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INTRODUCTION TO MATHEMATICAL PROGRAMMING



Częstochowa 2023



Czestochowa University of Technology

Anita Ciekot

Introduction to Mathematical Programming Part I



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INTRODUCTION

This textbook is intended for students of technical and economic universities. It is a result of my teaching of mathematical programming, optimization methods and operations research to students of Czestochowa University of Technology, including Erasmus+ Program students, for over ten years. This is the first part of a planned series (an intended series), limited to the presentation of issues related to linear programming. The second part will focus on non-linear programming problems.

The textbook is divided into five main chapters. Chapter 1 is a reminder of some mathematical topics (the basics of linear algebra and systems of linear equations) that will help the readers understand the material discussed. Chapter 2 considers linear programming problems from its standard form to practical, highly common, with various examples of applications at the intersection of technology and economics. In chapter 3, the transportation problem with applications will be considered. Chapter 4 deals with a special kind of linear programming, so-called integer programming. And finally, Chapter 5 shows how the Maple package can be used to solve any linear programming problems.

CHAPTER 1

REVIEW OF LINEAR ALGEBRA

This chapter will remind you of the basics of linear algebra and of systems of linear equations, which are used in the process of deriving a mathematical model and solving a given practical problem.

1.1. Matrices. Matrix operation

Definition 1. An $m \times n$ matrix \mathbf{A} is a rectangular array of elements (numbers or functions, real or complex) with m **rows** (horizontal) and n **columns** (vertical):

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix}$$

We say that matrix \mathbf{A} is of order $m \times n$ (where $m \times n$ is said “ m by n ”) if it has m rows and n columns. Usually, matrices are denoted by capital letters.

$\mathbf{A} = [a_{ij}]$ is shorthand notation for identifying the matrix whose element in row i and column j is a_{ij} for every $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. For example,

$\mathbf{A} = \begin{bmatrix} 0 & 8 \\ 2 & 6 \end{bmatrix}$ is a 2×2 matrix with elements $a_{11} = 0$, $a_{12} = 8$, $a_{21} = 2$, $a_{22} = 6$,

$\mathbf{B} = \begin{bmatrix} 1 & 2.4 & -1 & 0 \\ -4 & 2 & 0 & \sqrt{3} \end{bmatrix}$ is a 2×4 matrix.

Definition 2. Basic operations with matrices

Let \mathbf{A} and \mathbf{B} be two matrices having the same number of rows and columns.

- **Equal matrices:** two matrices \mathbf{A} and \mathbf{B} of the same order are equal ($\mathbf{A} = \mathbf{B}$) if and only if all the corresponding elements are equal ($a_{ij} = b_{ij}$ for all i and j).

- **Addition:** to add two matrices \mathbf{A} and \mathbf{B} , simply add the corresponding elements, so that $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.
- **Subtraction:** to subtract two matrices \mathbf{A} and \mathbf{B} , simply subtract the corresponding elements, so that $\mathbf{A} - \mathbf{B} = [a_{ij} - b_{ij}]$.
- **Multiplication by a scalar:** the operation of multiplying a matrix by a number (denote this number by k) is performed by multiplying each element of the matrix by k , so that $k\mathbf{A} = [ka_{ij}]$.

Matrix product requires an additional restriction related to the size of matrices \mathbf{A} and \mathbf{B} : matrix multiplication \mathbf{AB} is defined if and only if the number of columns of matrix \mathbf{A} is equal to the number of rows of matrix \mathbf{B} .

- **Matrix product:** to find the element in row i , column j of the matrix resulting from multiplying matrix \mathbf{A} times matrix \mathbf{B} , it is necessary to multiply each element in row i of \mathbf{A} by the corresponding element in column j of \mathbf{B} and then to add these products. For example, if \mathbf{A} is an $m \times n$ matrix, and \mathbf{B} is an $n \times s$ matrix, then their product is

$$\mathbf{AB} = \left[\sum_{k=1}^n a_{ik} a_{kj} \right]$$

where this product is $m \times s$ matrix. Even when both \mathbf{AB} and \mathbf{BA} are defined, $\mathbf{AB} \neq \mathbf{BA}$ in general.

- **Transposition:** this operation involves nothing more than interchanging the rows and columns of the matrix. Thus, for any $\mathbf{A} = [a_{ij}]$, its transpose is $\mathbf{A}^T = [a_{ji}]$.

Note that the operation matrix division is not defined.

To illustrate:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \sqrt{2^2} & |5-4| \\ \sqrt[3]{8} & 50^0 \end{bmatrix} \text{ are equal because corresponding elements are equal,}$$

$$\text{multiplication by a scalar: } 5 \cdot \begin{bmatrix} 2 & \frac{1}{5} & 1 \\ -5 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 1 & 5 \\ -25 & 0 & 15 \end{bmatrix},$$

$$\text{addition: } \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix},$$

$$\text{and subtraction: } \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -3 & -2 \end{bmatrix},$$

matrix product and transposition operation:

$$\begin{aligned} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 0 & 3 \end{bmatrix}^T \cdot \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \\ \begin{bmatrix} 1 \cdot 3 + 2 \cdot 5 & 1 \cdot 1 + 2 \cdot 1 \\ 4 \cdot 3 + 0 \cdot 5 & 4 \cdot 1 + 0 \cdot 1 \\ 2 \cdot 3 + 3 \cdot 5 & 2 \cdot 1 + 3 \cdot 1 \end{bmatrix} &= \begin{bmatrix} 10 & 3 \\ 12 & 4 \\ 21 & 6 \end{bmatrix}. \end{aligned}$$

Fact 1. The matrix operations described here satisfy the following laws:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C}$$

$$\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C}.$$

Some special matrices

1. A **column vector** or **column** is a matrix size $m \times 1$.

A **row vector** or **row** is a matrix size $1 \times n$.

For example:

$\mathbf{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a 3×1 matrix, and $\mathbf{D} = [e^x \quad \cos x]$ is a 1×2 matrix with elements

(as functions) $a_{11} = e^x$, $a_{12} = \cos x$.

2. **Square matrix** is a matrix size $n \times n$ or simply order n .

3. **Identity matrix \mathbf{I}** – is a square matrix whose elements are zeros except for the ones along the main diagonal. For any matrix \mathbf{A} , $\mathbf{I}\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}$, where the appropriate number of rows and columns is assigned in each case for the multiplication operation to be defined. Thus,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 \end{bmatrix}.$$

Let us now consider the set of n – vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ of the same type (they are either all row vectors or all column vectors).

Definition 3. A set of n – vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ is said to be **linearly dependent** if there exist m numbers (denoted by c_1, c_2, \dots, c_m), some of which are not zero, such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_m\mathbf{x}_m = \mathbf{0}$. Otherwise, the set is said to be **linearly independent**.

To exemplify, if $m = 3$ and $\mathbf{x}_1 = [1 \ 1 \ 1]$, $\mathbf{x}_2 = [0 \ 1 \ 1]$, $\mathbf{x}_3 = [2 \ 5 \ 5]$ then there exist three numbers: $c_1 = 2$, $c_2 = 3$ and $c_3 = -1$, such that

$$\begin{aligned} 2\mathbf{x}_1 + 3\mathbf{x}_2 + (-1)\mathbf{x}_3 &= [2 \ 2 \ 2] + [0 \ 3 \ 3] + [-2 \ -5 \ -5] \\ &= [0 \ 0 \ 0] \end{aligned}$$

therefore, these vectors are linearly dependent. Unfortunately, sometimes finding these numbers is not easy but this equation also implies that $\mathbf{x}_3 = 2\mathbf{x}_1 + 3\mathbf{x}_2$, and it means that they are linearly dependent because one of these three vectors is a linear combination of the others.

Definition 4. The **rank** of a set of vectors is the largest number of linearly independent vectors that can be chosen from the set.

Definition 5. A **basis** for a set of vectors is a collection of linearly independent vectors taken from the set such that every vector in the set is a linear combination of the vectors in the collection (every vector in the set equals the sum of certain multiples of the vectors in the collection). A collection of r linearly independent vectors chosen from a set of vectors is a **basis for the set** if and only if the set has rank r .

