SHAPE SENSITIVITY ANALYSIS OF TEMPERATURE DISTRIBUTION IN A NON-HOMOGENEOUS DOMAIN

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Abstract. The heated non-homogeneous domain from the two sub-domains compound is considered. The temperature distribution is described by the system of two Laplace equations. At the surface Γ_c between sub-domains the ideal contact is assumed, at the remaining surfaces the Dirichlet, Neumann and Robin conditions are taken into account. The problem is solved by means of the boundary element method. To estimate the changes of temperature due to the change of local geometry of internal boundary Γ_c the implicit variant of shape sensitivity analysis is applied. In the final part, the results of computations are shown and the conclusions are formulated.

Introduction

The system of two Laplace equations describing temperature distribution in non-homogeneous domain is considered

$$(x,y)\in\Omega_e: \quad \lambda_e \frac{\partial^2 T_e(x,y)}{\partial x^2}+\lambda_e \frac{\partial^2 T_e(x,y)}{\partial y^2}=0, \quad e=1,2 \tag{1}$$

where λ_e [W/(mK)] is the thermal conductivity of sub-domain Ω_e, T_e denotes the temperature and x, y are the geometrical co-ordinates.

On the contact surface between sub-domains the continuity of heat flux and the temperature field is assumed

$$(x,y)\in\Gamma_c: \quad \begin{cases} -\lambda_1 \dfrac{\partial T_1(x,y)}{\partial n}=\lambda_2 \dfrac{\partial T_2(x,y)}{\partial n} \\ T_1(x,y)=T_2(x,y) \end{cases} \tag{2}$$

where $\partial T_e(x,y)/\partial n$ is the normal derivative, $n=[n_x, n_y]$ is the normal outward vector.

On the remaining surfaces the Dirichlet, Neumann or Robin conditions can be taken into account.

The aim of the investigations is to estimate the changes of temperature due to change of local geometry of internal surface Γ_c.

1. Boundary element method

At first the homogeneous domain Ω is considered. In this case the boundary integral equation corresponding to the Laplace equation is the following [1-4]

$$(\xi,\eta)\in\Gamma: \quad B(\xi,\eta)T(\xi,\eta)+\int_{\Gamma}T^{*}(\xi,\eta,x,y)\,q(x,y)\mathrm{d}\Gamma = \int_{\Gamma}q^{*}(\xi,\eta,x,y)\,T(x,y)\mathrm{d}\Gamma \tag{3}$$

where $B(\xi,\eta)\in(0,1)$ is the coefficient connected with the local shape of boundary, (ξ,η) is the observation point, $q(x,y)=-\lambda\partial T(x,y)/\partial n$, $T^{*}(\xi,\eta,x,y)$ is the fundamental solution

$$T^{*}(\xi,\eta,x,y)=\frac{1}{2\pi\lambda}\ln\frac{1}{r} \tag{4}$$

where r is the distance between the points (ξ,η), (x,y) and

$$q^{*}(\xi,\eta,x,y) = -\lambda\frac{\partial T^{*}(\xi,\eta,x,y)}{\partial n} = \frac{d}{2\pi r^{2}} \tag{5}$$

while

$$d=(x-\xi)n_x+(y-\eta)n_y \tag{6}$$

In numerical realization of the BEM the boundary is divided into N boundary elements and integrals appearing in equation (3) are substituted by the sums of integrals over these elements

$$B(\xi_i,\eta_i)T(\xi_i,\eta_i)+\sum_{j=1}^{N}\int_{\Gamma_j}q(x,y)T^{*}(\xi_i,\eta_i,x,y)\mathrm{d}\Gamma_j = \sum_{j=1}^{N}\int_{\Gamma_j}T(x,y)q^{*}(\xi_i,\eta_i,x,y)\,\mathrm{d}\Gamma_j \tag{7}$$

For the linear boundary element Γ_j it is assumed that

$$(x,y)\in\Gamma_j: \quad \begin{cases} T(\theta)=N_pT_p^j+N_kT_k^j \\ q(\theta)=N_pq_p^j+N_kq_k^j \end{cases} \tag{8}$$

where $N_p=(1-\theta)/2$, $N_k=(1+\theta)/2$, $\theta\in[-1,1]$ are the shape functions.

