FINITELY PRESENTED MODULES OVER RIGHT HEREDITARY SPSD-RINGS

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Abstract. The full structure of right hereditary semiperfect semidistributive rings of bounded representation type are given. This structure is given in terms of special graphs which can be considered as some generalization of Coxeter-Dynkin diagrams.

Introduction

The theory of representations is an important tool for studying groups, algebras and ring by means of linear algebra. The results of this theory play a fundamental role in many recent developments of mathematics and theoretical physics.

One of the main problems of representation theory is to obtain information about the possible structure of indecomposable modules and to describe the isomorphism classes of all indecomposable modules. By the famous theorem on trichotomy for finite dimensional associative algebras over an algebraically closed field, obtained by Drozd, all such algebras are divided into three disjoint classes: finite, tame and wild type. Therefore the basic problems in the representation theory of associative algebras are that of obtaining necessary and sufficient conditions for an algebra to belong to one of these classes and classify the indecomposable representations in the finite and tame cases. Similar problems are considered for Artinian rings.

For the rings which are not Artinian (for instance, for Noetherian rings) the condition of finiteness of indecomposable finitely generated modules are not natural. In this case one of the fundamental problems is to characterize the rings of bounded representation type in terms of their structure.

Recall that A-module M is called **finitely presented** if there exists an epimorphism $\varphi: A^{(m)} \to M$ such that $\mathrm{Ker}(\varphi)$ is a finitely generated A-module.

Following to Warfield a ring A is said to be of **bounded representation type** if the length of finitely presented indecomposable right A-modules are bounded.

In the paper [1] Warfield, Jr. put the following question:

For what semiperfect rings is there an upper bound on the number of generators required for the indecomposable finitely presented modules?

The well known theorem of Warfield, Jr. in [1], and Drozd in [2] shows that for any finitely presented module over a serial ring is serial. So any serial ring is of bounded representation type.

The main goal of this paper is to give the full structure of right hereditary semiperfect semidistributive rings of bounded representation type. This structure is given in terms of special graphs which can be considered as some generalization of Coxeter-Dynkin diagrams.

All rings considered in this paper are associative with identity and all modules are unitary. We refer to [3] and [4] for the general material on theory of rings and modules.

1. Hereditary Artinian semidistributive rings of finite type

Recall that a module M is called **distributive** if $K\cap(L+N) = K\cap L+K\cap N$ for all submodules K, L, N. A module is called **semidistributive** if it is a direct sum of distributive modules. A ring A is called **right (left) semidistributive** if the right (left) regular module A_A ($_AA$) is semi-distributive. A right and left semidistributive ring is called **semidistributive.**

Semiperfect semidistributive rings (SPSD-rings, in short) were first considered by Tuganbaev in [5] and [6]. The properties and structure of these rings were studied in [7-9].

A ring A is called **right (left) hereditary** if each right ideal of A is projective. This is equivalent to the condition that any submodule of a projective right A-module is projective. A right and left hereditary ring is called **hereditary**.

Denote by $T(S,D)$ an incidence ring of a partial ordered set (poset, in short) S = = $\{\alpha_1, \alpha_2,\dots,\alpha_n\}$ over a division ring D, that is $T(S,D)$ is a subring of the generalized matrix ring $M_n(D)$ such that the (i,j)-entry of $T(S,D)$ is equal to 0 if $\alpha_i < \alpha_j$ in S. Then it holds the following theorem.

Theorem 1. *Every hereditary semidistributive Artinian ring A is Morita equivalent to a finite direct product of indecomposable rings of the form T(S, D), where S is a finite poset, D is a division ring, and the diagram of S contains no rhombuses. Conversely, every ring of this form is a hereditary semidistributive Artinian ring.*

Recall that an Artinian ring A is of **finite representation type** if there are (up to isomorphism) only a finite number of finitely generated indecomposable right A-modules.