Definition 6. Consider the matrix \mathbf{A} size $n \times n$. If there exists at least one nonsingular square submatrix in \mathbf{A} of order k , and if all square submatrices of order $k+i$, $i \geq 1$ are singular, then the number k is called the **rank of a matrix \mathbf{A}** and is denoted by $r(\mathbf{A})$.

The rank of a matrix is equal to the maximal number of its linearly independent rows (or columns).

A square matrix is **nonsingular** if its rank equals both the number of rows and the number of columns. Otherwise, it is **singular**.

Definition 7. If \mathbf{A} is nonsingular, there is a unique nonsingular matrix \mathbf{A}^{-1} , called **the inverse of \mathbf{A}** , such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$.

If \mathbf{A} is nonsingular and \mathbf{B} is a matrix for which either $\mathbf{A}\mathbf{B} = \mathbf{I}$ or $\mathbf{B}\mathbf{A} = \mathbf{I}$, then $\mathbf{B} = \mathbf{A}^{-1}$. Only nonsingular matrices have inverses.

1.2. Systems of linear equations. Methods of solution

Definition 8. An $m \times n$ system of linear equations is a set of m equations in n variables x_1, x_2, \dots, x_n of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_i are constants, and x_i are the unknown variables.

If b_i are all equal to zero, the system is called homogeneous.

Definition 9. A **solution** of the system of linear equations is a point in \mathbb{R}^n whose coordinates satisfy the system of equations.

This system of equations can also be written in matrix form as $\mathbf{A}\mathbf{x} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Definition 10. An **augmented matrix** is formed by appending the entries of the column vector \mathbf{b} (right-hand side of the equation) to those of the coefficient matrix

\mathbf{A} , creating a matrix that is now of order $m \times (n + 1)$. The augmented matrix of system is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} & b_m \end{array} \right].$$

Theorem 1. Kronecker's-Capelli Theorem

A set of linear equations $\mathbf{Ax} = \mathbf{b}$ has the solution if and only if the rank of matrix \mathbf{A} is equal to the rank of the augmented matrix $[\mathbf{A}|\mathbf{b}]$ of the set: $rk(\mathbf{A}) = rk[\mathbf{A}|\mathbf{b}]$.

Based on this theorem, the number of solutions to a system of equations can be determined. Let $\mathbf{Ax} = \mathbf{b}$ be the set of linear equations in n variables. Then:

- if $rk(\mathbf{A}) = rk[\mathbf{A}|\mathbf{b}]$, then the set has no solution (it is inconsistent);
- if $rk(\mathbf{A}) = rk[\mathbf{A}|\mathbf{b}] = n$, then the set has a single unique solution (it is determined);
- if $rk(\mathbf{A}) = rk[\mathbf{A}|\mathbf{b}] = r < n$, then the set has infinitely many solutions, which depend on $p = n - r$ parameters (it is underdetermined).

The solution to the system of equations can be obtained by the Gauss-Jordan method of elimination. The process begins by first expressing the system in matrix form, $\mathbf{Ax} = \mathbf{b}$, where \mathbf{A} represents the coefficient matrix, and \mathbf{b} the matrix of constant terms. The next step is to reduce it to an equivalent system by simple row operations. This row reduction continues until the system is expressed in what is called the **reduced row echelon form**. The matrix is in the reduced row echelon form if the first nonzero entry in each row is 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom. The solution is readily obtained from this form.

Definition 11. The **pivoting process** is a process of obtaining a 1 in a location, and then making all other entries zeros in that column. The number that is made a 1 is

called the **pivot element**, and the row that contains the pivot element is called the **pivot row**.

To summarize, the steps of Gauss-Jordan elimination are as follows:

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a non-zero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.
6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero;

and the three row operations the Gauss-Jordan method employs:

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

Example 1: Solve the following system of linear equations by the Gauss-Jordan method

$$\begin{cases} x - 2y + 3z = 7 \\ 2x + 5y + 2z = 1 \\ 3x + y + 5z = 10 \end{cases}$$

Solution

Step 1. Write the augmented matrix

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 2 & 5 & 2 & 1 \\ 3 & 1 & 5 & 10 \end{array} \right]$$

The element in the first row and first column is equal to 1, so you can move on to step 4.

Step 4. Use a row operation to get a 1 as the entry in the first row and first column, use row operations to make all other entries zeros in column one: to make the element (2) a zero in row 2, column 1, multiply row 1 by -2 and add it to the second row

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 2 & 5 & 2 & 1 \\ 3 & 1 & 5 & 10 \end{array} \right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 9 & -4 & -13 \\ 3 & 1 & 5 & 10 \end{array} \right]$$

to make the element (3) a zero in row 3, column 1, multiply row 1 by -3 and add it to the third row

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 9 & -4 & -13 \\ 3 & 1 & 5 & 10 \end{array} \right] \xrightarrow{-3R_1+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 9 & -4 & -13 \\ 0 & 7 & -4 & -11 \end{array} \right]$$

Step 5. Move to the next diagonal element, row 2, column 2. To make the element 9 a 1 in row 2, column 2, divide row 2 by 9

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 9 & -4 & -13 \\ 0 & 7 & -4 & -11 \end{array} \right] \xrightarrow{\frac{R_2}{9}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 1 & \frac{-4}{9} & \frac{-13}{9} \\ 0 & 7 & -4 & -11 \end{array} \right]$$

next, make all other elements zeros in the second column

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 1 & \frac{-4}{9} & \frac{-13}{9} \\ 0 & 7 & -4 & -11 \end{array} \right] \xrightarrow{\substack{2R_2+R_1 \\ -7R_2+R_3}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{9} & \frac{37}{9} \\ 0 & 1 & \frac{-4}{9} & \frac{-13}{9} \\ 0 & 0 & \frac{-8}{9} & \frac{-8}{9} \end{array} \right]$$

Step 6. Make the last diagonal element a 1, by dividing row 3 by $-8/9$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{9} & \frac{37}{9} \\ 0 & 1 & \frac{-4}{9} & \frac{-13}{9} \\ 0 & 0 & \frac{-8}{9} & \frac{-8}{9} \end{array} \right] \xrightarrow{R_3 \cdot \frac{-9}{8}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{9} & \frac{37}{9} \\ 0 & 1 & \frac{-4}{9} & \frac{-13}{9} \\ 0 & 0 & 1 & 1 \end{array} \right]$$

and finally, make all other elements zeros in column 3

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{9} & \frac{37}{9} \\ 0 & 1 & \frac{-4}{9} & \frac{-13}{9} \\ 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow[\frac{-19}{9}R_3+R_1]{\frac{4}{9}R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{18}{9} \\ 0 & 1 & 0 & \frac{-9}{9} \\ 0 & 0 & 1 & 1 \end{array} \right]$$

the solution is $x = 2, y = -1, z = 1$.

Example 2: Solve the following system of linear equations by using the Gauss-Jordan method

$$\begin{cases} x + y - 3z = 3 \\ 2x - 3y + 4z = -4 \\ x - y + z = -1 \end{cases}$$

Solution

Step 1. Write the augmented matrix

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 2 & -3 & 4 & -4 \\ 1 & -1 & 1 & -1 \end{array} \right]$$

By following the steps of the Gauss-Jordan method, we get:

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 2 & -3 & 4 & -4 \\ 1 & -1 & 1 & -1 \end{array} \right] \xrightarrow[\frac{-R_1+R_3}{-2R_1+R_2}]{\frac{-R_1+R_3}{-2R_1+R_2}} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & -5 & 10 & -10 \\ 0 & -2 & 4 & -4 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & -5 & 10 & -10 \\ 0 & -2 & 4 & -4 \end{array} \right] \xrightarrow[\frac{-2}{R_3}]{\frac{R_2}{-5}} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \end{array} \right]$$

and finally

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \end{array} \right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

because of the last row (which has only the elements equal to zero) this system of equations has infinitely many solutions, which depend on $p = 3 - 2 = 1$ parameters, $p \in \mathbb{R}$, according to Kronecker's-Capelli Theorem.