After the mathematical manipulations [2, 5] one obtains the following system of equations ($i = 1, 2, \ldots, R$)

$$B_i T_i + \sum_{r=1}^{R} G_{i\,r} q_r = \sum_{r=1}^{R} \hat{H}_{i\,r} T_r \tag{9}$$

where for the single node r being the end of the boundary element Γ_j and being the beginning of the boundary element Γ_{j+1} one has

$$G_{i\,r} = G^{k}_{i\,j} + G^{p}_{i\,j+1}, \qquad \hat{H}_{i\,r} = \hat{H}^{k}_{i\,j} + \hat{H}^{p}_{i\,j+1} \tag{10}$$

while for double node r, $r + 1$

$$\begin{aligned} G_{i\,r} &= G^{k}_{i\,j}, \quad G_{i\,r+1} = G^{p}_{i\,j-1} \\ \hat{H}_{i\,r} &= \hat{H}^{k}_{i\,j}, \quad \hat{H}_{i\,r+1} = \hat{H}^{p}_{i\,j-1} \end{aligned} \tag{11}$$

In dependencies (10), (11):

$$G^{p}_{i\,j} = \frac{l_j}{4\pi\lambda} \int_{-1}^{1} N_p \ln \frac{1}{r_{i\,j}} \, \mathrm{d}\theta \tag{12}$$

$$G^{k}_{i\,j} = \frac{l_j}{4\pi\lambda} \int_{-1}^{1} N_k \ln \frac{1}{r_{i\,j}} \, \mathrm{d}\theta \tag{13}$$

and

$$\hat{H}^{p}_{i\,j} = \frac{1}{4\pi} \int_{-1}^{1} N_p \frac{r^j_x l^j_y - r^j_y l^j_x}{r^2_{i\,j}} \mathrm{d}\theta \tag{14}$$

$$\hat{H}^{k}_{i\,j} = \frac{1}{4\pi} \int_{-1}^{1} N_k \frac{r^j_x l^j_y - r^j_y l^j_x}{r^2_{i\,j}} \mathrm{d}\theta \tag{15}$$

where

$$r_{i\,j} = \sqrt{(N_p x^p_j + N_k x^k_j - \xi_i)^2 + (N_p y^p_j + N_k y^k_j - \eta_i)^2} = \sqrt{(r^j_x)^2 + (r^j_y)^2} \tag{16}$$

and

$$l_j = \sqrt{(x^k_j - x^p_j)^2 + (y^k_j - y^p_j)^2} = \sqrt{(l^j_x)^2 + (l^j_y)^2} \tag{17}$$

is the length of element Γ_j.

It should be pointed out that if (ξ_i,η_i) is the beginning of boundary element Γ_j, this means $(\xi_i,\eta_i) = (x_j^p, y_j^p)$ then

$$G_{i\,j}^{p} = \frac{l_j(3-2\ln l_j)}{8\pi\lambda}, \quad G_{i\,j}^{k} = \frac{l_j(1-2\ln l_j)}{8\pi\lambda}, \quad \hat{H}_{i\,j}^{p} = \hat{H}_{i\,j}^{k} = 0 \tag{18}$$

while if (ξ_i,η_i) is the end of boundary element Γ_j: $(\xi_i,\eta_i) = (x_j^k, y_j^k)$ then

$$G_{i\,j}^{p} = \frac{l_j(1-2\ln l_j)}{8\pi\lambda}, \quad G_{i\,j}^{k} = \frac{l_j(3-2\ln l_j)}{8\pi\lambda}, \quad \hat{H}_{i\,j}^{p} = \hat{H}_{i\,j}^{k} = 0 \tag{19}$$

