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ANALYSIS OF SOLUTIONS OF THE 1D FRACTIONAL CATTANEO HEAT TRANSFER EQUATION

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Abstract. In this paper, a solution of the single-phase lag heat conduction problem is presented. The research concerns the generalized 1D Cattaneo equation in a whole-space domain, where a second order time derivative is replaced by the fractional Caputo derivative. The Fourier-Laplace transform technique is used to determine a solution of the considered problem. The numerical inversion of the Laplace transforms is applied. The effect of the order of the fractional derivative on the temperature distribution is investigated.

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1. Introduction

The basis of the classical theory of heat transfer is Fourier's law described by a parabolic partial differential equation [1]. In this model, however, a certain unreality appears, namely the infinite speed of heat propagation. One way to avoid this is to go to a fractional calculus [2, 3] and/or introducing a phase-lag parameter to heat flux in heat conduction law [4, 5]. However, analytical solutions to fractional differential equations exist only in the case of certain types of equations [6]. In most cases, the only way is a numerical solution [7, 8].

In this paper, heat conduction described by the fractional Cattaneo equation is presented. The object of the consideration is the whole-space 1D domain. In addition, a relaxation time is introduced which delays the heat flux. The primary aim of the research is to investigate the effect of the fractional order derivative on the temperature distribution. A solution of the problem is determined by using the Fourier-Laplace technique, and the final solution is obtained by numerical inversion of the Laplace transform.

2. Problem formulation

We consider the following 1D fractional Cattaneo equation without the capacity of internal heat sources in the form

$$c\rho\left(\frac{\partial T(x,t)}{\partial t}+\frac{\tau_q^{\alpha_q}}{\Gamma(\alpha_q+1)}\frac{{}^C\partial^{\alpha_q+1}T(x,t)}{\partial t^{\alpha_q+1}}\right)=\lambda\frac{\partial^2 T(x,t)}{\partial x^2} \tag{1}$$

where T [K] is the temperature, x [m], t [s] denote the geometrical co-ordinate and time, c [J/(kg K)], ρ [kg/m^3] and λ [W/(m K)] are the specific heat capacity of the medium, the density of the material and the thermal conductivity, respectively. The parameter τ_q [s] is the thermodynamic property of material called the thermal relaxation time. In Eq. (1), the fractional derivative of function $T(x, t)$ of order α_q+1, for $\alpha_q \in (0,1]$ occurs, which is defined in the Caputo sense [2, 6] as

$$\frac{{}^C\partial^{\alpha_q+1}T(x,t)}{\partial t^{\alpha_q+1}}:=\begin{cases}\dfrac{1}{\Gamma(1-\alpha_q)}\displaystyle\int_0^t\frac{1}{(t-u)^{\alpha_q}}\frac{\partial^2 T(x,u)}{\partial u^2}\,\mathrm{d}u & \text{for } \alpha_q+1\in(1,2)\\ \dfrac{\partial^2 T(x,t)}{\partial t^2} & \text{for } \alpha_q+1=2\end{cases} \tag{2}$$

where Γ denotes the Gamma function.

We solve Equation (1) in a whole-space domain taking into account the following initial-boundary conditions (the so-called Cauchy problem)

$$T(x,0)=\delta(x),\quad \left.\frac{\partial T(x,t)}{\partial t}\right|_{t=0}=0,\quad \text{for } -\infty<x<\infty \tag{3}$$

$$\lim_{|x|\to\infty}T(x,t)=0,\quad \text{for } t>0 \tag{4}$$

where δ is the Dirac's delta function.