So, the solution is

$$z = p, \quad y = 2 + 2p, \quad x = 1 + p, \quad p \in \mathbb{R}.$$

Example 3: Solve the following system of linear equations by using the Gauss-Jordan method

$$\begin{cases} 2x + 5y = 9 \\ x + 2y - z = 3 \\ -3x - 4y + 7z = 1 \end{cases}$$

Solution

Step 1. Write the augmented matrix

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 5 & 0 & 9 \\ 1 & 2 & -1 & 3 \\ -3 & -4 & 7 & 1 \end{array} \right]$$

By following the steps of the Gauss-Jordan method, we get:

$$\left[\begin{array}{ccc|c} 2 & 5 & 0 & 9 \\ 1 & 2 & -1 & 3 \\ -3 & -4 & 7 & 1 \end{array} \right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 9 \\ -3 & -4 & 7 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 9 \\ -3 & -4 & 7 & 1 \end{array} \right] \xrightarrow{\substack{-2R1+R2 \\ 3R1+R3}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 10 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 10 \end{array} \right] \xrightarrow{\substack{-2R2+R1 \\ -2R2+R3}} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{array} \right]$$

this system is inconsistent (because in the last row is: $0 \neq 4$), which means that the system has no solution.

Exercises

1. For given matrices **A**, **B**, **C**, add **AB** to **AC** and compare them with **A(B+C)**:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix}.$$

2. Analyze the order of the product matrices or the factor matrices indicated in parts 1) – 3):
- 1) If **AB** is a 4×5 matrix, how many columns does **B** have?
 - 2) If **AB** is a 4×9 matrix, how many rows does **A** have?
 - 3) If matrix **A** is a 2×6 matrix, and **AB** is a 2×4 matrix, what is the order of matrix **B**?
3. Use the Gauss-Jordan elimination method to find the solution for the given system of equations:

a)
$$\begin{cases} 3x - y = 1 \\ 4x + 5y = 14 \end{cases}$$

b)
$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

c)
$$\begin{cases} 4x + y - 4z = 2 \\ 2x - 3y + 2z = 2 \\ x - y + z = 2 \end{cases}$$

d)
$$\begin{cases} 3x - y + z = -2 \\ -x + y + z = 4 \\ 5x + 2y + 9z = 15 \end{cases}$$

CHAPTER 2

LINEAR PROGRAMMING

Mathematical programming is a field of knowledge that came into existence after World War II. There is a set of methods for finding the point that minimizes or maximizes the value of a real function in a subset of the space \mathbb{R}^n . It is one of the most used parts of mathematical modeling – a mathematical technique that applies advanced analytical methods to help make better decisions.

Mathematical programming is an effective tool widely used in science and engineering. First, through observation, we form hypotheses about the behavior of the world around us. Next, we develop mathematical models of those hypotheses that we can evaluate, improve, verify and validate. Mathematical models and methods provide a rigorous, systematic, and quantitative description of various real-world phenomena. There are many real-world problems in such different areas as industrial production, transport, telecommunications, finance, or personnel planning that may be transformed into the form of a mathematical programming problem. The problem of mathematical programming can be formulated in a general way as follows:

$$\begin{aligned} f(\mathbf{x}) &\rightarrow \text{maximized (or minimized)} \\ \text{subject to } g_i(\mathbf{x}) &\leq 0; \quad i = 1, 2, \dots, m \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $f(\mathbf{x})$, $g_i(\mathbf{x})$, $i = 1, 2, \dots, m$ are real valued functions of \mathbf{x} .

The most crucial part of mathematical programming is to start with the right mathematical model to represent the problem. Generally, they consist of a set of variables, restrictions over these variables, and a function to be maximized or minimized. Mathematical programming problems are ordinarily classified according to the types of decision variables, constraints, and the function. If the functions $f(\mathbf{x})$, $g_i(\mathbf{x})$ are all linear, the problem is known as a linear programming problem. It is the most common type of mathematical programming – a well understood case for

which efficient algorithms are known. If these functions are non-linear, one of them or all of them, then the problems are called non-linear mathematical programming problems. They are more difficult, and there is no well-known method to find a solution to a given problem. Very often, approximation and heuristic methods are used to find good solutions for non-linear problems.

2.1. Basic concepts and notation

In this section, the important terms that are used to describe any linear programming problems will be introduced and defined.

Definition 1. A **decision variable** is a physical quantity represented by a mathematical symbol and describes the decision to be made. It represents the unknowns to be determined by the solution to the model.

Definition 2. A **linear programming problem** (also called LP) is an optimization problem that consists of the following components:

1. A linear real-valued function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, of n decision variables, x_1, x_2, \dots, x_n . The function that is to be maximized or minimized is called the **objective function**. This function is linear and can be written in the form:

$$z = f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

here, each $c_i \in \mathbb{R}$ – the coefficients in the objective function are called the **objective function coefficients**.

2. A **set of m constraints**, functional equalities, or inequalities, which describes the limitation that restrict our choices for decision variables. Each constraint is linear in that it takes the form:

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \end{cases}$$

where $1 \leq i \leq m$ and where each a_{ij} and b_i belongs to \mathbb{R} .

3. The **sign restrictions** are placed on any of the decision variables. If a decision variable can only assume nonnegative values, then we add the sign restriction $x_1, x_2, \dots, x_n \geq 0$ (non-negativity constraints). If a variable can assume both posi-

tive and negative (or zero) values, then we say that this variable is unrestricted in sign (often abbreviated: urs).

Definition 3. The **feasible solution** of the linear programming problem is the vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

which satisfies all constraints and sign restrictions of LP.

Definition 4. An **optimal solution** is a feasible solution whose corresponding objective function value is greater than or equal to that of any other feasible solution for a maximization problem and less than or equal to that of any other feasible solution for a minimization problem.

Definition 5. The **feasible region** is the set of all feasible solutions.

It can be shown that in linear programming problems, the feasible region is always a convex set.

Definition 6. A subset K of \mathbb{R}^n is **convex** if K is empty or K is a single point, or if for each two points x and y in K , the line segment connecting the two points lies entirely in the subset: $\lambda x + (1 - \lambda)y \in K$ for $0 \leq \lambda \leq 1$.

A convex feasible set for a linear problem will have a shape with edges that are straight lines and corners where the edges meet.

Definition 7. A point x is a **corner point** of convex set K if x is not an interior point for any segment line contained in K .

Definition 8. If the convex feasible region corresponding to the linear programming problem is nonempty, it must have at least one corner point. The corners of the feasible set are called **extreme points**.

These properties of the feasible region and feasible solutions allow the formulation of the fundamental theorem of linear programming (Corner point theorem).

Theorem 1. Let us consider the linear programming problem

$$z = f(x_1, x_2, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \rightarrow \max (\min)$$

$$\text{subject to } \begin{cases} a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \\ x_1, x_2, \dots, x_n \geq 0 \\ a_{ij}, b_i \in \mathbb{R}, i = 1, 2, \dots, m \end{cases}$$

If a linear programming problem has a solution, then there exists an extreme point of the feasible region which is optimal.

The fundamental theorem of linear programming reduces to a finite value the number of feasible solutions that need to be evaluated.

The solution of any Linear Programming (LP) takes one of four general forms:

1. The LP has a **unique optimal solution**.
2. The LP has more than one optimal solution (an infinite number), so we say that it has **alternative (multiple) optimal solutions**.
3. The LP is **infeasible**, meaning that its feasible region is empty set (the feasible set contains no points).
4. The LP is **unbounded**. For a problem in which one seeks to maximize the objective function, this means that the objective function can be made as large as one likes, as there are points in the feasible region with an arbitrary large value of an objective function. For a minimization problem, this means that the objective function can be made as small as one likes.

The matrix form of a Linear programming problem

Typically, we describe linear models by writing them out fully. This is acceptable for small linear programs, but it does not work when the linear models are very large. In this case, it helps to use an algebraic form (matrix form). It is a more compact form.

$$z = \mathbf{c} \cdot \mathbf{x} \rightarrow \mathbf{max}$$

subject to

$$\mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

where:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}, \quad \mathbf{c} = [c_1 \quad c_2 \quad \dots \quad c_n]$$

2.2. Linear programming problem formulation

There are a few steps that can be helpful in the process of formulating the mathematical model for a linear programming problem:

- step 1 – identify and define the decision variables precisely,
 step 2 – definite the objective function,
 step 3 – identify and express the right constraints mathematically.