The system of equations (9) can be written in the form

$$\sum_{r=1}^{R} G_{i\,r} q_r = \sum_{r=1}^{R} H_{i\,r} T_r, \quad i = 1,2,\ldots,R \tag{20}$$

or

$$\mathbf{Gq} = \mathbf{HT} \tag{21}$$

where

$$H_{i\,r} = \begin{cases} \hat{H}_{i\,r} & i \neq r \\ \hat{H}_{i\,r} - B_i & i = r \end{cases} \tag{22}$$

In the case of non-homogeneous domain $\Omega = \Omega_1 \cup \Omega_2$ two systems of equations for each sub-domain, should be taken into account separately. So, the condition (2) can be written in the form

$$(x,y) \in \Gamma_c: \quad \begin{cases} \mathbf{q}_{c1} = -\mathbf{q}_{c2} = \mathbf{q} \\ \mathbf{T}_{c1} = \mathbf{T}_{c2} = \mathbf{T} \end{cases} \tag{23}$$

and then one obtains the following systems of equations

$$(x,y) \in \Gamma_1 \cup \Gamma_c: \quad [\mathbf{G}_1 \;\; \mathbf{G}_{c1}] \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_{c1} \end{bmatrix} = [\mathbf{H}_1 \;\; \mathbf{H}_{c1}] \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_{c1} \end{bmatrix} \tag{24}$$

and

$$(x,y) \in \Gamma_2 \cup \Gamma_c: \quad [\mathbf{G}_2 \;\; \mathbf{G}_{c2}] \begin{bmatrix} \mathbf{q}_2 \\ \mathbf{q}_{c2} \end{bmatrix} = [\mathbf{H}_2 \;\; \mathbf{H}_{c2}] \begin{bmatrix} \mathbf{T}_2 \\ \mathbf{T}_{c2} \end{bmatrix} \tag{25}$$

Coupling of these system gives

$$(x,y)\in\Gamma_1\cup\Gamma_c\cup\Gamma_2:-\begin{bmatrix}\mathbf{G}_1 & -\mathbf{H}_{c1} & \mathbf{G}_{c1} & \mathbf{0}\\ \mathbf{0} & -\mathbf{H}_{c2} & -\mathbf{G}_{c2} & \mathbf{G}_2\end{bmatrix}\begin{bmatrix}\mathbf{q}_1\\ \mathbf{T}\\ \mathbf{q}\\ \mathbf{q}_2\end{bmatrix}=\begin{bmatrix}\mathbf{H}_1 & \mathbf{H}_2\end{bmatrix}\begin{bmatrix}\mathbf{T}_1\\ \mathbf{T}_2\end{bmatrix} \tag{26}$$

The remaining boundary conditions should be introduced, of course. Finally, the system of equations (26) can be written in the form

$$\mathbf{AZ}=\mathbf{B} \tag{27}$$

where **A** is the main matrix, **Z** is the unknown vector and **B** is the vector of the right-hand side.

2. Implicit differentiation method of shape sensitivity analysis

We assume that b is the shape parameter, this means b corresponds to the x or y coordinate of one of boundary node located at the contact surface between sub-domains. The implicit differentiation method [5-8] starts with the algebraic system of equations (27). The differentiation of (27) with respect to b leads to the following system of equations

$$\frac{\partial \mathbf{A}}{\partial b}\mathbf{Z}+\mathbf{A}\frac{\partial \mathbf{Z}}{\partial b}=\frac{\partial \mathbf{B}}{\partial b} \tag{28}$$

or

$$\mathbf{A}\frac{\partial \mathbf{Z}}{\partial b}=\frac{\partial \mathbf{B}}{\partial b}-\frac{\partial \mathbf{A}}{\partial b}\mathbf{Z} \tag{29}$$

So, this approach of shape sensitivity analysis is connected with the differentiation of elements of matrices **G** and **H** (c.f. equations (12)-(15)).