2.1. Derivation of considered problem

One can find many laws of the heat conduction in the different materials in literature [7, 9-10]. Typically, they are described by the relation between the heat flux vector $\mathbf{q}(\mathbf{x},t)$ and gradient of temperature $\nabla T(\mathbf{x},t)$ in the point $\mathbf{x}$ of the considered domain and at the moment of time t. In the case of the classical Fourier law, the relation $\mathbf{q}(\mathbf{x},t)=-\lambda\nabla T(\mathbf{x},t)$ occurs. One of the non-Fourier constitutive models is the Cattaneo heat transfer model [4] expressed by the following relation

$$\mathbf{q}\left(\mathbf{x},t+\tau_q\right)=-\lambda\,\nabla T\left(\mathbf{x},t\right) \tag{5}$$

Here, one can notice that the heat flux and temperature gradient occur at different times, wherein the heat flux is delayed by the relaxation time τ_q. If $\tau_q = 0$, the relation (5) becomes the aforementioned classical Fourier law.

The second important equation is the energy conservation equation

$$c\rho\frac{\partial T\left(\mathbf{x},t\right)}{\partial t}=-\nabla\cdot\mathbf{q}\left(\mathbf{x},t\right)+Q\left(\mathbf{x},t\right) \tag{6}$$

where the function $Q\left(\mathbf{x},t\right)$ [W/m^3] is a capacity of internal heat sources.

In order to derive the heat transfer equation, we need to combine Eqs. (5) and (6) by eliminating the heat flux **q**. To do this, we expand the left side of Eq. (5) using the Taylor series. Here, we apply the fractional expansion of function **q** near τ_q in the Taylor series of the fractional order $\alpha_q \in (0,1]$ [11]. Hence, the left side of Eq. (5) can be expressed as follows

$$\mathbf{q}\left(\mathbf{x},t+\tau_q\right)=\sum_{k=0}^{\infty}\frac{\tau^{\alpha_q k}}{\Gamma\left(\alpha_q k+1\right)}\frac{{}^{C}\partial^{\alpha_q k}\mathbf{q}\left(\mathbf{x},t\right)}{\partial t^{\alpha_q k}} \tag{7}$$

where

$$\frac{{}^{C}\partial^{\alpha_q k}\mathbf{q}\left(\mathbf{x},t\right)}{\partial t^{\alpha_q k}}:=\underbrace{\frac{{}^{C}\partial^{\alpha_q}}{\partial t^{\alpha_q}}\frac{{}^{C}\partial^{\alpha_q}}{\partial t^{\alpha_q}}\cdots\frac{{}^{C}\partial^{\alpha_q}}{\partial t^{\alpha_q}}}_{k\ \text{times}}\mathbf{q}\left(\mathbf{x},t\right),\quad \text{for } k\in\mathbb{N} \tag{8}$$

and

$$\frac{{}^{C}\partial^{\alpha_q}}{\partial t^{\alpha_q}}\mathbf{q}\left(\mathbf{x},t\right):=\begin{cases}\dfrac{1}{\Gamma\left(1-\alpha_q\right)}\displaystyle\int_0^t\frac{1}{\left(t-u\right)^{\alpha_q}}\frac{\partial\mathbf{q}\left(\mathbf{x},u\right)}{\partial u}\mathrm{d}u & \text{for } \alpha_q\in\left(0,1\right)\\ \dfrac{\partial\mathbf{q}\left(\mathbf{x},t\right)}{\partial t} & \text{for } \alpha_q=1\end{cases} \tag{9}$$

In general, any number of terms of the fractional Taylor series can be considered. Here, we take into account only the first two terms of this series. Thus, the approximation for the heat flux **q** can be written as follows

$$\mathbf{q}\left(\mathbf{x},t+\tau_q\right)=\mathbf{q}\left(\mathbf{x},t\right)+\frac{\tau_q^{\alpha_q}}{\Gamma\left(\alpha_q+1\right)}\frac{{}^{C}\partial^{\alpha_q}\mathbf{q}\left(\mathbf{x},t\right)}{\partial t^{\alpha_q}} \tag{10}$$

and next after putting Eq. (10) into Eq. (5), one obtains the following heat conduction constitutive equation

$$\mathbf{q}(\mathbf{x},t)+\frac{\tau_q^{\alpha_q}}{\Gamma(\alpha_q+1)}\frac{{}^C\partial^{\alpha_q}\mathbf{q}(\mathbf{x},t)}{\partial t^{\alpha_q}}=-\lambda\nabla T(\mathbf{x},t) \tag{11}$$