For semi-primary rings the conditions in this definition were shown to be right-left symmetric by Eisenbud and Griffith [10], and Dlab, Ringel [11]. Ringel and Tachikawa [12] proved that if A is an Artinian ring of finite representation type then all A-modules are direct sums of finitely generated modules. In particular, in this case all indecomposable A-modules are finitely generated. From their result it also follows that for a semi-primary ring A the number of finitely generated indecomposable left A-modules equals the number of finitely generated indecomposable right A-modules.

The first important class of such rings are Artin algebras. Let R be a commutative Artinian ring. A ring A is said to be an R-algebra if $ra = ar$ for each $r \in R$ and $a \in A$. An R-algebra A over a commutative Artinian ring R is called an **Artin algebra** if A is finitely generated as an R-module. It is clear that an Artin algebra is both a left and a right Artinian ring. Important examples of Artin algebras are finite dimensional algebras over a field.

The modules over Artin algebras were studied by Auslander, Reiten, Smalø in [13-17], Dlab and Ringel in [11, 18]. The classification of Artinian hereditary rings of finite representation type was studied by Dowbor, Ringel, Simson in [19]. The classification of these rings are given in terms of Dynkin diagrams.

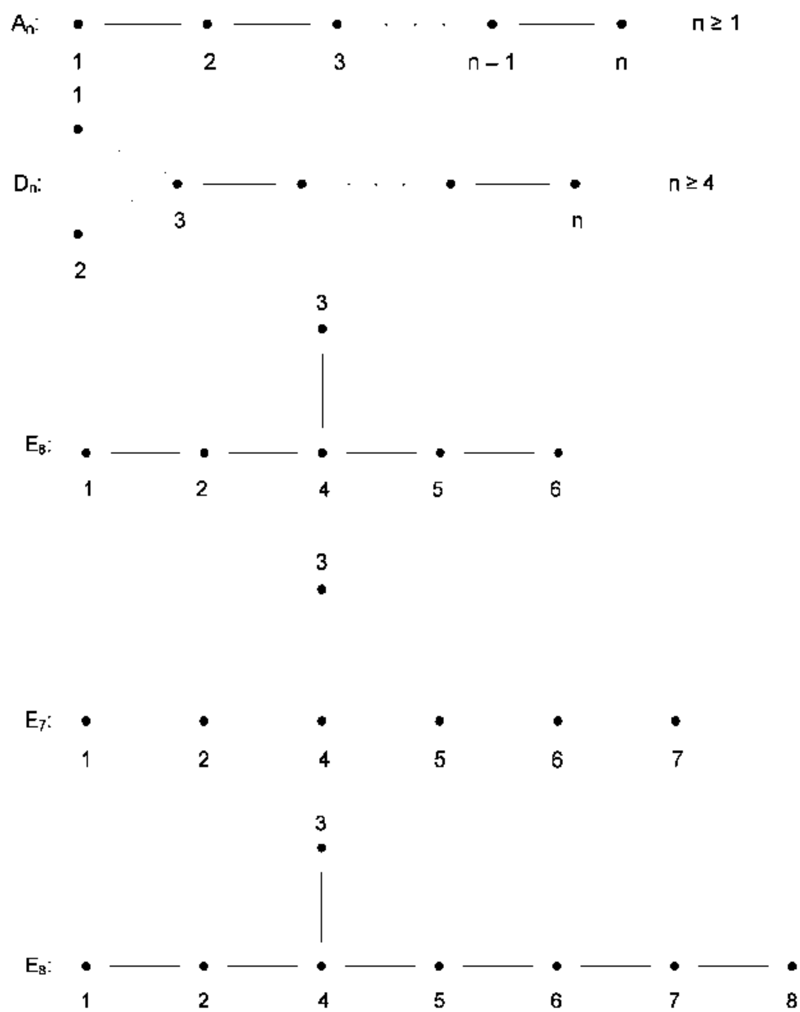
The next class of rings of finite representation type which had been described by means of Coxeter-Dynkin diagrams are hereditary pure semisimple rings. Simson in [5] proved the theorem which gives the classification of all indecomposable hereditary right pure semisimple rings of finite representation type in terms of the Coxeter valued diagrams.