Taking into account the dependencies (10), (11) one has

– for a single boundary node

$$\frac{\partial G_{ir}}{\partial b}=\frac{\partial G_{ij}^{k}}{\partial b}+\frac{\partial G_{i,j+1}^{p}}{\partial b},\qquad \frac{\partial \hat{H}_{ir}}{\partial b}=\frac{\partial \hat{H}_{ij}^{k}}{\partial b}+\frac{\partial \hat{H}_{i,j-1}^{p}}{\partial b} \tag{30}$$

– for a double boundary node

$$\begin{aligned}\frac{\partial G_{ir}}{\partial b}&=\frac{\partial G_{ij}^{k}}{\partial b},\quad \frac{\partial G_{i,r+1}}{\partial b}=\frac{\partial G_{i,j-1}^{p}}{\partial b}\\ \frac{\partial \hat{H}_{ir}}{\partial b}&=\frac{\partial \hat{H}_{ij}^{k}}{\partial b},\quad \frac{\partial \hat{H}_{i,r+1}}{\partial b}=\frac{\partial \hat{H}_{i,j+1}^{p}}{\partial b}\end{aligned} \tag{31}$$

A differentiation of (12), (13) gives

$$\frac{\partial G_{ij}^{p}}{\partial b}=\frac{1}{4\pi\lambda}\left[\frac{\partial l_j}{\partial b}\int_{-1}^{1}N_p\ln\frac{1}{r_{ij}}\,\mathrm{d}\theta+l_j\int_{-1}^{1}N_p\frac{\partial}{\partial b}\left(\ln\frac{1}{r_{ij}}\right)\mathrm{d}\theta\right] \tag{32}$$

and

$$\frac{\partial G_{ij}^{k}}{\partial b}=\frac{1}{4\pi\lambda}\left[\frac{\partial l_j}{\partial b}\int_{-1}^{1}N_k\ln\frac{1}{r_{ij}}\,\mathrm{d}\theta+l_j\int_{-1}^{1}N_k\frac{\partial}{\partial b}\left(\ln\frac{1}{r_{ij}}\right)\mathrm{d}\theta\right] \tag{33}$$

where

$$\frac{\partial l_j}{\partial b}=\frac{1}{l_j}\left(l_x^j\frac{\partial l_x^j}{\partial b}+l_y^j\frac{\partial l_y^j}{\partial b}\right),\quad \frac{\partial l_x}{\partial b}=\frac{\partial}{\partial b}\left(x_j^k-x_j^p\right),\ \frac{\partial l_y}{\partial b}=\frac{\partial}{\partial b}\left(y_j^k-y_j^p\right) \tag{34}$$

and

$$\frac{\partial}{\partial b}\left(\ln\frac{1}{r_{ij}}\right)=-\frac{1}{r_{ij}}\frac{\partial r_{ij}}{\partial b} \tag{35}$$

where

$$\begin{gathered}\frac{\partial r_{ij}}{\partial b}=\frac{1}{r_{ij}}\left(r_x^j\frac{\partial r_x^j}{\partial b}+r_y^j\frac{\partial r_y^j}{\partial b}\right)\\ \frac{\partial r_x^j}{\partial b}=N_p\frac{\partial x_j^p}{\partial b}+N_k\frac{\partial x_j^k}{\partial b}-\frac{\partial \xi_i}{\partial b},\qquad \frac{\partial r_y^j}{\partial b}=N_p\frac{\partial y_j^p}{\partial b}+N_k\frac{\partial y_j^k}{\partial b}-\frac{\partial \eta_i}{\partial b}\end{gathered} \tag{36}$$

Next, using the formulas (14), (15) one obtains

$$\begin{gathered}\frac{\partial \hat{H}_{ij}^{p}}{\partial b}=\frac{1}{4\pi}\int_{-1}^{1}N_p\left[\frac{1}{r_{ij}^2}\left(\frac{\partial r_x^j}{\partial b}l_y^j+r_x^j\frac{\partial l_y^j}{\partial b}-\frac{\partial r_y^j}{\partial b}l_x^j-r_y^j\frac{\partial l_x^j}{\partial b}\right)-\right.\\ \left.\frac{2}{r_{ij}^4}\left(r_x^j\frac{\partial r_x^j}{\partial b}+r_y^j\frac{\partial r_y^j}{\partial b}\right)\left(r_x^j l_y^j-r_y^j l_x^j\right)\right]\mathrm{d}\theta\end{gathered} \tag{37}$$