Now, we combine Eq. (11) with the energy conservation equation (6) by elimination of the heat flux vector. If we apply the divergence operator to both sides of Eq. (11), then we have

$$\nabla\cdot\mathbf{q}(\mathbf{x},t)+\frac{\tau_q^{\alpha_q}}{\Gamma(\alpha_q+1)}\frac{{}^C\partial^{\alpha_q}\nabla\cdot\mathbf{q}(\mathbf{x},t)}{\partial t^{\alpha_q}}=-\nabla\cdot\left[\lambda\nabla T(\mathbf{x},t)\right] \tag{12}$$

and next from Eq. (6) we determine $\nabla\cdot\mathbf{q}(\mathbf{x},t)$ and put it into Eq. (12). Hence, we derived the Fractional Cattaneo Equation (FCE) of the form

$$\begin{aligned} c\rho\left(\frac{\partial T(\mathbf{x},t)}{\partial t}+\frac{\tau_q^{\alpha_q}}{\Gamma(\alpha_q+1)}\frac{{}^C\partial^{\alpha_q+1}T(\mathbf{x},t)}{\partial t^{\alpha_q+1}}\right)&=\nabla\cdot\left[\lambda\nabla T(\mathbf{x},t)\right]\\ +Q(\mathbf{x},t)+\frac{\tau_q^{\alpha_q}}{\Gamma(\alpha_q+1)}\frac{{}^C\partial^{\alpha_q}Q(\mathbf{x},t)}{\partial t^{\alpha_q}}& \end{aligned} \tag{13}$$

The FCE (13) we supplement by two initial conditions

$$T(\mathbf{x},t)\big|_{t=0}=T_0(\mathbf{x}),\quad \left.\frac{\partial T(\mathbf{x},t)}{\partial t}\right|_{t=0}=T_1(\mathbf{x}) \tag{14}$$

and by the boundary conditions depending on the considered problem.

It should be noted that for $\tau_q = 1$, Eq. (13) becomes the classical Cattaneo equation (as the hyperbolic-type partial differential equation) in the form

$$c\rho\left(\frac{\partial T(\mathbf{x},t)}{\partial t}+\tau_q\frac{\partial^2 T(\mathbf{x},t)}{\partial t^2}\right)=\nabla\cdot\left[\lambda\nabla T(\mathbf{x},t)\right]+Q(\mathbf{x},t)+\tau_q\frac{\partial Q(\mathbf{x},t)}{\partial t} \tag{15}$$

and for $\tau_q = 0$, Eq. (13) reduces to the classical Fourier heat transfer equation (as the parabolic-type partial differential equation)

$$c\rho\frac{\partial T(\mathbf{x},t)}{\partial t}=\nabla\cdot\left[\lambda\nabla T(\mathbf{x},t)\right]+Q(\mathbf{x},t) \tag{16}$$

whose solution is characterized by an infinite speed of heat propagation.

If we take into consideration the 1D space domain in Eq. (13) then $\mathbf{x} = \{x\}$, and the infinite space domain, i.e. $-\infty < x < \infty$ (for the Cauchy problem: $T_0(x) = \delta(x)$, $T_1(x) = 0$), and we additionally assume that the parameters c, ρ and λ are constants and $Q(x,t) = 0$ then we obtain the governing equation (1) supplemented by the initial-boundary conditions (3) and (4).

