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# A GENERALIZED VERSION OF THE LIONS-TYPE LEMMA

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**Abstract.** In this short paper, I recall the history of dealing with the lack of compactness of a sequence in the case of an unbounded domain and prove the vanishing Lions-type result for a sequence of Lebesgue-measurable functions. This lemma generalizes some results for a class of Orlicz–Sobolev spaces. What matters here is the behavior of the integral, not the space.

## 1. Introduction

In 1984 P.L. Lions published his celebrated article [10], in which he introduced a concentration-compactness method for solving minimization problems on unbounded domains. One of the main tool provided by [10] is lemma I.1. A variety of formulations of this lemma has been widely used to deal with the lack of compactness on unbounded domains for different types of equations. In [7, p. 102] we can find the following version of the Lions Lemma:

Lemma 1. *Suppose* $\{u_n\} \in \mathbf{H}^1(\mathbb{R}^N)$ *is a bounded sequence satisfying*

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B_r(y)}|u_n|^p\right)=0$$

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*for some* $p \in [2, 2^*]$ *and* $r > 0$*, where* $B_r(y)$ *denotes the open ball of radius* $r$ *centered at* $y \in \mathbb{R}^N$*. Then* $u_n \to 0$ *strongly in* $\mathbf{L}^q(\mathbb{R}^N)$ *for all* $2 < q < 2^*$*, where* $2^*$ *is the limiting exponent in the Sobolev embedding* $\mathbf{H}^1(\mathbb{R}^N) \hookrightarrow \mathbf{L}^p(\mathbb{R}^N)$*.*

This version of lemma has been used for solving semilinear elliptic equation in the whole space $\mathbb{R}^N$, i.e.

$$-\Delta u + u = h(u), \ u \in \mathbf{H}^1(\mathbb{R}^N).$$

In [8] and [12] you can find a comprehensive description of lack of compactness in Sobolev spaces

The Lions Lemma has been generalized in some ways, for example in [3] we can find the formulation of the lemma for isotropic Orlicz–Sobolev spaces $\mathbf{W}_0^1\,\mathbf{L}^A(\mathbb{R}^N)$, i.e. spaces obtained by the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{\mathbf{W}^1\,\mathbf{L}^A(\mathbb{R}^N)} = \||\nabla u|\|_{\mathbf{L}^A(\mathbb{R}^N)} + \|u\|_{\mathbf{L}^A(\mathbb{R}^N)}$, where

$$\|u\|_{\mathbf{L}^A(\mathbb{R}^N)} = \inf\left\{k > 0 : \int_{\mathbb{R}^N} A\left(\frac{|u|}{k}\right)dt \le 1\right\}$$

is a Luxemburg norm, $A\colon \mathbb{R} \to [0,\infty)$ is an $N$-function (i.e. is convex, even, coercive and vanishes only at 0) satisfying $\Delta_2\nabla_2$ condition (i.e. there exist $K_1, K_2 > 0$, such that $K_1 A(v) \le A(2v) \le K_2 A(v)$ for all $v \in \mathbb{R}^n$).

Lemma 2 ([3, Theorem 1.3]). *Assume that* $a(t)t$ *is increasing in* $(0,+\infty)$ *and that there exist* $l, m \in (1, N)$ *such that*

$$l \le \frac{a(|t|)t^2}{A(t)} \le m \quad \text{for all } t \ne 0, \tag{1}$$

*where* $A(t) = \int_0^{|t|} a(s)s\,ds$, $l \le m < l^* = \frac{lN}{N-l}$*. Let* $\{u_n\} \subset \mathbf{W}^1\,\mathbf{L}^A(\mathbb{R}^N)$ *be a bounded sequence such that there exists* $r > 0$ *satisfying:*

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B_r(y)} A(|u_n|)\right) = 0. \tag{$L_1$}$$

*Then, for any N-function B verifying* $\Delta_2$*-condition (i.e. there exsists* $K > 0$ *such that* $B(2t) \le KB(t)$ *for all* $t > 0$*) and satisfying*

$$\lim_{t\to 0}\frac{B(t)}{A(t)} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{B(t)}{A^*(t)} = 0,$$

*where* $A^*$ *is a Sobolev measurete of* $A$ *(defined by* $(A^*)^{-1}(t) = \int_0^t \frac{A^{-1}(s)}{s^{(N+1)/N}}\,ds$*), we have*

$$u_n \to 0 \quad \text{in } \mathbf{L}^B(\mathbb{R}^N).$$

In [3] the authors use Lemma 2 to prove the existence of solutions to some isotropic quasilinear problems.