and

$$\begin{gathered}\frac{\partial \hat{H}_{ij}^{k}}{\partial b}=\frac{1}{4\pi}\int_{-1}^{1}N_k\left[\frac{1}{r_{ij}^2}\left(\frac{\partial r_x^j}{\partial b}l_y^j+r_x^j\frac{\partial l_y^j}{\partial b}-\frac{\partial r_y^j}{\partial b}l_x^j-r_y^j\frac{\partial l_x^j}{\partial b}\right)-\right.\\ \left.\frac{2}{r_{ij}^4}\left(r_x^j\frac{\partial r_x^j}{\partial b}+r_y^j\frac{\partial r_y^j}{\partial b}\right)\left(r_x^j l_y^j-r_y^j l_x^j\right)\right]\mathrm{d}\theta\end{gathered} \tag{38}$$

In the case when shape parameter b corresponds to the node $(\xi_i, \eta_i) = (x_j^p, y_j^p)$ or to the node $(\xi_i, \eta_i) = (x_j^k, y_j^k)$ then the formulas (18), (19) should be differentiated with respect to b.

It should be pointed out that using the Taylor expansion

$$\begin{aligned} T(x,y,b+\Delta b) &= T(x,y,b) + U(x,y,b)\Delta b \\ T(x,y,b-\Delta b) &= T(x,y,b) - U(x,y,b)\Delta b \end{aligned} \tag{39}$$

one has

$$\Delta T(x,y) = T(x,y,b+\Delta b) - T(x,y,b-\Delta b) = 2U(x,y,b)\Delta b \tag{40}$$

where $U = \partial T/\partial b$ is the sensitivity function and Δb is the perturbation of parameter b. So, on the basis of formula (40) the change of temperature due to the change of parameter b can be estimated.

3. Results of computations

The non-homogeneous domain from two sub–domains compound as shown in Figure 1 is considered. On the upper boundary the Dirichlet condition T = 40°C has been assumed, but the temperature from the node 11 to the node 23 is changing according to the quadratic function (T_{max} = 60°C). On the bottom boundary T = 40°C has been accepted, on the remaining parts of boundary the Neumann condition q = 0 W/m^2 has been established. At the surface between sub-domains the ideal contact (c.f. equation (2)) is taken into account. The discretization of the domain is shown in Figure 1, while Figure 2 illustrates the temperature distribution in the domain considered.