Let us look at some typical, practical examples, to better understand the idea of formulating linear programming problems.

Example 4: Outdoor furniture factory “Patio Accessories” sells tables and chairs. They have two resources devoted to the production of tables and chairs: wood and labor. “Patio Accessories” makes \$200 worth of profit from every table they sell, and \$50 worth of profit from every chair they sell. Each table requires 10 units of wood and 3 hours of labor, and each chair requires 5 units of wood and 4 units of labor. Let us also assume that this factory has 100 units of wood and 60 units of labor available. For the sake of simplicity, let us assume that it is not possible to get more of any of these two resources.

“Patio Accessories” wants to know how many tables and how many chairs to produce each month in order to maximize their profits.

Solution: let us start by gathering all the information in table form:

	Tables	Chairs	Available
Wood	10	5	100
Labor	3	4	60
Profits [\$]	200	50	

First step: determination of the decision variables

let us denote by x_1 – number of tables produced
 x_2 – number of chairs produced

Second step: identification of the formula for the objective function

“Patio Accessories” total profit will be the sum of the profit from selling tables and chairs. The profit from selling tables will be the product of the profit from each table times the number of tables sold. Similarly, we can define the profit from selling chairs. Because “Patio Accessories” wants to maximize their profit, the objective function will take the following form

$$z = f(x_1, x_2) = 200x_1 + 50x_2 \rightarrow \max$$

Third step: determination of the constraints:

the wood constraint: if the production of tables is x_1 and chairs is x_2 , “Patio Accessories” should use $10x_1 + 5x_2$ units of wood

$$10x_1 + 5x_2 \leq 100$$

the labor constraint: if the production of tables is x_1 and chairs is x_2 , “Patio Accessories” should use $3x_1 + 4x_2$ units of labor

$$3x_1 + 4x_2 \leq 60$$

and **non-negativity constraint or sign restriction:** $x_1, x_2 \geq 0$

Finally, the mathematical model of the “Patio Accessories” problem is:

$$z = f(x_1, x_2) = 200x_1 + 50x_2 \rightarrow \max$$

$$\text{s.t.} \quad \begin{cases} 10x_1 + 5x_2 \leq 100 \\ 3x_1 + 4x_2 \leq 60 \\ x_1, x_2 \geq 0 \end{cases}$$

and in matrix form as:

$$z = \mathbf{c} \cdot \mathbf{x} \rightarrow \mathbf{max}$$

$$\text{s.t.} \quad \begin{aligned} \mathbf{Ax} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 10 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 60 \\ 100 \end{bmatrix}, \quad \mathbf{c} = [200 \quad 50].$$

Example 5: A company makes two products P1 and P2, using two machines, M1 and M2. Each unit of P1 that is produced requires 50 minutes of processing time on machine M1 and 30 minutes of processing time on machine M2. Each unit of P2 is produced, requiring 24 minutes of processing time on machine M1 and 33 minutes of processing time on machine M2.

Machine M1 is going to be available for 40 hours, and machine M2 is available for 35 hours. The profit per unit of P1 is \$25, and the profit per unit of P2 is \$30. Company policy is to determine the production quantity of each product in such a way as to maximize the total profit given so that the available resources will not be exceeded.

Solution: table form of the problem:

	P1 [min]	P2 [min]	Available [hour]
machine M1	50	21	40
machine M2	30	33	35
profit per unit [\$]	25	30	

First step: identify the decision variables

x_1 – the number of units of P1

x_2 – the number of units of P2

Second step: determination of the objective function

The total profit will be:

$$z = f(x_1, x_2) = 25x_1 + 30x_2$$

where \$25 is the profit per each unit of product P1, \$30 is a profit per unit of product P2, the production of product P1 is x_1 units, and the production of product P2 is x_2 units.

Third step: determination of the constraints

the machine M1 constraint: if the production of P1 is x_1 , and P2 is x_2 , the company should use $50x_1 + 24x_2$ minutes of M1 machine work, on the other hand, machine M1 is available for 40 hours $50x_1 + 24x_2 \leq 40$

the machine M2 constraint: if the production of P1 is x_1 and P2 is x_2 the company should use $30x_1 + 33x_2$ minutes of M2 machine work, and M2 is available for 35 hours $30x_1 + 33x_2 \leq 35$, and non-negativity constraint or sign restriction $x_1, x_2 \geq 0$.

In addition, this example requires matching units (they are minutes and hours), so let us change all time units to minutes. Therefore, 40 hours is 2400 minutes, and 35 hours is 2100 minutes.

Finally, this gives us the following mathematical model:

$$z = f(x_1, x_2) = 25x_1 + 30x_2 \rightarrow \max$$

$$\text{s.t.} \begin{cases} 50x_1 + 24x_2 \leq 2400 \\ 30x_1 + 33x_2 \leq 2100 \\ x_1, x_2 \geq 0 \end{cases}$$

and in matrix form as:

$$z = \mathbf{c} \cdot \mathbf{x} \rightarrow \mathbf{max}$$

$$\text{s.t.} \begin{cases} \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

where

$$\mathbf{A} = \begin{bmatrix} 50 & 24 \\ 30 & 33 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2400 \\ 2100 \end{bmatrix}, \quad \mathbf{c} = [25 \ 30].$$

Example 6: Peter is in a hot-dog eating contest that lasts 1 hour. Each hot-dog that he eats takes 2 minutes. Each hot-dog with toppings that he eats takes 3 minutes. He receives 4 points for each hot-dog and 5 points for each hot-dog with toppings. What should Peter eat so as to get the most points?

Solution: table form of the problem:

	Hot-dog	Hot-dog with toppings	Available [min]
time [min]	2	3	60
points	4	5	

First step: the decision variables

x_1 – the number of hot-dogs eaten by Peter

x_2 – the number of hot-dogs with toppings eaten by Peter

Second step: the objective function $z = f(x_1, x_2) = 4x_1 + 5x_2$

Third step: the constraints $2x_1 + 3x_2 \leq 60$

so, the mathematical model is:

$$z = f(x_1, x_2) = 4x_1 + 5x_2 \rightarrow \max$$

$$\text{s.t.} \quad \begin{aligned} 2x_1 + 3x_2 &\leq 60 \\ x_1, x_2 &\geq 0 \end{aligned}$$

and in matrix form as: $z = \mathbf{c} \cdot \mathbf{x} \rightarrow \mathbf{max}$

$$\text{s.t.} \quad \begin{aligned} \mathbf{Ax} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

where

$$\mathbf{A} = [2 \ 3], \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = [60], \quad \mathbf{c} = [4 \ 5].$$

One of the most real problems solved by using linear programming is the so-called “**diet problem**”. It appears in the choice of healthy food. The problem of

the diet is to provide minimum daily nutritional demands by defining the mix of foods in the diet that minimizes total cost per day. It is an example of a linear programming problem, in which the objective function is to be minimized, and the constraints sign is “ \geq ”.

Define:

m – number of nutrients

n – number of sorts of food

a_{ij} – number of units of nutrient i in food j ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$)

b_i – number of units of nutrient i required per day ($i = 1, 2, \dots, m$)

c_j – cost per day of food j ($j = 1, 2, \dots, n$)

x_j – number of units of food j in the diet per day ($j = 1, 2, \dots, n$)

The goal is to find the values of the unknown variables to minimize the total cost per day. Therefore, the mathematical model for diet problem is:

$$z = f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots + c_nx_n \rightarrow \min$$

$$\text{s.t.} \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m \\ x_1, x_2, \dots, x_n \geq 0 \end{cases}$$

Example 7: Alina’s diet consists of four basic food groups: a chocolate muffin, strawberry ice cream, beetroot juice and cheesecake. Each chocolate muffin costs \$0.5, each scoop of strawberry ice cream costs \$0.2, each glass of beetroot juice costs \$0.3, and each piece of cheesecake costs \$0.8. Each day, she must take in at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The content of nutrients in each of the food types (ounces per unit) is given in the table below. Formulate a mathematical model that can allow her to satisfy her daily nutritional requirements at minimum cost.