3. Solution of the considered problem

In order to simplify the further calculations, we introduce the dimensionless variables t' and x'

$$t' = \frac{1}{t_{ref}} t \quad , \qquad x' = \sqrt{\frac{c\rho}{\lambda} \frac{1}{t_{ref}}} \, x \tag{17}$$

where t_{ref} is reference time. The particular partial derivatives in Eq. (1) after replacement of the variables are as follows

$$\frac{\partial T(x,t)}{\partial t} = \frac{1}{t_{ref}} \frac{\partial T(x',t')}{\partial t'} \tag{18}$$

$$\frac{{}^{C}\partial^{\alpha_q+1} T(x,t)}{\partial t^{\alpha_q+1}} = \frac{1}{t_{ref}^2} \frac{{}^{C}\partial^{\alpha_q+1} T(x',t')}{\partial t'^{\alpha_q+1}} \tag{19}$$

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{c\rho}{\lambda} \frac{1}{t_{ref}} \frac{\partial^2 T(x',t')}{\partial x'^2} \tag{20}$$

and after inserting them into Eq. (1) and simplifying this equation we obtain

$$\frac{\partial T(x',t')}{\partial t'} + \frac{\tau_q^{\alpha_q}}{\Gamma(\alpha_q+1)} \frac{1}{t_{ref}} \frac{{}^{C}\partial^{\alpha_q+1} T(x',t')}{\partial t'^{\alpha_q+1}} = \frac{\partial^2 T(x',t')}{\partial x'^2} \tag{21}$$

Additionally, if we assume $t_{ref} = \tau_q^{\alpha_q} / \Gamma(\alpha_q + 1)$ then Eq. (21) simplifies again and can be written as

$$\frac{\partial T(x',t')}{\partial t'} + \frac{{}^{C}\partial^{\alpha_q+1} T(x',t')}{\partial t'^{\alpha_q+1}} = \frac{\partial^2 T(x',t')}{\partial x'^2} \tag{22}$$

while the initial-boundary conditions have the following forms

$$T(x',0)=\delta(x'),\quad \left.\frac{\partial T(x',t')}{\partial t'}\right|_{t'=0}=0,\quad -\infty<x'<\infty \tag{23}$$

$$\lim_{|x'|\to\infty} T(x',t')=0,\quad t'>0 \tag{24}$$

3.1. Solution using the Laplace-Fourier technique

It should be pointed out that four constant thermophysical coefficients c, ρ, λ and τ_q in Eq. (22) have been eliminated. This approach not only allows us to derive a simplified form of the equation solution, but more importantly, it makes it possible to eliminate the influence of these variables on the solutions. In the calculation examples, the influence of the parameter α_q will be mainly investigated.

We solve the initial-boundary problem described by Eqs. (22)-(24) using the Fourier-Laplace transform technique. Let's introduce the notations: the Laplace transform of $T(x',t')$ is $\mathcal{L}\{T(x',t')\}(s)=\overline{T}(x',s)$ where s is a complex parameter, and the Fourier transform of $\overline{T}(x',s)$ is $\mathcal{F}\{\overline{T}(x',s)\}(\omega)=\hat{\overline{T}}(\omega,s)$ where ω is angular frequency. Hence, (ω,s) denotes the Fourier-Laplace space. In addition to the well-known properties of the Laplace transform, we use the transform of the Caputo fractional derivative [2, 9] in the form

$$\mathcal{L}\left\{\frac{{}^{C}\partial^{\alpha}T(x',t')}{\partial t'^{\alpha}}\right\}(s)=s^{\alpha}\overline{T}(x',s)-\sum_{k=0}^{m-1}s^{\alpha-1-k}\left.\frac{\partial^{k}T(x',t')}{\partial t'^{k}}\right|_{t'=0} \tag{25}$$

for $m-1<\alpha\le m$, $m\in\mathbb{N}$.

Therefore, the Laplace transformation of Eq. (22) can be written as

$$\begin{aligned}&s\overline{T}(x',s)-T(x',0)+s^{\alpha_q+1}\overline{T}(x',s)-s^{\alpha_q}T(x',0)-s^{\alpha_q-1}\left.\frac{\partial T(x',t')}{\partial t'}\right|_{t'=0}\\&=\frac{\partial^{2}\overline{T}(x',s)}{\partial x'^{2}}\end{aligned} \tag{26}$$

and taking into account the initial conditions (23), we obtain

$$s\overline{T}(x',s)-\delta(x')+s^{\alpha_q+1}\overline{T}(x',s)-s^{\alpha_q}\delta(x')=\frac{\partial^{2}\overline{T}(x',s)}{\partial x'^{2}} \tag{27}$$

or in the following form

$$\left(s+s^{\alpha_q+1}\right)\overline{T}\left(x',s\right)=\frac{\partial^2\overline{T}\left(x',s\right)}{\partial x'^2}+\left(s^{\alpha_q}+1\right)\delta\left(x'\right) \tag{28}$$