It is worth noticing, that in the proof of the lemma above authors essentially use the fact that function $A$ satisfies $\Delta_2\nabla_2$ condition, which is guaranteed by condition (1). Isotropic Young function satisfying globally $\Delta_2\nabla_2$ condition is bounded by some power functions with power $1 < p < \infty$ (see e.g [6, Lemma C.4]). If $A$ satisfies $\Delta_2\nabla_2$ then $\mathbf{W}^1\,\mathbf{L}^A$ is a reflexive, separable Banach space (see e.g. [1, Theorem 8.31]).

There are also papers, where authors consider non-reflexive spaces, e.g. [2]. In this case instead of condition $(L_1)$ authors use the assumption $(L_2)$ (see [9]) and assume that the sequence $\left\{\int_{\mathbb{R}^N} A^*(|u_n|)\,dx\right\}$ is bounded.

Lemma 3 ([2, Theorem 1.3]). *Let $A, B$ be an N-functions, $A^*$ be a Sobolev conjugate of $A$ and*

$$\lim_{t\to 0}\frac{B(t)}{A(t)} = 0 \quad \textit{and} \quad \lim_{t\to\infty}\frac{B(t)}{A^*(t)} = 0.$$

*If $\{u_n\} \subset \mathbf{W}^1\,\mathbf{L}^A(\mathbb{R}^N)$ is a sequence such that $\left\{\int_{\mathbb{R}^N} A(|u_n|)\,dx\right\}$ and $\left\{\int_{\mathbb{R}^N} A^*(|u_n|)\,dx\right\}$ are bounded, and for each $\varepsilon > 0$ we have*

$$(L_2) \qquad \operatorname{meas}(|u_n| > \varepsilon) \to 0 \quad \textit{as } n\to\infty,$$

*then*

$$\int_{\mathbb{R}^N} B(u_n) \to 0 \quad \textit{as } n\to\infty.$$

In [13] the author uses the lemma similar to Lemma 2, but for sequences from anisotropic Orlicz–Sobolev spaces, to find solutions of the anisotropic quasilinear problem

$$-\operatorname{div}(\nabla\Phi(\nabla u)) + V(x)N'(u) = f(u), \qquad \text{where } u \in \mathbf{W}^1\,\mathbf{L}^\Phi(\mathbb{R}^n),$$

where $\Phi$ is an anisotropic $n$-dimensional $N$-function (see more in [5]), satisfying $\Delta_2\nabla_2$ condition and $N$ is a differentiable $N$-function, such that $N \approx \Phi_0$, where $\Phi_0\colon [0,\infty) \to [0,\infty)$ is the left-continuous increasing function obeying

$$|\{v \in \mathbb{R}^n\colon \Phi_0(|v|) \le t\}| = |\{v \in \mathbb{R}^n\colon \Phi(v) \le t\}| \qquad \text{for } t \ge 0,$$

where $|\cdot|$ stands for Lebesgue measure.

In [11] the authors prove the Lions type lemma for reflexive fractional Orlicz–Sobolev spaces, while in [4] the authors prove it for non-reflexive fractional Orlicz–Sobolev spaces.

## 2. Main Theorem

In this paper, we generalize the Lions-type Lemmas 1, 2, 3, we mentioned in the introduction. We do not assume that functions $\Psi$, $\Phi_1$, and $\Phi_2$, from the theorem below, are $N$-functions.

We need only the fact that they are locally essentially bounded, non-negative, essential supremum of $\Psi$ is greater than zero and $\Phi_1$ vanishes only at zero (assumption (2)). It is worth noticing that they can have growth not bounded by polynomials, so it will be possible to use this lemma in non-reflexive spaces. In the proof of the following lemma we will use some techniques from [11].

THEOREM 4. *Assume that* $\Phi_1, \Phi_2, \Psi \in \mathbf{L}^{\infty}_{loc}(\mathbb{R}^n, [0,\infty))$,

$$\forall_{R>0} \quad \operatorname{ess\,sup}_{B_R(0)} \Psi > 0, \qquad \Phi_1(x) = 0 \iff x = 0, \tag{2}$$

$$\lim_{|v|\to 0} \frac{\Psi(v)}{\Phi_1(v)} = 0, \tag{$\Psi_1$}$$

$$\lim_{|v|\to \infty} \frac{\Psi(v)}{\Phi_2(v)} = 0. \tag{$\Psi_2$}$$

*Let* $\{u_k\}$ *be a sequence of Lebesgue-measurable functions* $u_k\colon \mathbb{R}^N \to \mathbb{R}^n$ *such that* $\int_{\mathbb{R}^N} \Phi_1(u_k)$, $\int_{\mathbb{R}^N} \Phi_2(u_k)$ *exist,*

$$M_1 = \sup_k \int_{\mathbb{R}^N} \Phi_1(u_k) < \infty, \quad M_2 = \sup_k \int_{\mathbb{R}^N} \Phi_2(u_k) < \infty,$$

*and*

$$\lim_{k\to\infty} \left[ \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} \Phi_1(u_k) \right] = 0, \tag{3}$$

*for some* $r > 0$. *Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \Psi(u_k) = 0.$$

Proof. We let $|A|$ denote the Lebesgue measure of a subset $A$. Let $\{u_k\}$, $\Phi_1$, $\Phi_2$, $\Psi$ satisfy the above assumptions.