	Calories	Chocolate [ounces]	Sugar [ounces]	Fat [ounces]
Chocolate muffin	400	3	2	2
Strawberry ice cream [scoop]	200	2	2	4
Beetroot juice [1 glass]	150	0	4	1
Cheesecake [1 piece]	500	0	4	5

First step: the decision variables

x_1 – the number of chocolate muffins eaten daily

x_2 – the number of scoops of strawberry ice cream eaten daily

x_3 – glasses of beetroot juice drunk daily

x_4 – pieces of cheesecake eaten daily

Second step: the objective function

$$z = 0.5x_1 + 0.2x_2 + 0.3x_3 + 0.8x_4$$

Third step:

the calorie constraints $400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500$

the chocolate constraint $3x_1 + 2x_2 \geq 6$

the sugar constraint $2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10$

the fat constraint $2x_1 + 4x_2 + x_3 + 5x_4 \geq 8$

non-negativity constraint $x_1, x_2, x_3, x_4 \geq 0$

so, the mathematical model is:

$$z = 0.5x_1 + 0.2x_2 + 0.3x_3 + 0.8x_4 \rightarrow \min$$

$$\text{s.t.} \begin{cases} 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \\ 3x_1 + 2x_2 \geq 6 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \\ 2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

and in matrix form as:

$$z = \mathbf{c} \cdot \mathbf{x} \rightarrow \mathbf{max}$$

$$\text{s.t.} \begin{cases} \mathbf{Ax} \geq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

where

$$\mathbf{A} = \begin{bmatrix} 400 & 200 & 150 & 500 \\ 3 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 \\ 2 & 4 & 1 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 500 \\ 6 \\ 10 \\ 8 \end{bmatrix}, \quad \mathbf{c} = [0.5 \quad 0.2 \quad 0.3 \quad 0.8]$$

2.3. Graphical solutions to linear problems

Linear programming problems with two or three decision variables can be solved using the graphical method. It is enough to draw a set of feasible solutions and then choose the optimal solution among the corner points.

The graphical solution to linear problems is best explained by using an example. Let us recall **Example 4**:

$$z = f(x_1, x_2) = 200x_1 + 50x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 10x_1 + 5x_2 \leq 100 \\ 3x_1 + 4x_2 \leq 60 \\ x_1, x_2 \geq 0 \end{cases}$$

To find the optimal solution to this problem, we must first find the **feasible set** that is a set of all points $(x_1, x_2) \in \mathbb{R}^2$ which satisfies the constraints. For this, we have to graph each of inequalities separately and the final feasible set will be the intersection of all these sets.

We begin with a system of coordinates with x_1 on the horizontal axis, and x_2 on the vertical axis. Since both x_1 and x_2 are constrained to be non-negative, we are only interested in the first quadrant. Let us start with the first constraint $10x_1 + 5x_2 \leq 100$. Geometrically, it is a half-space with the border given by this constraint which is the straight line $10x_1 + 5x_2 = 100$. There is a simple procedure for plotting the line by simply finding the points where it intersects the axes, the so-called “intercept”, and drawing a straight line through the points. To sketch the graph of straight line, two points are enough. The point where the line intersects the horizontal axis (x – intercept) has $x_2 = 0$, so $x_1 = 10$. The point where the line intersects the vertical axis (y – intercept) has $x_1 = 0$, so $x_2 = 20$. This line divides the plane into two regions, the upper and the lower one. The way to test whether the feasible area is below or above the line is to take a point and see whether the constraint is satisfied. Similarly, the second constraint gives us the constraint area for labor. The areas representing the constraints is shown in Figure 2.1.

We want the constraints to be satisfied simultaneously, so they need to be represented in the same graph. Figure 2.2 shows the feasible area for the model from **Example 4**. One or more of the points in the feasible area will result in the max profit. Therefore, the next step is finding that point, or points.

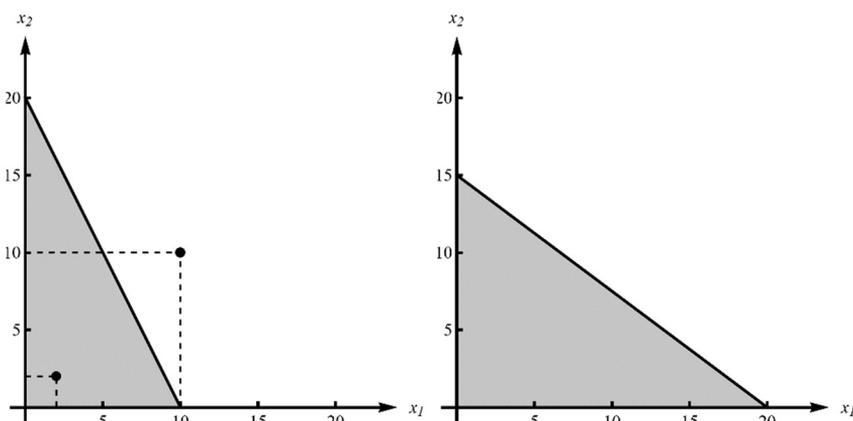


Figure 2.1. First and second constraint for “Patio Accessories”

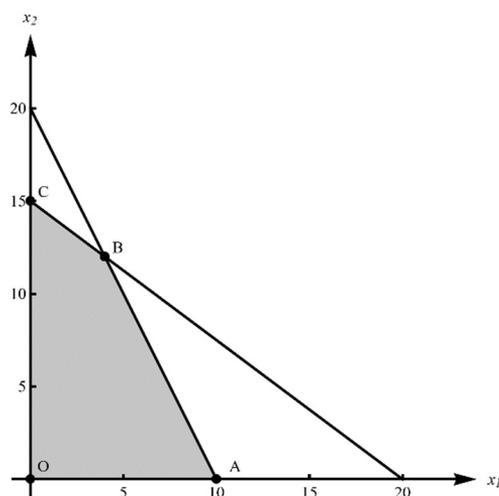


Figure 2.2. Feasible region for “Patio Accessories”

According to Theorem 1, the optimal solution is one of the corner points of the feasible region that gives the largest value of the objective function (the criterion is maximum). In **Example 4**, the feasible region has four corner points denoted by $O(0,0)$, $A(10,0)$, $B(4,12)$, $C(0,15)$, which are located at the intersection of the constraint boundary lines. Examining each corner point separately and finding the optimal solution is not too difficult since there are only four corner points in the problem under study (two decision variables and two constraints):

$$z_0 = f(0,0) = 0,$$

$$z_A = f(10,0) = 200 \cdot 10 + 50 \cdot 0 = 2000,$$

$$z_B = f(4,12) = 200 \cdot 4 + 50 \cdot 12 = 1400,$$

$$z_C = f(0,15) = 200 \cdot 0 + 50 \cdot 15 = 750.$$

The largest value is for point B with coordinates $(x_1, x_2) = (10, 0)$, so the optimum is reached at this point, and the value of the objective function is $z = 2000$.

In problems with more constraints, the number of extreme points can be quite large, and calculating the extreme points and the objective function values for these points can be inconvenient. In those cases, it would be preferable to use a visual method by which we also need to graphically represent the objective function.

The objective function is a linear function of two variables so that its contours form parallel lines referred to as the “**isoprofit lines**” – for maximization problems or “**isocost lines**” for minimization problems. The isoprofit line is shown as a dashed line in Figure 2.3.

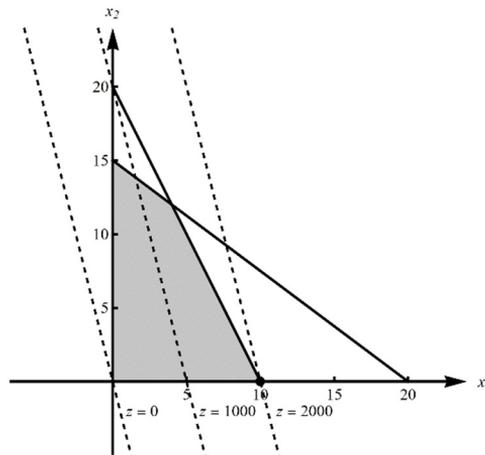


Figure 2.3. Feasible region with isoprofit lines for $z = 0, z = 1000, z = 2000$

To determine the optimal solution, the isoprofit line is moved parallel in the direction that increases value of the objective function until the last point intersecting the feasible region is obtained.