If we use the Fourier transform for the above equation, taking into account the boundary condition (24) and the following known properties [14]

$$\mathcal{F}\left\{\frac{\partial^2\overline{T}\left(x',s\right)}{\partial x'^2}\right\}(\omega)=-\omega^2\hat{\overline{T}}\left(\omega,s\right) \quad \text{and} \quad \mathcal{F}\left\{\delta\left(x'\right)\right\}(\omega)=1 \tag{29}$$

then we obtain

$$\left(s+s^{\alpha_q+1}\right)\hat{\overline{T}}\left(\omega,s\right)=-\omega^2\hat{\overline{T}}\left(\omega,s\right)+s^{\alpha_q}+1 \tag{30}$$

which the solution with respect to $\hat{\overline{T}}\left(\omega,s\right)$ is

$$\hat{\overline{T}}\left(\omega,s\right)=\frac{s^{\alpha_q}+1}{\omega^2+s\left(s^{\alpha_q}+1\right)} \tag{31}$$

In order to find the inverse Fourier transform (31), we apply the known inverse transformation [14]

$$\mathcal{F}^{-1}\left\{\frac{1}{\omega^2+a^2}\right\}\left(x'\right)=\frac{\exp\left(-a\left|x'\right|\right)}{2a} \quad \text{for} \quad a>0 \tag{32}$$

Hence, we have

$$\overline{T}\left(x',s\right)=\mathcal{F}^{-1}\left\{\hat{\overline{T}}\left(\omega,s\right)\right\}\left(x'\right)=\left(s^{\alpha_q}+1\right)\frac{\exp\left(-\left|x'\right|\sqrt{s\left(s^{\alpha_q}+1\right)}\right)}{2\sqrt{s\left(s^{\alpha_q}+1\right)}} \tag{33}$$

In the next step, to get the solution of Eq. (22) in space $\left(x',t'\right)$, we must apply the inverse Laplace transform

$$T\left(x',t'\right)=\mathcal{L}^{-1}\left\{\overline{T}\left(x',s\right)\right\}\left(t'\right)=\mathcal{L}^{-1}\left\{\left(s^{\alpha_q}+1\right)\frac{\exp\left(-\left|x'\right|\sqrt{s\left(s^{\alpha_q}+1\right)}\right)}{2\sqrt{s\left(s^{\alpha_q}+1\right)}}\right\}\left(t'\right) \tag{34}$$

Expressing the above solution containing the inverse Laplace transform by the form of analytical function seems to be a complex problem. One can expand the

Laplace transform into a series and then find the inverse transform of the particular components of the sum [5], but from a computational point of view, such a series may slowly converge. Here, the use of numerical methods to find the inverse Laplace transform in Eq. (34) comes in handy. In the next Section, in order to determine values of functions presented in the plots, we applied the numerical method described in [15] by de Hoog et. al., which is characterized by high precision of computations.

3.2. Analytical solution of the particular case (classical Cattaneo equation)

Let's consider a particular case. If $\alpha_q = 1$, then Eq. (34) takes the form

$$T(x',t') = \mathcal{L}^{-1}\left\{\frac{1}{2}(s+1)\frac{\exp\left(-|x'|\sqrt{s^2+s}\right)}{\sqrt{s^2+s}}\right\}(t') \tag{35}$$