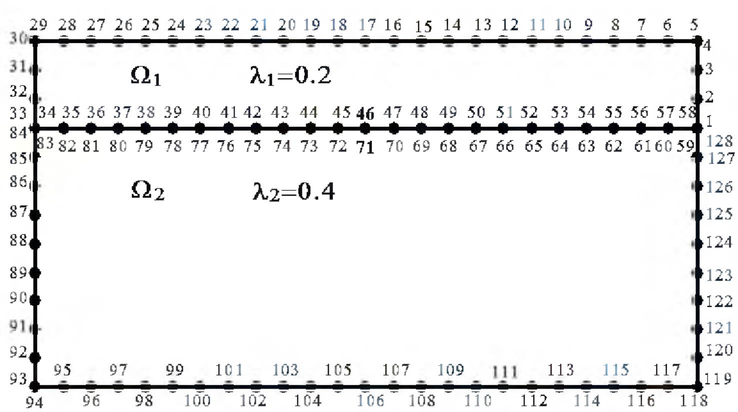

Fig. 1. Discretization

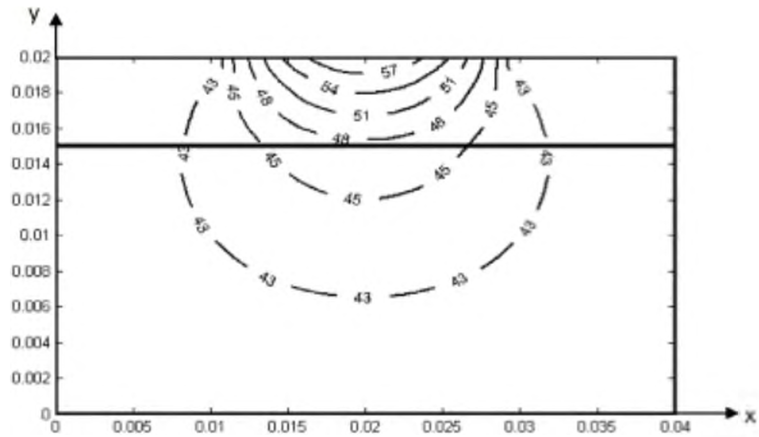

Fig. 2. Temperature distribution

The distribution of the sensitivity function $U = \partial T/\partial b$ under the assumption that $b = y_{46} = y_{71}$ (c.f. Figure 1) is the shape parameter is shown in Figure 3. The temperature at the node 46 = 71 equals 47.825, while the sensitivity function at this node equals 1268.99. So, using the formula (40) for Δb = 0.0001 m the change of temperature at the node 46 = 71 due to the change of parameter b is equal to 0.25°C. In Figure 4 the changes of temperature at the nodes located on the contact surface between sub-domains are presented.

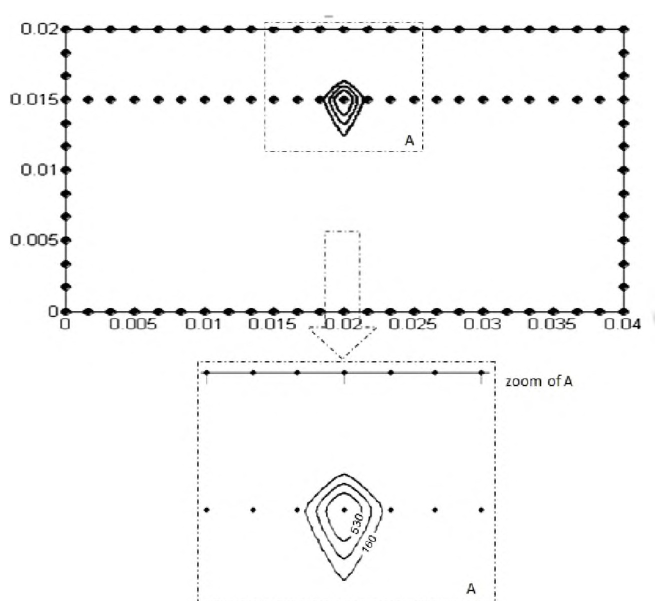

Fig. 3. Distribution of function U

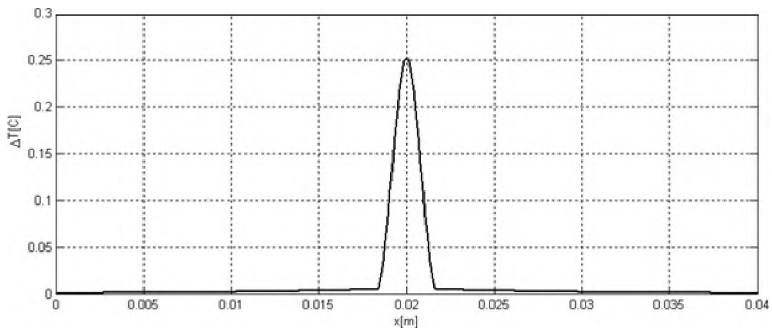

Fig. 4. Change of temperature at the contact surface due to the change of parameter $b = y_{46} = y_{71}$

Conclusions

The non-homogeneous domain from two sub-domains compound has been considered and the temperature distribution has been described by the system of two Laplace equations supplemented by boundary conditions. The problem has been solved using the boundary element method. The implicit method of shape sensitivity analysis has been discussed. To estimate the changes of temperature in the case when the local geometry of the boundary is changed the Taylor series containing the sensitivity function has been applied.

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