KURATOWSKI LIMITS OF SEQUENCES OF YOUNG MEASURES IN CLASSICAL VARIATIONAL PROBLEMS

Piotr Puchała

Institute of Mathematics, Częstochowa University of Technology, Poland
piotr.puchala@im.pcz.pl

Abstract. We calculate Kuratowski limits of sequences of supports of Young measures associated to the minimizing sequences in two classical variational problems.

Introduction

Before formulating the problem we will recall some notions from functional analysis and set theory.

By $L^p(\Omega)$ we denote the space of functions g on Ω with values in $\mathbb{R}$ and satisfying the condition

$$\int_\Omega |g(x)|^p dx < \infty$$

for $1 \leq p < \infty$ where integration is with respect to the Lebesgue measure. This space endowed with a norm

$$\|g\|_p := \left(\int_\Omega |g(x)|^p dx\right)^{\frac{1}{p}}$$

is a Banach space. All the functions from this space having first weak derivative g' in L^p belong to the *Sobolev space* $W^{1,p}(\Omega)$ which is a Banach space under the norm

$$\|g\|_{1,p} := \left(\|g\|_p^p + \|g'\|_p^p\right)^{\frac{1}{p}}.$$

The letter A will denote the subset of $W^{1,p}(\Omega)$ consisting of functions which fulfil suitable initial conditions.

Let S be a set and $A_n \subset S, n = 1,2,\ldots$.

We define

$$\mathrm{Liminf}_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k,$$

and

$$\mathrm{Limsup}_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

If the two above limits are equal we call this common limit *the Kuratowski limit of the sequence* (A_n) of sets (see [1], eqs. (1.1), (1.2) p. 21).
We will consider the minimization problem of the form

$$\inf\{J(u): u \in A\} =: m,$$

$$J(u) := \int_\Omega f\big(x, u(x), u'(x)\big)\,dx,$$

where

$$f: \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$$

can be interpreted as the density of the internal energy and the function $u \in A$ can be understood as the deformation of the body Ω.

It is well known that if $J(u)$ is not identically equal to $+\infty$ and inf $\{J(u): u \in A\} = m > -\infty$ then there exists sequence (u_n) of functions from $W^{1,p}(\Omega)$, such that $\lim_{n\to\infty} J(u_n) = m$ (i.e. there exists the minimizing sequence for the functional J). However, in many interesting cases (both from theoretical and practical points of view) functional J does not attain its infimum (even when the minimizing sequence is convergent). In these cases the minimizing sequences are divergent in the strong topology but are convergent in the weak topology in the suitable Sobolev space. The elements of (u_n) oscillate more and more wildly around the weak limit the sequence (u_n). These oscillations describe the so called *microstructure*. We can observe such microstructures for instance in some alloys like Cu-Al-Ni or In-Th.

L.C. Young [2] introduced Young measures (called by himself „generalized trajectories") as a tool for analyzing such problems. Today Young measures are widely used in analyzing problems arising for instance in the theory of partial differential equations or in the nonlinear elasticity (see for instance [3-5] and the references cited therein).

In this paper, using the method introduced in [6], we calculate the Kuratowski limit of sequences of Young measures associated to minimizing sequences of the functionals appearing in the classical variational problems.

A very brief introduction to Young measures is in [6]. For more details we refer to the literature cited there.

Examples of computations

2a. We are seeking the minimum of the functional

$$J(u) = \int_0^1 \left[u^2 + \left(\left(\frac{du}{dx}\right)^2 - 1\right)^2\right] dx$$

with boundary conditions

$$u(0) = 0 = u(1).$$

This problem was first formulated by Oskar Bolza in 1902.

In this case we have $\Omega = (0,1)$ and

$$A = \{u \in W^{1,4}(\Omega) : u(0) = 0 = u(1)\}.$$

It can be seen that $\inf J = 0$, but there exists no function u such that $u \equiv 0$ and $\frac{du}{dx} = \pm 1$ almost everywhere with respect to the Lebesgue measure. Thus there are no classical minimizers.