Further transformations of the above equation lead to

$$\begin{aligned} T(x',t') &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\sqrt{\left(s+\frac{1}{2}\right)^2-\frac{1}{4}}}\exp\left(-|x'|\sqrt{\left(s+\frac{1}{2}\right)^2-\frac{1}{4}}\right)\right\}(t') \\ &= \frac{1}{2}\exp\left(-\frac{t'}{2}\right)\mathcal{L}^{-1}\left\{\left(s+\frac{1}{2}\right)\frac{1}{\sqrt{s^2-\frac{1}{4}}}\exp\left(-|x'|\sqrt{s^2-\frac{1}{4}}\right)\right\}(t') \end{aligned} \tag{36}$$

Let us recall the known inverse Laplace transforms [12]

$$\mathcal{L}^{-1}\left\{\frac{\exp\left(-|x'|\sqrt{s^2-a^2}\right)}{\sqrt{s^2-a^2}}\right\}(t') = I_0\left(a\sqrt{t'^2-|x'|^2}\right)H\left(t'-|x'|\right) \tag{37}$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{s\frac{\exp\left(-|x'|\sqrt{s^2-a^2}\right)}{\sqrt{s^2-a^2}}\right\}(t') &= \frac{d}{dt'}\left(I_0\left(a\sqrt{t'^2-|x'|^2}\right)H\left(t'-|x'|\right)\right) \\ &= \frac{at'}{\sqrt{t'^2-|x'|^2}}I_1\left(a\sqrt{t'^2-|x'|^2}\right)H\left(t'-|x'|\right)+I_0\left(a\sqrt{t'^2-|x'|^2}\right)\delta\left(t'-|x'|\right) \end{aligned} \tag{38}$$

If we put Eqs. (37) and (38) into Eq. (36), assuming that $I_0\left(a\sqrt{t'^2-|x'|^2}\right)\delta\left(t'-|x'|\right)=\delta\left(t'-|x'|\right)$ and $a=1/2$ then we obtain

$$
\begin{aligned}
T(x',t') = \frac{1}{4}\exp\left(-\frac{t'}{2}\right)\left(I_0\left(\frac{1}{2}\sqrt{t'^2-|x'|^2}\right)+\frac{t'}{\sqrt{t'^2-|x'|^2}}I_1\left(\frac{1}{2}\sqrt{t'^2-|x'|^2}\right)\right)\times \\
H\left(t'-|x'|\right)+\frac{1}{2}\exp\left(-\frac{t'}{2}\right)\delta\left(t'-|x'|\right)
\end{aligned}
\tag{39}
$$

4. Examples of solutions

On the basis of the solution presented in the previous section, we performed sample calculations that are presented in the several plots. We investigated the effect of the time-fractional orders of the Caputo derivative α_q on the temperature distribution $T(x', t')$ for the non-dimensional variables in the 1D domain. At first, in Figure 1, the sample solutions as temperature distributions over space (at the selected moments of time $t' = 0.2; 0.5; 1; 2; 5; 10; 20$) and over time (at the selected points of the domain $x' = 0.2; 0.5; 1; 2; 5; 10; 20$) for $\alpha_q = 1$ are presented. This case corresponds to the solution of the classical Cattaneo equation and can be treated as a reference to the solutions of the fractional Cattaneo equation for $0 < \alpha_q < 1$.

The next two Figures, 2 and 3, show the effect of the time-fractional orders of the Caputo derivative α_q on the temperature distribution over space at the selected moments of time and over time at the selected points of the domain, respectively. As we expected (according to the assumed boundary condition), for $|x'| \to \infty$, the temperature decreases to zero for all values α_q, and the solutions are symmetrical with respect to the point $x = 0$. It can be seen that largest differences of the temperature occur in the initial period of time t'. The temperature in a whole-space domain decreases with $|t'| \to \infty$. As we can see, for $\alpha_q = 1$ (i.e. the classical Cattaneo model), the sharp front of thermal wave propagation occurs which is related to the assumed Dirac's delta function in the initial condition. For $\alpha_q < 1$, the sharp front flatten more if the parameter α_q decreases more.

