Dr Marian Podhorodyński
Wyższa Szkoła Zarządzania i Marketingu w Sosnowcu
Instytut Informatyki i Matematyki

PROBABILISTIC PROPERTIES OF THE SETS OF DETEMINISTIC SEQUENCES

The paper shows the consequences of frequential definition of probability which is natural from the intuitive point of view. In particular: probability isn't a σ-additive function and the family of events isn't a σ-field and even it isn't a field.

Key words: probability, average, event, field, π-system, Dynkin system.

A family $\mathcal{Z} \subset 2^{\Lambda}$ ($\Lambda \neq \emptyset$) will be called a *half -field* in Λ if

$$\mathcal{Z} \neq \emptyset,$$
$$A \in \mathcal{Z} \Rightarrow A' = \Lambda \setminus A \in \mathcal{Z},$$
$$A,B \in \mathcal{Z},\ A \cap B = \emptyset \Rightarrow A \cup B \in \mathcal{Z}$$

and a σ-*half-field* in Λ (Dynkin system [2]) if

$$\Lambda \in \mathcal{Z},$$
$$A \in \mathcal{Z} \Rightarrow A' \in \mathcal{Z},$$
$$A_n \in \mathcal{Z},\quad n \in \mathbf{N},\quad A_i \cap A_j = \emptyset,\ i \neq j \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{Z}.$$

Corollary 1. A family $\mathcal{Z}$ is a half-field in Λ if

$$\Lambda \in \mathcal{Z},$$
$$A,B \in \mathcal{Z},\ A \subset B \Rightarrow B \setminus A \in \mathcal{Z},$$
$$A,B \in \mathcal{Z},\ A \cap B = \emptyset \Rightarrow A \cup B \in \mathcal{Z}.$$

Moreover for every half-field $\mathcal{Z}$ in Λ we have

$$\emptyset \in \mathcal{Z},$$
$$A,B,\ A \cap B \in \mathcal{Z} \Rightarrow A \cup B,\quad A \setminus B \in \mathcal{Z},$$
$$A_1,\dots,A_n \in \mathcal{Z},\ A_i \cap A_j = \emptyset,\ i \neq j \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{Z}.$$

Corollary 2. A family $\mathcal{Z}$ is a field in Λ (i.e. $\mathcal{Z} \neq \emptyset$; $A \in \mathcal{Z} \Rightarrow A' \in \mathcal{Z}$; $A,B \in \mathcal{Z} \Rightarrow A \cup B \in \mathcal{Z}$) iff $\mathcal{Z}$ is a half-field in Λ and $\mathcal{Z}$ is a π-system (i.e. $A,B \in \mathcal{Z} \Rightarrow A \cap B \in \mathcal{Z}$) [2].

Corollary 3. A σ-half-field is a half-field and a family $\mathcal{Z}$ is a σ-field (i.e. $\mathcal{Z} \neq \emptyset$; $A \in \mathcal{Z} \Rightarrow A' \in \mathcal{Z}$; $A_n \in \mathcal{Z}$, $n \in \mathbf{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{Z}$) iff $\mathcal{Z}$ is a σ-half-field and $\mathcal{Z}$ is a π-system.

Now let $\emptyset \neq \lambda \subset \Lambda^{\mathbf{N}}$ ($\Lambda \neq \emptyset$) and for a function $S : \Lambda \to \mathbf{R}$ we define

$$S(\lambda) = \{(S(\lambda_i) : i \in \mathbf{N}) : (\lambda_i : i \in \mathbf{N}) \in \lambda\}$$

which will be called an *operation* on λ and

$$\langle S(\lambda) \rangle = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} S(\lambda_i)$$

if the above limit exists for every $(\lambda_i : i \in \mathbf{N}) \in \lambda$ and it is identic for all these sequences. $\langle S(\lambda) \rangle$ will be called an *average* of the operation S on λ.