Fix $\varepsilon > 0$. From $(\Psi_1)$, there exists $\delta > 0$, such that

$$\frac{\Psi(v)}{\Phi_1(v)} < \frac{\varepsilon}{3M_1} \tag{4}$$

for all $|v| \leq \delta$.

Similarly from $(\Psi_2)$, there exists $T > 0$, such that

$$\frac{\Psi(v)}{\Phi_2(v)} < \frac{\varepsilon}{3M_2} \tag{5}$$

for all $|v| \geq T$. Let us denote:

$$A_k = \left\{x \in \mathbb{R}^N : |u_k(x)| < \delta\right\}, \quad B_k = \left\{x \in \mathbb{R}^N : \delta \leq |u_k(x)| \leq T\right\},$$
$$C_k = \left\{x \in \mathbb{R}^N : |u_k(x)| > T\right\}.$$

Then

$$\int_{\mathbb{R}^N} \Psi(u_k) = \int_{A_k} \Psi(u_k) + \int_{B_k} \Psi(u_k) + \int_{C_k} \Psi(u_k).$$

By (4), we obtain

$$\int_{A_k} \Psi(u_k) \leq \frac{\varepsilon}{3M_1} \int_{\mathbb{R}^N} \Phi_1(u_k) < \frac{\varepsilon}{3}$$

and by (5), we get

$$\int_{C_k} \Psi(u_k) \leq \frac{\varepsilon}{3M_2} \int_{\mathbb{R}^N} \Phi_2(u_k) < \frac{\varepsilon}{3}.$$

We need to show that

$$\int_{B_k} \Psi(u_k) < \frac{\varepsilon}{3}.$$

We will show that $|B_k| \to 0$ as $k \to \infty$.

Assume, by contradiction, that (up to subsequence)

$$|B_k| \to L > 0. \tag{6}$$

First of all, we will show (just as in [11, p. 506]), that for some subsequence $\{u_k\}$, there exist $y_0 \in \mathbb{R}^N$ and $\sigma > 0$, such that

$$|B_k \cap B_r(y_0)| \geq \sigma > 0. \tag{7}$$

Assume, again by contradiction, that for all $\varepsilon > 0$, $m \in \mathbb{N}$, $y \in \mathbb{R}^N$ we have

$$|B_k \cap B_r(y)| < \frac{\varepsilon}{2^m}. \tag{8}$$

The last estimate holds for any subsequence of $\{u_k\}$, and WLOG we can take just $\{u_k\}$. Let us choose $\{y_m\} \subset \mathbb{R}^N$, such that

$$B := \bigcup_{m=1}^{\infty} B_r(y_m) = \mathbb{R}^N.$$

Using (8), for arbitrary $\varepsilon$ we obtain

$$|B_k| = |B_k \cap B| \leq \sum_{m=1}^{\infty} |B_k \cap B_r(y_m)| < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon,$$

which contradicts (6).

Let

$$C_\Psi = \text{ess sup}_{\delta \leq |v| \leq T} \Psi(v), \quad c_\Phi = \text{ess inf}_{\delta \leq |v| \leq T} \Phi_1(v),$$
$$C_\Phi = \text{ess sup}_{\delta \leq |v| \leq T} \Phi_1(v).$$

We observe that

$$\int_{B_r(y_0)} \Phi_1(u_k) \geq \int_{B_r(y_0) \cap B_k} \Phi_1(u_k) \geq c_\Phi |B_k \cap B_r(y_0)|.$$

Hence, by assumption (3), we have that

$$|B_k \cap B_r(y_0)| \to 0 \quad \text{as } k \to \infty$$

which contradicts (7) and as a result $|B_k| \to 0$ as $k \to \infty$. Hence we have that there exists $k_0$ such that for all $k \geq k_0$

$$|B_k| < c_\Phi \left(3 C_\Phi C_\Psi\right)^{-1} \varepsilon.$$

Then

$$|B_k| \leq (c_\Phi)^{-1} \int_{B_k} \Phi_1(u_k) \leq C_\Phi (c_\Phi)^{-1} |B_k|$$

and

$$\int_{B_k} \Psi(u_k) \leq C_\Psi (c_\Phi)^{-1} \int_{B_k} \Phi_1(u_k) \leq C_\Psi C_\Phi (c_\Phi)^{-1} |B_k| < \frac{\varepsilon}{3}. \qquad \square$$

REMARK 5. Note that what matters in this theorem (just as in the concentration-compactness lemma of Lions in [9]) is the behavior of the integral, not the space.

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