From theorem 1 and [20] we obtain the following theorem.

Theorem 2. *A hereditary Artinian semidistributive ring $T(S, D)$, where S is a finite poset and D is a division ring, is a ring of finite representation type if and only if the undirected graph* $\overline{\Gamma(S)}$ *of the Hasse diagram* $\Gamma(S)$ *of the poset S is of finite type.*

A central role in the theory of representations of finite dimensional algebras and rings is played by quivers which were introduced by Gabriel in 1972 [21]. In this paper he gave a full description of quivers of finite type over an algebraically closed field. The Gabriel theorem was proved by Bernstein, Gel'fand, Ponomarev for an arbitrary field [22]. Taking into account that the quiver $Q(T)$ of a hereditary Artinian semidistributive ring $T(S, D)$ coincides with the undirected graph $\overline{\Gamma(S)}$ of the Hasse diagram $\Gamma(S)$ of S we obtain the following theorem.

Theorem 3. *A hereditary Artinian semidistributive ring $T(S, D)$ is of finite type if and only if the undirected graph* $\overline{\Gamma(S)}$ *of the Hasse diagram* $\Gamma(S)$ *is a disjoint union of the Dynkin diagrams of the form* A_n, D_n, E_6, E_7, E_8, *where*



2. Right hereditary semiperfect semidistributive rings of bounded representation type

Following to Bass [23] a ring A is called **semiperfect** if any finitely generated A-module has a projective cover. Müller in [24] proved that a ring A is semiperfect if and only if the identity of A can be decomposed into a sum of a finite number of pairwise orthogonal local idempotents.

Let $\{O_i\}$ be a family of discrete valuation rings (not necessary commutative) with the Jacobson radical M_i and a common division ring of fractions D for $i = 1$,

$2,\ldots,k$; $S = \{\alpha_1, \alpha_2,\ldots,\alpha_n\}$ a finite poset with a partial order $\leq$; $S_0 = \{\alpha_1, \alpha_2,\ldots,\alpha_k\}$ a subset of minimal points of S ($k \leq n$), and $S = S_0 \cup S_1$, where $S_1 = \{\alpha_{k+1}, \alpha_{k+2},\ldots,\alpha_n\}$.

According with this partition of S consider the **poset S with weights** so that the point i has the weight $H_{n_i}(O_i)$, $i = 1, 2,\ldots,k$; $n_i \in \mathbf{N}$; and all other points j have the weight D.

Construct a ring $A = A(S, S_0, S_1, O_1,\ldots,O_k, D, n_1, n_2,\ldots,n_k)$ (or $A(S,O)$, in short) which is a subring of $M_s(D)$, $s = n_1+n_2+\ldots+n_k+(n-k)$ by the following way. Let the identity of A be decomposed into a sum of pairwise orthogonal idempotents $1 = f_1 + f_2 +\ldots+ f_n$, and the two-sided Peirce decomposition have the following form:

$$A = \bigoplus_{i,j=1}^{n} f_i A f_j$$

where $f_i A f_i = H_{n_i}(O_i)$ for $i = 1, 2,\ldots,k$; $fAf = T(S_1)$ for $f = f_{k+1} + \ldots + f_n$; and $A_{ij} = f_i A f_j$ is an (A_{ii}, A_{jj})-bimodule, for $i,j = 1, 2,\ldots,n$. Moreover, $A_{ij} = 0$ if $\alpha_i < \alpha j$ in S. If $\alpha_i < \alpha j$ in S and $\alpha_i \in S_0$, $\alpha_j \in S_1$, then $eAf_j = D$ for any $e \in f_i$. So the two-sided Peirce decomposition have the following form:

$$A=\begin{pmatrix} H_{n_1}(O_1) & \cdots & O & M_1 \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & H_{n_k}(O_k) & M_k \\ O & \cdots & O & T(S_1) \end{pmatrix}$$

where M_i is a $(H_{n_i}(O_i), T(S_1))$-bimodule for $i=1, 2,\ldots, k$; $T(S_1)$ is the incidence ring of a poset S_1 over a division ring D. These rings were first considered in [20].