Definition.
A set $A \subset \Lambda$ will be called an *event* under the set $\emptyset \neq \lambda \subset \Lambda^{\mathbf{N}}$ (shortly λ-event) if there exists the average $\langle 1_A(\lambda) \rangle$, (where $1_A(x) = 1$ for $x \in A$ and $1_A(x) = 0$ for $x \notin A$) i.e. if for all $(\lambda_i : i \in \mathbf{N}) \in \lambda$ there exists common limit of the sequences

$$\left(\frac{1}{n} \sum_{i=1}^{n} 1_A(\lambda_i) : n \in \mathbf{N}\right)$$

of frequences of the set A. This average will be called *λ-probability* of the λ-event A and will be noticed $P_\lambda(A)$.

For the family $\mathcal{Z}_\lambda$ of all λ-events we have

$$P_\lambda(A) = \langle 1_A(\lambda) \rangle \text{ for } A \in \mathcal{Z}_\lambda.$$

Proposition.
The family $\mathcal{Z}_\lambda$ is a half-field in Λ and P_λ is nonnegative ($P_\lambda(A) \geq 0$ for $A \in \mathcal{Z}_\lambda$) normed ($P_\lambda(\Lambda) = 1$) and additive function on $\mathcal{Z}_\lambda$
(i.e. $P_\lambda(A \cup B) = P_\lambda(A) + P_\lambda(B)$ for $A, B \in \mathcal{Z}_\lambda$ such that $A \cap B = \emptyset$).

Corollary 4. λ-probability has the following properties:
$P_\lambda(B \setminus A) = P_\lambda(B) - P_\lambda(A)$, $P_\lambda(B) \geq P_\lambda(A)$ for $A, B \in \mathcal{Z}_\lambda$ such that $A \subset B$;

$$P_\lambda(A') = 1 - P_\lambda(A) \text{ for } A \in \mathcal{Z}_\lambda;$$

$$P_\lambda(A \cup B) = P_\lambda(A) + P_\lambda(B) - P_\lambda(A \cap B), \quad P_\lambda(A \setminus B) = P_\lambda(A) - P_\lambda(A \cap B)$$

for $A, B \in \mathcal{Z}_\lambda$ such that $A \cap B \in \mathcal{Z}_\lambda$.

Corollary 5. The family $\mathcal{Z}_\lambda$ is complet under P_λ i.e.

$$B \in \mathcal{Z}_\lambda, A \subset B, P_\lambda(B) = 0 \Rightarrow A \in \mathcal{Z}_\lambda (P_\lambda(A) = 0)$$

and it has the following properties:

$$B \in \mathcal{Z}_\lambda, A \supset B, P_\lambda(B) = 1 \Rightarrow A \in \mathcal{Z}_\lambda (P_\lambda(A) = 1);$$
$$B \in \mathcal{Z}_\lambda, P_\lambda(B) = 0 \Rightarrow A \cap B \in \mathcal{Z}_\lambda (P_\lambda(A \cap B) = 0) \text{ for all } A \subset \Lambda;$$
$$A, B \in \mathcal{Z}_\lambda, P_\lambda(B) = 1 \Rightarrow A \cap B \in \mathcal{Z}_\lambda (P_\lambda(A \cap B) = P_\lambda(A)).$$

For $\emptyset \neq \lambda \subset \Lambda^{\mathbf{N}}$ we define the set R_λ of all realizations of λ by

$$R_\lambda = \{ x \in \Lambda : \exists (\lambda_i : i \in \mathbf{N}) \in \lambda \ \exists\, i \in \mathbf{N} \text{ such that } x = \lambda_i \}.$$

Corollary 6. The family $\mathcal{Z}_\lambda$ has the following properties:

$$R_\lambda \in \mathcal{Z}_\lambda (P_\lambda(R_\lambda) = 1),$$
$$A \cap R_\lambda = \emptyset \Rightarrow A \in \mathcal{Z}_\lambda (P_\lambda(A) = 0),$$
$$R_\lambda \subset A \Rightarrow A \in \mathcal{Z}_\lambda (P_\lambda(A) = 1),$$
$$A \in \mathcal{Z}_\lambda \Rightarrow A \cap R_\lambda \in \mathcal{Z}_\lambda (P_\lambda(A) = P_\lambda(A \cap R_\lambda)).$$

Remark 1. There exist the set Λ and $\lambda \subset \Lambda^{\mathbf{N}}$ such that the λ-probability isn't σ-additive function on $\mathcal{Z}_\lambda$.