In this example, the optimum point is reached as single point $x_1 = 10, x_2 = 0$, and the value of the objective function is $z = 2000$. This means that “Patio Accessories” will need to produce all tables (10 of them), and no chairs in order to maximize profits. The profit in this case is $z = 2000$.

Example 8: Graphically solve the following problem

$$z = f(x_1, x_2) = -x_1 + 4x_2 \rightarrow \min$$

$$\text{s.t.} \begin{cases} x_1 + 3x_2 \geq 5 \\ x_1 + x_2 \geq 4 \\ x_1 - x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{cases}$$

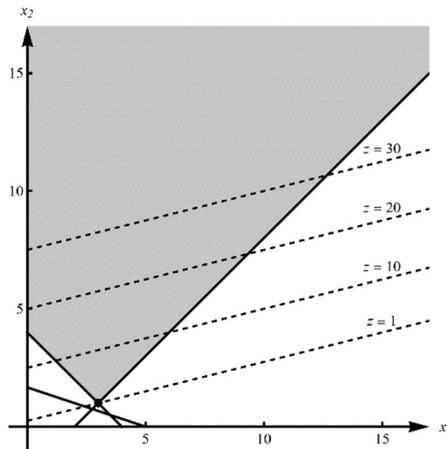


Figure 2.4. Feasible region with isocost lines for $z = 1, z = 10, z = 20, z = 30$

The objective function line (in minimization problem called “isocost”) minimizes its value by moving parallel in the down-right direction. The optimal solution is the intersection point of two constraint lines, the second and the third, with coordinates: $x_1 = 3, x_2 = 1$, and the minimum value of the objective function $z = f(3,1) = -1 \cdot 3 + 4 \cdot 1 = 1$ (Fig. 2.4).

Example 9: Graphically solve the following problem

$$z = f(x_1, x_2) = 6x_1 + 4x_2 \rightarrow \max$$

$$\text{s.t.} \begin{cases} 3x_1 + 2x_2 \leq 18 \\ x_1 \leq 4 \\ x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{cases}$$

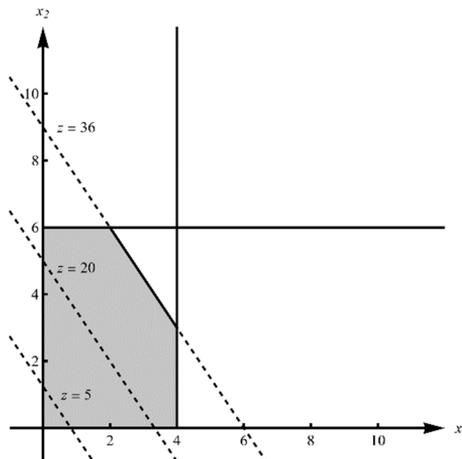


Figure 2.5. Feasible region with isoprofit lines for $z = 5, z = 20, z = 36$ – alternative optimal solution

The isoprofit line (maximization problem) is parallel to the entire line segment joining the corner points $x_1 = 4, x_2 = 3$ and $x_1 = 2, x_2 = 6$. This means that any point on this line segment is an optimal solution. The problem has a so-called “alternative optimal solution” that belongs to the segment line $t(4,3) + (1-t)(2,6) = (2+2t, 6-3t)$; $0 \leq t \leq 1$, and gives the same value of the objective function $z = 36$ (Fig. 2.5).

Example 10: Graphically solve the following problem

$$z = f(x_1, x_2) = -2x_1 + 3x_2 \rightarrow \max$$

$$\text{s.t.} \begin{cases} 4x_1 + 6x_2 \geq 24 \\ x_1 \leq 6 \\ x_1, x_2 \geq 0 \end{cases}$$

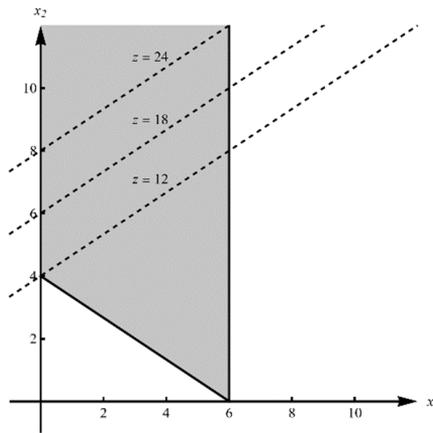


Figure 2.6. Feasible region with isoprofit line for $z = 12, z = 18, z = 24$ – unbounded problem

Moving parallel to the isoprofit line in the direction of an increasing value of the objective function, we see that any isoprofit line we graph will never lose contact with the feasible region. Thus, there are points in the feasible area that have an arbitrary large value of the objective function. Then the linear programming problem is unbounded (Fig. 2.6).

In summary, the steps of the graphical method used to solve linear programming problems are as follows:

1. Draw all the model constraints as equations on the graph.
2. Determine the feasible region.
3. Represent the objective function graphically, then move this line parallel in the direction of the increasing value of the objective function to find the optimum.

An optimal solution to the LP is the last point in the feasible region that contacts an objective function line.

4. Find the coordinates of the optimum point by solving a system of two equations with two unknown variables.

2.4. The Simplex Method

Considered to be the classical approach for solving linear problems, the simplex method was developed in 1947 by George Dantzing, who used it to solve programming problems for the Air Force. It is a theoretically and practically efficient technique that enables us to solve any linear problems relatively easily.

2.4.1. Introduction to the Simplex Method – the algebraic approach

It is known from previous considerations that a feasible region of the linear programming problem has “corner points” or “extreme points”, one of which is the optimal solution. The simplex method is a linear, algebra – based method that graphically corresponds to starting at one extreme point on the boundary of the feasible region, typically (0,0), and moving to neighboring (adjacent) extreme points through the iterative process until an optimal solution is obtained.

Before the simplex method can be applied, the linear program must be converted into “standard form”.

Definition 1. The linear programming problem is said to be in **standard form** if a linear model fulfills the following condition: all constraints are equations with a non-negative right-hand side and all variables are nonnegative.

The process of converting a linear program to its standard form requires a few steps: multiplying by (-1) both sides of the constraint (note that the multiplying by (-1) changes the sign of the inequality to the opposite sign), if the right-hand side of a constraint is negative, and adding a so-called “slack variable” to the left-hand side of “ \leq ” constraint.

Definition 2. A **slack variable** s_i is an additional variable that represents the amount not used in the i -th “ \leq ” constraint.

It is usually assumed that the slack variables make no contribution to the objective function; they appear in the objective function with zero coefficients.

Example 11: Convert the given LP to its standard form:

$$z = f(x_1, x_2) = 120x_1 + 160x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 \leq 10 \\ -3x_2 \geq -11 \\ x_1 + x_2 \leq 5 \end{cases} \quad x_1, x_2 \geq 0$$

Solution: first multiply by (-1) the second constraint and then add the appropriate slack variable. Finally, the standard form of given LP problem is

$$z = f(x_1, x_2) = 120x_1 + 160x_2 + 0s_1 + 0s_2 + 0s_3 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 + s_1 = 10 \\ 3x_2 + s_2 = 11 \\ x_1 + x_2 + s_3 = 5 \end{cases} \quad x_1, x_2, s_1, s_2, s_3 \geq 0$$

In matrix notation, a linear programming problem in standard form can be written as:

$$z = \mathbf{c} \cdot \mathbf{x} \rightarrow \mathbf{max}$$

$$\text{s.t. } [\mathbf{A} | \mathbf{I}_m] \cdot \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}; \quad \mathbf{x}, \mathbf{s} \geq 0$$

where \mathbf{I}_m denotes the “ m by m ” identity matrix, and

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \\ s_1 \\ \dots \\ s_m \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}, \quad \mathbf{c} = [c_1 \quad c_2 \quad \dots \quad c_n]$$

then the constraints of the standard form of an LP can be simply represented by a system of simultaneous equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, involving m equations and $r = m + n$ unknown variables (the vector \mathbf{x} contains the original set of decision variables and a new collection of variables – slack variables introduced to convert the inequalities to equalities).