The main aim of the research was to investigate the effect of the fractional order derivative on the temperature distribution, so the numerical calculations in all cases were done and presented in all Figures for the non-dimensional variables given by Equation (17). Of course, one can easily rescale horizontal axes in those Figures for original variables x and t by using the following relations which contain the given thermophysical parameters of materials

$$x = \sqrt{\frac{\tau_q^{\alpha_q}}{\Gamma(\alpha_q + 1)} \cdot \frac{\lambda}{c\rho}} \cdot x' \quad \text{and} \quad t = \frac{\tau_q^{\alpha_q}}{\Gamma(\alpha_q + 1)} \cdot t' \tag{40}$$

but the analysis of the influence of these parameters on the solutions was not the subject of this manuscript.

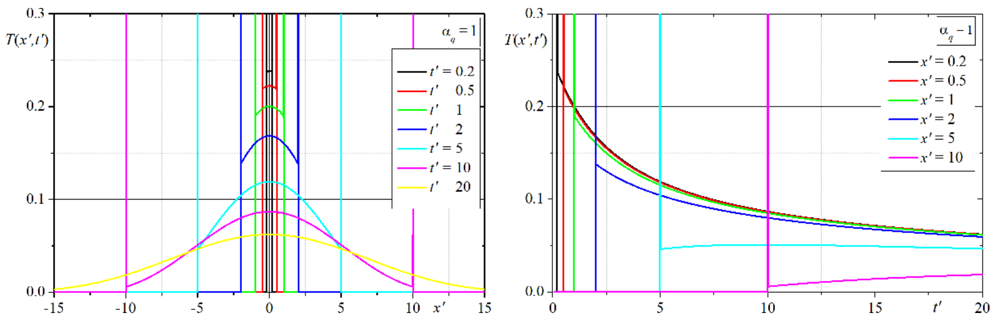

Fig. 1. Temperature distribution $T(x',t')$ over space and over time for $\alpha_q = 1$ (classical Cattaneo equation)

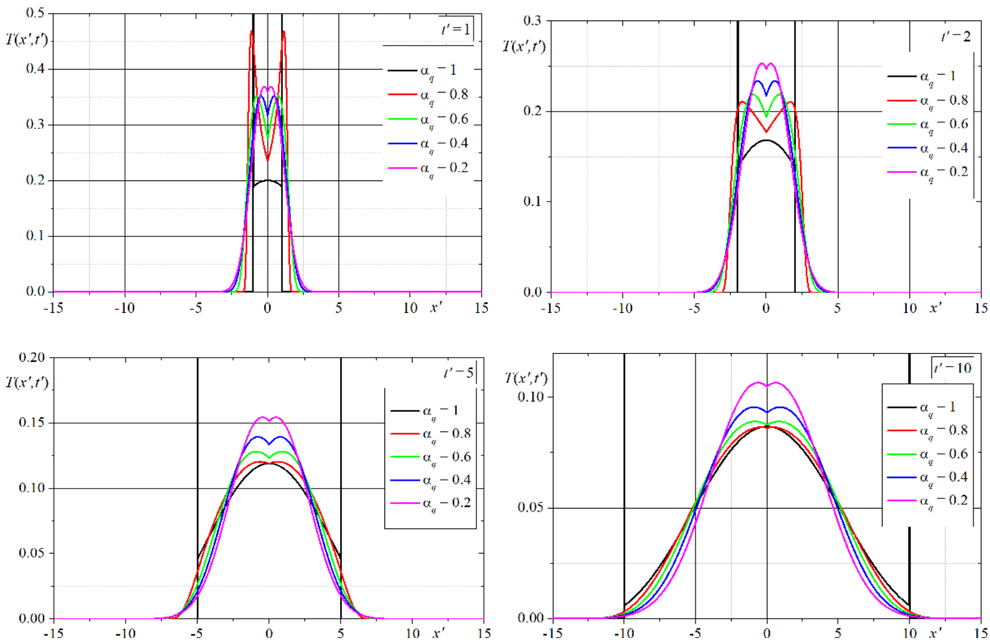

Fig. 2. Effect of fractional order α_q on temperature distribution $T(x',t')$ over space at the selected moments of time $t' \in \{1, 2, 5, 10\}$