Proof. Let Λ be an infinite set, say $\{\lambda_1, \lambda_2, \ldots\} \subset \Lambda$.
Let $\lambda = \{(\lambda_1, \lambda_2, \ldots)\}$ and $A_n = \{\lambda_n\}$, $n \in \mathbf{N}$. It is easy to see that

$$\bigcup_{n=1}^{\infty} A_n, \ A_n \in \mathcal{Z}_\lambda, \ n \in \mathbf{N}, A_i \cap A_j = \emptyset, i \neq j$$

and

$$P_\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) \neq \sum_{n=1}^{\infty} P_\lambda(A_n)$$

because

$$P_\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 \text{ and } P_\lambda(A_n) = 0 \text{ for all } n \in \mathbf{N}.$$

Remark 2. There exist the set Λ and $\lambda \subset \Lambda^{\mathbf{N}}$ such that the family $\mathcal{Z}_\lambda$ isn't a σ-half-field.

Proof. Let Λ be an infinite set and

$$\{\lambda_1, \lambda_2, \ldots\} \cup \{\lambda_1', \lambda_2', \ldots\} \subset \Lambda, \ \{\lambda_1, \lambda_2, \ldots\} \cap \{\lambda_1', \lambda_2', \ldots\} = \emptyset.$$

For $\lambda = \{(\lambda_n : n \in \mathbf{N}), (\lambda_n' : n \in \mathbf{N})\}$ and $A_n = \{\lambda_n\}$, $n \in \mathbf{N}$ we have $A_n \in \mathcal{Z}_\lambda$ for $n \in \mathbf{N}$ because $P_\lambda(A_n) = 0$ for all $n \in \mathbf{N}$ and $A = \bigcup_{n=1}^{\infty} A_n \notin \mathcal{Z}_\lambda$ because

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1_A(\lambda_i) = 1$$

but

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1_A(\lambda_i') = 0.$$

Remark 3. There exist the set Λ and $\lambda \subset \Lambda^{\mathbf{N}}$ such that the family $\mathcal{Z}_\lambda$ isn't a field (π-system).

Proof. Let $\Lambda = \mathbf{R}$ and $\{\lambda_n : n \in \mathbf{N}\}$, $\{\lambda_n' : n \in \mathbf{N}\}$ be the sets such that $\{\lambda_n : n \in \mathbf{N}\} \cup \{\lambda_n' : n \in \mathbf{N}\} \subset [0,1]$, $\{\lambda_n : n \in \mathbf{N}\} \cap \{\lambda_n' : n \in \mathbf{N}\} = \emptyset$

and

$$\frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty, x)}(\lambda_i) \to F(x), \quad \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty, x)}(\lambda_i') \to F(x),$$

where

$$F(x) = \int_{-\infty}^{x} 1_{[0,1]}(t)dt, \quad x \in \mathbf{R}$$

(see Glivenko-Cantelli theorem [2]).

For $A = ((-\infty, ½) \cap \{\lambda_n' : n \in \mathbf{N}\}) \cup ([½, +\infty) \setminus \{\lambda_n' : n \in \mathbf{N}\})$, $B = (-\infty, ½)$ and $\lambda = \{(\lambda_n : n \in \mathbf{N}), (\lambda_n' : n \in \mathbf{N})\}$ we have

$$A, B \in \mathcal{Z}_\lambda \text{ and } A \cap B \notin \mathcal{Z}_\lambda$$

because

$$\langle 1_A(\lambda) \rangle = \frac{1}{2} = \langle 1_B(\lambda) \rangle$$

but

$$\lim_{n \to \infty} \frac{1}{n}\sum_{i=1}^{n} 1_{A \cap B}(\lambda_i) = 0$$

and

$$\lim_{n \to \infty} \frac{1}{n}\sum_{i=1}^{n} 1_{A \cap B}(\lambda_i') = \frac{1}{2}.$$

REFERENCES

[1] H. Bauer, *Probability Theory and Elements of Measure Theory*, Academic Press, London 1981.

[2] P Billingsley, *Probability and Measure*, J. Wiley and Sons, New York 1986.

[3] P.R. Halmos, *Measure Theory*, GTM 18, Springer Verlag, New York 1974.

[4] R. Mises, *Mathematical Theory of Probability and Statistics*, Academic Press, New York 1964.