Consider the minimizing sequence for J of the form

$$u_n(x) := \begin{cases} x - \frac{k}{n}, & x \in \left(\frac{k}{n}, \frac{2k+1}{2n}\right] \\ -x + \frac{k+1}{n}, & x \in \left(\frac{(2k+1)a}{2n}, \frac{(k+1)a}{n}\right) \end{cases},$$

where $n = 1, 2, \dots,\ k = 0, 1, \dots, n-1$. It is divergent in the strong topology in $W^{1,4}(\Omega)$, but it weakly converges to 0 in this space. It is also easy to see its „oscillation nature". Using method described in [6] we compute the Young measure ν_x^n associated with u_n for fixed n and obtain

$$\nu_x^n = 2n dz,$$

where dz stands for the Lebesgue measure on Ω. As $u_n(\Omega) = \left[0, \frac{1}{2n}\right]$ for each n the support of each ν_x^n is the set $\left[0, \frac{1}{2n}\right]$. Furthermore,

$$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left[0, \frac{1}{2k}\right] = \bigcup_{n=1}^{\infty} \{0\} = \{0\},$$

and

$$\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left[0,\frac{1}{2k}\right]=\bigcap_{n=1}^{\infty}\left[0,\frac{1}{2n}\right]=\{0\}.$$

This means that the set $\{0\}$ is the Kuratowski limit of the sequence of supports of Young measures associated with the elements of the minimizing sequence. The limit set contains the element which is (not attained) minimum of the functional J.

Consider another minimizing sequence for J (see [5]). Namely, let

$$W_n(x)=\begin{cases} x-\dfrac{4k}{4n}, & x\in\left[\dfrac{4k}{4n},\dfrac{4k+1}{4n}\right) \\ -x-\dfrac{4k+2}{4n}, & x\in\left[\dfrac{4k+1}{4n},\dfrac{4k+3}{4n}\right), \\ x-\dfrac{4k+4}{4n}, & x\in\left[\dfrac{4k+3}{4n},\dfrac{4k+4}{4n}\right] \end{cases}$$

where $n = 1,2, \ldots, k = 0,1,\ldots, n-1$. Again this is highly oscillatory sequence divergent in the strong topology in $W^{1,4}(\Omega)$, but weakly convergent to 0 in this space. Here the Young measures for the elements of (W_n) are of the same form as for (u_n), but for each n the element of the sequence of supports of corresponding Young measures is equal to $\left[-\frac{1}{4n},\frac{1}{4n}\right]$. Calculating the Kuratowski limit of this sequence again gives the set $\{0\}$.

2b. Functional to be minimized has the form

$$J(u)=\int_0^1\left(\left|\frac{du}{dx}\right|-\frac{1}{2}u\right)dx,$$

with the admissible set $A=\{u\in W^{1,1}(\Omega): u(0)=0, u(1)=1\}$. We have $\inf J=1$ and again it is not attained (see [1]). If we take into account the minimizing sequence

$$u_n(x):=\begin{cases} 0, & x\in\left[0,1-\dfrac{1}{n}\right) \\ nx-n+1, & x\in\left[1-\dfrac{1}{n},1\right] \end{cases}$$

and do the same calculations as above, we will get the Young measures $\nu_x^n=\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}dz$. Each of this measures has support $\{0\}\cup\left[1-\frac{1}{n},1\right]$. The

Kuratowski limit of the sequence $(\{0\} \cup [1-\frac{1}{n}, 1])$ equals $\{0\} \cup \{1\}$. Again, this limit contains the element, which is (not attained) minimum of J.

Remark. It is of interest to know what is the nature of the relationship between the infimum of the functional J and the Kuratowski (or maybe some other type) limit of the sequence of supports of the Young measures associated to the elements of the minimizing sequence. It seems that this limit contains as its element the infimum of J, even when it is not attained. But this is to be proved or disproved in the general case.

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