frac{\rho_3}{s}}$, for nonhomogeneous membrane shown in Figure 1 for various values of σ_1 and σ_2

σ_1	Ω_{01}	Ω_{02}	Ω_{03}	Ω_{04}	Ω_{05}
	$\sigma_2 = 0.1$				
0.1	7.22575	19.09385	31.38584	43.66073	55.38157
1.0	6.12412	9.09007	18.70560	20.28878	32.31386
5.0	1.43392	3.82080	6.26935	7.97735	8.94993
10.0	0.71866	1.91728	3.15893	4.40742	5.65656
	$\sigma_2 = 1.0$				
0.1	4.48688	10.11479	16.13784	22.28729	28.48738
1.0	4.09769	8.32377	12.52867	16.72628	20.92059
5.0	1.40561	3.70511	5.74119	6.78442	8.83797
10.0	0.71516	1.90563	3.13651	4.36547	5.54419
	$\sigma_2 = 5.0$				
0.1	1.01913	2.89382	5.18590	7.56466	9.91070
1.0	1.01416	2.86149	5.11402	7.43692	9.68276
5.0	0.89200	2.01103	3.20914	4.43146	5.66214
10.0	0.62994	1.43690	2.08907	3.04102	3.82215
	$\sigma_2 = 10.0$				
0.1	0.51144	1.45959	2.62192	3.83689	5.06853
1.0	0.51081	1.45556	2.61350	3.82366	5.04984
5.0	0.49536	1.33866	2.20651	2.87151	3.78985
10.0	0.44705	1.00976	1.61248	2.22774	2.84796

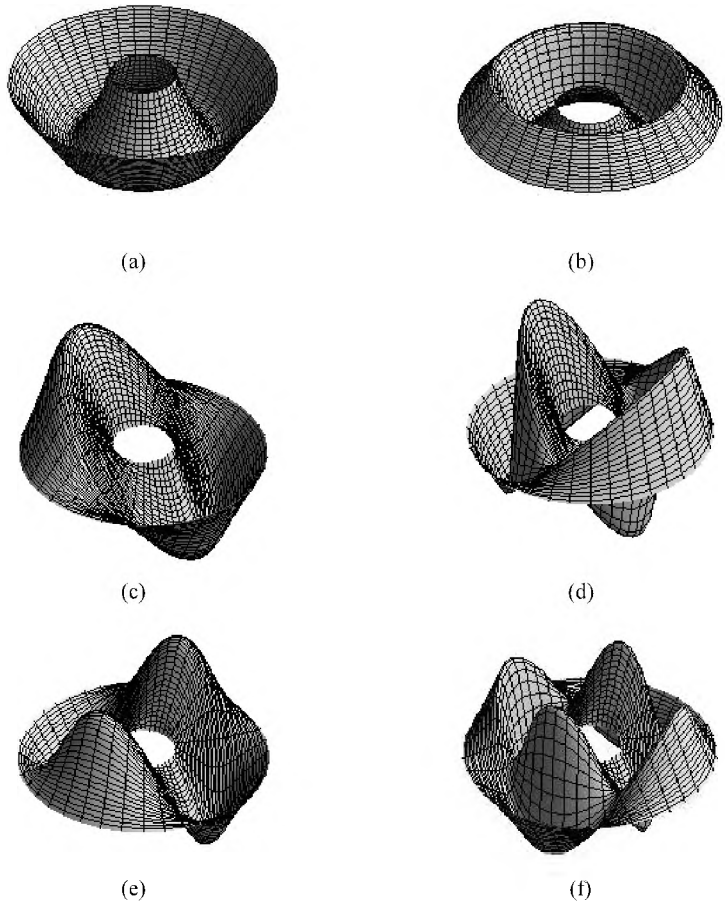

Fig. 2. Mode shapes for an annular membrane: a) $\Omega_{01} = 1.01416$, b) $\Omega_{02} = 2.86149$, c) $\Omega_{11} = 1.09474$, d) $\Omega_{12} = 2.90734$, e) $\Omega_{21} = 1.29468$, f) $\Omega_{22} = 3.03193$

Mode shapes of the annular membrane shown in Figure 1 for $r_0 = 0.2$, $r_1 = 0.4$, $r_2 = 0.6$, $r_3 = 0.8$, and $\sigma_1 = 1.0$; $\sigma_2 = 5.0$ are presented in Figure 2. The surfaces are plotted on the basis of formulas (17) and (18). The first two surfaces (Fig. 2a, b) obtained for $n = 0$, represent the axisymmetric modes of vibration and the remained for $n = 1$ and $n = 2$, present the antisymmetric modes of vibrations.

Conclusions

In this paper the solution to the problem of free vibration of an annular membrane consisting of three concentric segments, each of constant density is present-

ed. The frequency equation and mode shapes are expressed by Green's functions corresponding to relevant Helmholtz problems. Numerical results presented in the table show the effect of change the densities of the membrane segments on the eigenfrequencies of the system: increase of the ratios of densities causes a decrease of the free vibration frequencies. The presented here the method to solution of the free vibration problem of a membrane consisting of the three segments can be generalized on the annular membrane consisting of an arbitrary number of concentric segments.

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