Thus, the optimal solution to the optimization problem must be a solution to this set of linear equations. But, since $m < r$, this system will have an infinite number

of solutions. However, in the simplex method, only a finite number of these solutions will be of interest; it is the so-called “**basic solutions set**”.

Definition 3. The **basis matrix** is any m by m nonsingular submatrix of \mathbf{A} , selected for a given system of equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.

Definition 4. A **basic solution** \mathbf{x}_B to the system of m linear equations with r unknowns is found by setting to zero $r - m$ variables (referred to as the **non-basic variables**) and solving the system of m equations for remaining m variables (referred to as the **basic variables**).

Definition 5. A **basic feasible solution** is a basic solution where all basic variables are nonnegative. The **optimal solution** will be one of the basic feasible solutions with the best value of the objective function (the largest for the maximization problem, and the smallest for the minimization problem).

The number of basic solutions is limited by the formula:

$$\frac{r!}{m!(r-m)!}$$

Denoting the basis matrix by \mathbf{B} , the resulting system after setting $r - m$ variables to zero may be written in matrix form as: $\mathbf{B} \cdot \mathbf{x}_B = \mathbf{b}$ or $\mathbf{x}_B = \mathbf{B}^{-1} \cdot \mathbf{b}$.

Example 12: Determine all basic solutions and basic feasible solutions for the linear problem in **Example 11**.

Let us start the solution with the standard form of the problem:

$$z = f(x_1, x_2) = 120x_1 + 160x_2 + 0s_1 + 0s_2 + 0s_3 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 + s_1 = 10 \\ 3x_2 + s_2 = 11 \\ x_1 + x_2 + s_3 = 5 \end{cases}$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

the coefficients matrix for this system of equations is as follows:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}; \quad r = 5; \quad m = 3$$

There are 10 possible basic solutions:

$$\frac{r!}{m!(r-m)!} = \frac{5!}{3! \cdot (5-3)!} = \frac{5!}{3! \cdot 2!} = \frac{4 \cdot 5}{2 \cdot 1} = \frac{20}{2} = 10.$$

The basic solution is determined by setting to zero $r - m = 2$ variables and solving the system of $m = 3$ equations for the remaining m variables.

The first possible basis can be for example:

$$\mathbf{B}_1 = [a_1 \ a_2 \ a_3] = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix};$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 1 & 0 \end{vmatrix} = -3 \neq 0;$$

Meaning that \mathbf{B}_1 is a nonsingular matrix, so \mathbf{B}_1 is a basis matrix, and $x_1, x_2, s_1 \Rightarrow$ are basic variables, $s_2 = s_3 = 0 \Rightarrow$ are non-basic variables.

For this basis matrix the basic variable can be obtained by using the formula $\mathbf{x}_B = \mathbf{B}^{-1} \cdot \mathbf{b}$

$$\mathbf{B}_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{B}_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & 0 \\ 1 & \frac{2}{3} & -2 \end{bmatrix},$$

and

$$\mathbf{x}_{B_1} = \mathbf{B}_1^{-1} \cdot \mathbf{b} = \begin{bmatrix} 0 & \frac{-1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \\ 1 & \frac{2}{3} & -2 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 11 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{11}{3} \\ \frac{22}{3} \end{bmatrix}$$

or by solving the system of equations:

$$\begin{cases} 2x_1 + s_1 = 10 \\ 3x_2 = 11 \\ x_1 + x_2 = 5 \end{cases} \Rightarrow s_1 = \frac{23}{3}; x_1 = \frac{4}{3}; x_2 = \frac{11}{3}$$

and the basic feasible solution is:

$$\mathbf{x}_{\mathbf{B}_1}^T = \left[\frac{4}{3} \quad \frac{11}{3} \quad \frac{8}{3} \quad 0 \quad 0 \right]; \quad z(\mathbf{x}_{\mathbf{B}_1}) = \frac{2240}{3}$$

the next possible basis is:

$$\mathbf{B}_2 = [a_1 \quad a_2 \quad a_4] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -2 \neq 0 \Rightarrow \mathbf{B}_2 \text{ - non-singular, is a basis matrix}$$

$x_1, x_2, s_2 \Rightarrow$ basic variable

$s_1 = s_3 = 0 \Rightarrow$ non-basic variable

for this basis matrix the basic variables are obtained as follows:

$$\begin{cases} 2x_1 = 10 \\ 3x_2 + s_2 = 11 \\ x_1 + x_2 = 5 \end{cases} \Rightarrow s_2 = 11; x_1 = 5; x_2 = 0$$

and the basic feasible solution is:

$$\mathbf{x}_{\mathbf{B}_2}^T = [5 \quad 0 \quad 0 \quad 11 \quad 0]; \quad z(\mathbf{x}_{\mathbf{B}_2}) = 600$$

$$\mathbf{B}_3 = [a_1 \quad a_2 \quad a_5] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 6 \neq 0 \Rightarrow \mathbf{B}_3 \text{ - non-singular, is a basis matrix}$$

$x_1, x_2, s_3 \Rightarrow$ basic variable

$s_1 = s_2 = 0 \Rightarrow$ non-basic variable

For this basis matrix the basic variables are obtained as follows:

$$\begin{cases} 2x_1 = 10 \\ 3x_2 = 11 \\ x_1 + x_2 + s_3 = 5 \end{cases} \Rightarrow x_1 = 5; x_2 = \frac{11}{3}; s_3 = \frac{-11}{3}$$

$\mathbf{x}_{B_3} = \left[5 \quad \frac{11}{3} \quad 0 \quad 0 \quad \frac{-11}{3} \right]$, infeasible basic solution, because of the negative value

$$\mathbf{B}_4 = [a_1 \quad a_3 \quad a_4] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow \mathbf{B}_4 - \text{non-singular, is a basis matrix}$$

$x_1, s_1, s_2 \Rightarrow$ basic variable

$x_2 = s_3 = 0 \Rightarrow$ non-basic variable

For this basis matrix the basic variables are obtained as follows:

$$\begin{cases} 2x_1 + s_1 = 10 \\ s_2 = 11 \\ x_1 = 5 \end{cases} \Rightarrow x_1 = 5; s_1 = 0; s_2 = 11$$

and the basic feasible solution is:

$$\mathbf{x}_{B_4}^T = [5 \quad 0 \quad 0 \quad 11 \quad 0]; \quad z(\mathbf{x}_{B_4}) = 600$$

$$\mathbf{B}_5 = [a_1 \quad a_3 \quad a_5] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0 \Rightarrow \mathbf{B}_5 - \text{singular, is not a basis matrix}$$

$$\mathbf{B}_6 = [a_1 \quad a_4 \quad a_5] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2 \neq 0 \Rightarrow \mathbf{B}_6 - \text{non-singular, is a basis matrix}$$

$x_1, s_2, s_3 \Rightarrow$ basic variable

$x_2 = s_1 = 0 \Rightarrow$ non-basic variable

For this basis matrix the basic variables are obtained as follows:

$$\begin{cases} 2x_1 = 10 \\ s_2 = 11 \\ x_1 + s_3 = 5 \end{cases} \Rightarrow x_1 = 5; s_2 = 11; s_3 = 0$$

and the basic feasible solution is:

$$\mathbf{x}_{\mathbf{B}_6}^T = [5 \ 0 \ 0 \ 11 \ 0]; \quad z(\mathbf{x}_{\mathbf{B}_6}) = 600$$

$$\mathbf{B}_7 = [a_2 \ a_3 \ a_4] = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow \mathbf{B}_7 - \text{non-singular, is a basis matrix}$$