A ring $A(S,O)$ possesses a right classical ring of fractions $\tilde{A}$ which is an Artinian right hereditary semidistributive ring and has the form:

$$\tilde{A} = \begin{pmatrix} M_{n_1}(D) & \cdots & \mathbf{O} & \tilde{M}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \cdots & M_{n_k}(D) & \tilde{M}_k \\ \mathbf{O} & \cdots & \mathbf{O} & T(S_1) \end{pmatrix}$$

where $\tilde{M}_i = \tilde{H}_i \otimes_{\tilde{H}_i} M_i$, and $\tilde{H}_i = M_{n_i}(D)$ for $i = 1, 2, \ldots, k$.

For Artinian rings along with the notion of finite representation type there is considered the notion of bounded representation type. Recall that a right Artinian ring A is said to be of bounded representation type if there is a bound on the length of finitely generated indecomposable right A-modules. The first Brauer-Thrall conjecture says that these notions are the same. Roiter in [25] proved that this conjecture true for finite dimen-sional algebras and Auslander generalized this theorem for Artin algebras and one-sided Artinian rings (see [14, 15]).

Warfield, Jr. introduced in [1] the notion of bounded representation type for semiperfect rings (not necessary Artinian).

Definition. A ring A is called to be of a **right bounded representation type** if there is an upper bound on the number of generators required for indecomposable finitely presented right A-modules.

Note that since for a right Artinian ring A any finitely generated module is finitely presented and a bound of the length of indecomposable modules implies a bound of the number of generators required for indecomposable finitely presented right A-modules and visa verso, these notions are the same for right Artinian rings.

Proposition 4. *If a right hereditary ring $A = A(S,O)$ is of bounded representation type then its classical right ring of fractions $\tilde{A}$ is a ring of finite representation type.*

Proof. Suppose that A is a ring of bounded representation type, but $\tilde{A}$ is of infinite representation type. Then from [13] it follows that $\tilde{A}$ is of unbounded representation type.

Assume the following notations: if P is a right A-module, then

$$P' = P \otimes_O D \cong P \otimes_A \tilde{A};$$

on the other hand if P is a right $\tilde{A}$ -module then P' is a module P considered as an A-module.

Prove that for any $\tilde{A}$-module M there is an $\tilde{A}$-module X such that $M'' = M \oplus X$. We have

$$M'' = M' \otimes_O D = (M \otimes_O D) \otimes_O D = M \otimes_O (D \otimes_O D)$$

Since O is an integral domain, and D is its division ring of fractions, D is an injective torsion-free O-module. Therefore the map $D \to D \otimes_O D$ with $d \to 1 \otimes d$, for any $d \in D$, is a monomorphism, i.e., there exists an exact sequence of O-modules:

$$0 \to D \to D \otimes_O D \to \mathrm{Coker}(\varphi) \to 0$$

Since D is injective, this sequence splits, i.e., $D \otimes_O D = D \oplus Y$, where $Y \otimes$ Coker(φ). Therefore

$$M'' = M \otimes_O (D \otimes_O D) = M \otimes_O (D \oplus Y) = M \oplus X$$

Since $\tilde{A}$ is a ring of infinite representation type, for any $N > 0$ there is an indecomposable finitely generated $\tilde{A}$-module M such that $l(M) > N$. Consider an A-module M'. It is finitely generated and it is decomposed in a direct sum of indecomposable submodules:

$$M' = N_1 \oplus \cdots \oplus N_t \oplus X_1 \oplus \cdots \oplus X_r,$$

where N_i are torsion O-modules ($i = 1, 2,\ldots,t$), X_j are torsion-free O-modules ($j = 1, 2,\ldots,r$). Then $N_i' = N_i \otimes_O D = 0$ ($i = 1,2, \ldots, t$), and $M'' = X_1' \oplus \ldots \oplus X_s'$. Since $M'' = M \oplus X$, we have $M \oplus X = X_1' \oplus \ldots \oplus X_r'$. From uniqueness of decomposition it follows that there is a number i such that M is a direct summand of X_i', i.e., there is an $\tilde{A}$-module P such that $X_i' = M \oplus P$. Then we have a chain of inequalities:

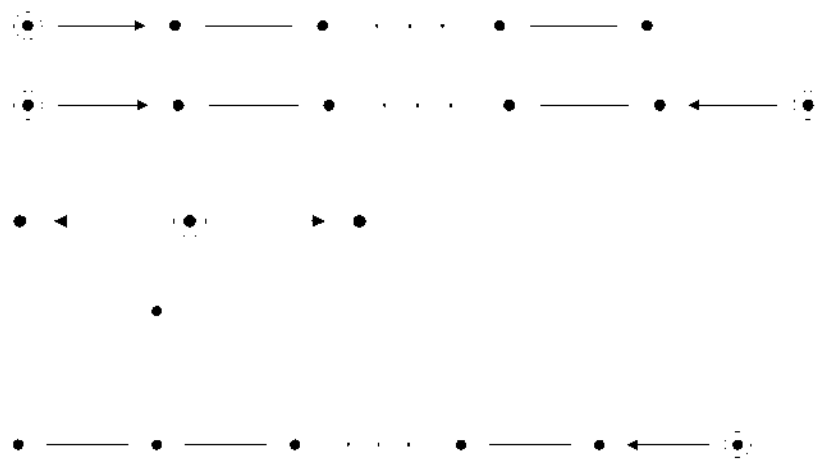
$$\mu_A(X_i) = \mu_{\tilde{A}}(X_i^{\cdot}) \geq l(X_i^{'}) = l(M) + l(P) \geq l(M) > N$$

which contradicts the assumption that A is of bounded representation type. This proves the proposition.

Now consider the finite poset S = $\{\alpha_1, \alpha_2,\ldots,\alpha_n\}$ with a partial order $\leq$; $S_0 = \{\alpha_1, \alpha_2,\ldots,\alpha_k\}$ a subset of minimal points of S ($k \leq n$), and $S = S_0 \cup S_1$, where $S_1 = \{\alpha_{k+1}, \alpha_{k+2},\ldots,\alpha_n\}$. For the Hasse diagram of S write to each vertex α_i of S_0 a weight 1 and to each vertex of S_1 a weight 0. Such Hasse diagram we shall call the **Hasse diagrams with weights**. Write a vertex with a weight 1 by ⊙ and a vertex with a weight 0 by •.

The following theorem gives the full classification of hereditary semiperfect semidistributive rings of bounded representation type in terms of Dynkin diagrams and Hasse diagrams with weights.

Theorem 6. *Let $\{O_i\}$ be a family of discrete valuation rings with a common division ring of fractions D, and let $S = S_0 \cup S_1$ be a disjoint union of subposets. A right hereditary SPSD-ring $A = A(S,O)$ is of bounded representation type if and only if the undirected graph $\overline{\Gamma(S)}$ of the Hasse diagram $\Gamma(S)$ of the poset S is a finite disjoint union of Dynkin diagrams of the form A_n, D_n, E_6, E_7, E_8 and the following diagrams with weights:*



where all vertices with weight 1 correspond to the minimal elements of the poset S.

Note that theorem 8 is a generalization of results obtained in [20].

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