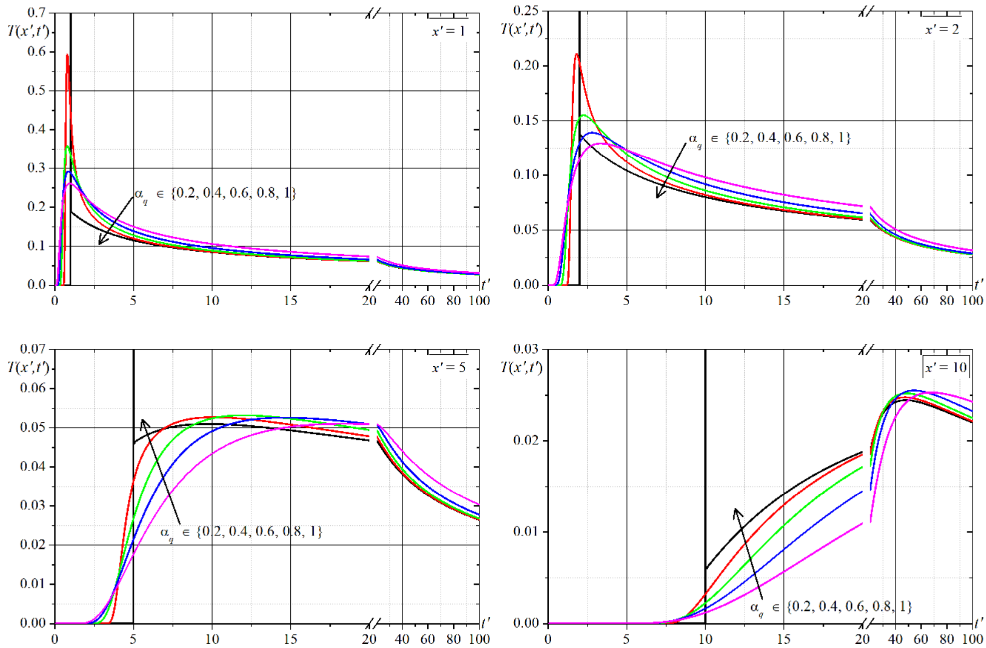

Fig. 3. Effect of fractional order α_q on temperature distribution $T(x',t')$ over time at the selected points $x' \in \{1, 2, 5, 10\}$

5. Conclusions

The non-Fourier model of heat conduction, being a generalization of the Cattaneo model, has been considered. Here, the Cattaneo law of heat conduction has been expanded into the first-order fractional Taylor series, which enabled us to define the fractional Cattaneo heat transfer equation. Thus, the additional parameter $\alpha_q \in (0, 1]$ appeared in the model. Such an approach allows us to obtain new feature solutions of the model compared to the solutions of the classical (also called: integer order) Cattaneo model.

In this work, we only investigated solutions related to the one-dimensional Cauchy problem, however, in a more general case, it can also be extended to other initial conditions. We investigated the effect of the time-fractional derivative order α_q on the temperature distribution in the whole space domain. As can be seen in the example results, the sharp front (with peaks) of thermal wave propagation occurs only in the classical Cattaneo model (the case for $\alpha_q = 1$) – such a front is related to the assumed initial condition (the Dirac's delta function), of course. In the case of the solutions for values of parameter $\alpha_q < 1$, one can notice that the sharp front flats, as the value of parameter α_q decreases more. It seems to us that such solutions can be more realistic in the case of modeling real thermal problems. Another generalization of the considered model may be the fractional dual-phase-lag model [5].

The considerations in this paper concerned only the expansion of the formula (7) into the fractional Taylor series, which was truncated to the first two terms. Of course, one can take more terms of this Taylor series, derive the related fractional differential equations and find solutions for them with the appropriate boundary initial conditions. We plan to conduct such research in the future.

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