$x_2, s_1, s_2 \Rightarrow$ basic variable

$x_1 = s_3 = 0 \Rightarrow$ non-basic variable

For this basis matrix the basic variables are obtained as follows:

$$\begin{cases} s_1 = 10 \\ 3x_2 + s_2 = 11 \\ x_1 + x_2 = 5 \end{cases} \Rightarrow s_1 = 10; x_2 = 5; s_2 = -4$$

$$\mathbf{x}_{\mathbf{B}_7}^T = [0 \ 5 \ 10 \ -4 \ 0]; \text{ basic, but not basic feasible solution}$$

$$\mathbf{B}_8 = [a_2 \ a_3 \ a_5] = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -3 \neq 0 \Rightarrow \mathbf{B}_8 - \text{non-singular, is a basis matrix}$$

$$x_2, s_1, s_3 \Rightarrow \text{basic variable}$$

$$x_1 = s_2 = 0 \Rightarrow \text{non-basic variable}$$

For this basis matrix the basic variables are obtained as follows:

$$\begin{cases} s_1 = 10 \\ 3x_2 = 11 \\ x_1 + s_3 = 5 \end{cases} \Rightarrow s_1 = 10; x_2 = \frac{11}{3}; s_3 = \frac{4}{3}$$

and the basic feasible solution is:

$$\mathbf{x}_{\mathbf{B}_8}^T = \left[0 \quad \frac{11}{3} \quad 10 \quad 0 \quad \frac{4}{3} \right]; \quad z(\mathbf{x}_{\mathbf{B}_8}) = \frac{1760}{3}$$

$$\mathbf{B}_9 = [a_2 \quad a_4 \quad a_5] = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0 \Rightarrow \mathbf{B}_9 - \text{singular, is not a basis matrix}$$

and the last one

$$\mathbf{B}_{10} = [a_3 \quad a_4 \quad a_5] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \mathbf{B}_{10} - \text{non-singular, is a basis matrix}$$

$$s_1, s_2, s_3 \Rightarrow \text{basic variable}$$

$$x_1 = x_2 = 0 \Rightarrow \text{non-basic variable}$$

For this basis matrix the basic variables are obtained as follows:

$$\begin{cases} s_1 = 10 \\ s_2 = 11 \\ s_3 = 5 \end{cases} \Rightarrow s_1 = 10; s_2 = 11; s_3 = 5$$

and the basic feasible solution is:

$$\mathbf{x}_{\mathbf{B}_{10}}^T = [0 \ 0 \ 10 \ 11 \ 5]; \quad z(\mathbf{x}_{\mathbf{B}_{10}}) = 0.$$

Finally, the eight basic feasible solutions were obtained.

The optimal solution is the basic feasible solution with the best value of the objective function, so in this example it is the following solution

$$\mathbf{x}_{opt}^T = \left[\frac{4}{3} \ \frac{11}{3} \ \frac{22}{3} \ 0 \ 0 \right] \text{ with } z(\mathbf{x}_{opt}) = \frac{2240}{3}.$$

2.4.2. The Simplex Algorithm – step by step

Up to now, in the search for the optimal solution, all basic solutions were analyzed, and the optimal solution was obtained by selecting the point with the best value of the objective function, the largest for the maximization problem and the smallest for the minimization problem. The simplex algorithm examines only a few basic solutions, selected according to the rule defined in this method. It always starts with basic feasible solutions and moves through other basic feasible solutions that successively improve the value of the objective function. The algorithm terminates once the optimal value is reached. This procedure is best executed with the help of simply tableau with the most important information about the analyzed linear program. Sometimes this tableau is referred to as “simplex tableau”.

Example 13: A carpentry company builds and sells wooden tables. They produce two types of tables using a combination of three types of wood, acacia, chestnut and larch. To construct the type 1 table, the worker requires 2 board foot (bf) acacia and 1 bf chestnut. To construct the type 2 table, they require 3 bf of larch and 1 bf of chestnut. Given that they have 10 bf of acacia, 5 bf of chestnut and 11 bf of larch and they can sell type 1 of table for \$120 and type 2 table for \$160, how many of each table type should they make to maximize revenue?

Solution: let us start by gathering all the information in table form:

	table type 1	table type 2	Available
acacia	2	0	10
chestnut	1	1	5
larch	0	3	11
Profits [\$]	120	160	

next, formulate the mathematical model:

determination of the decision variables:

let us denote by x – number of type 1 tables to be built
 y – number of type 2 tables to be built

determination of the objective function:

the revenues are a function of the number of tables of each type sold, the objective function will take the following form:

$$z = f(x, y) = 120x + 160y$$

determination of the constraints:

the acacia constraint: if the production of type 1 tables is x , and type 2 tables is y , the woodworker should use $3x$ units of acacia wood, so the first constraint is $3x \leq 10$

the chestnut constraint: if the production of type 1 tables is x , and type 2 tables is y , the woodworker should use $3y$ units of chestnut wood, so the second constraint is $3y \leq 11$

the larch constraint: if the production of type 1 tables is x , and type 2 tables is y , the woodworker should use $x + y$ units of larch wood, so the third constraint is $x + y \leq 5$

and sign restriction $x, y \geq 0$.

Finally, the mathematical model of the problem is:

$$z = f(x, y) = 120x + 160y \rightarrow \max$$

$$\text{s.t.} \begin{cases} 2x \leq 10 \\ 3y \leq 11 \\ x + y \leq 5 \\ x, y \geq 0 \end{cases}$$

and in matrix form as:

$$z = \mathbf{c} \cdot \mathbf{x} \rightarrow \max$$

$$\text{s.t.} \begin{cases} \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 11 \\ 5 \end{bmatrix}, \quad \mathbf{c} = [120 \quad 160]$$

Step 1. Convert the linear program into standard form. The linear program in standard form is:

$$z = f(x, y) = 120x + 160y + 0s_1 + 0s_2 + 0s_3 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x + s_1 = 10 \\ 3y + s_2 = 11 \\ x + y + s_3 = 5 \end{cases}$$

$$x, y, s_1, s_2, s_3 \geq 0$$

The initial simplex tableau should fulfill the following conditions: one basic variable per row, the coefficients of all basic variables are +1, and the coefficients above and below the basic variables are zero, the objective function should be written in the form where all variables are on the left-hand side $z - 120x - 160y = 0$. It will be the first row of the table (also called **row 0**). If the table is in proper form, then the solution can be read directly by looking at the table and the right-hand side value.

To determine whether a basic feasible solution is optimal, we need to determine if any of the non-basic variables (who have a value of zero) can be increased to improve the value of the objective function.

Theorem 1. In a maximization problem the basic feasible solution is optimal if all coefficients of non-basic variable in row 0 in initial simplex tableau are more than or equal to zero – it is known as the **optimality condition**. This means that there are (in row 0) no negative signs, indicating that the maximum value of the objective function had been found.

In a minimization problem, the optimality condition is fulfilled when none of the coefficients in row 0 are positive. Furthermore, in a minimization problem, the entering variable is chosen to be the non-basic variable with the highest positive coefficient in row 0.

Definition 1. The non-basic variable with the most negative coefficient in row 0 is called the **entering variable** and is always selected as the non-basic variable that becomes a basic variable. The entering variable column is called the **pivot column** (see Chapter 1.2).

Definition 2. The **leaving variable** is determined by using a “**ratio test**”. The “ratio” values are simply the ratio of the right-hand side values divided by the element in the pivot column (entering variable column) of the same row, but only by positive coefficients. A negative and zero are not important for the current value of the

objective function. The leaving variable is selected to be the basic variable in the row with the smallest ratio. This row is called the **pivot row**. The intersection of the pivot column and pivot row is called the **pivot element**.

In other words, the smallest value of ratio is the largest value of the entering variable that will keep all current basic variables nonnegative.

In the next step, just use **elementary row operations** to make the entering variable a basic variable, (coefficients 1 in that row and 0 in all other rows), in the row of the leaving variable. In other words, the pivot column should contain 1 in the pivot element and 0 in every other constraint row. These operations result in a new, basic feasible solution with the better value of the objective function.

The new tableau shows a new feasible solution. After that, the algorithm returns to step 3 to check if the new basic feasible solution is optimal. These steps are repeated until the objective function value cannot be improved by increasing the value of any of the non-basic variables.

Step 2. Obtain the basic feasible solution from the standard form (decision variable – non-basic variables – equal to zero – Table 2.1).

Table 2.1. The initial simplex tableau

Basic	z	x	y	s_1	s_2	s_3	RHS	ratio
z	1	-120	-160	0	0	0	0	
s_1	0	2	0	1	0	0	10	none
s_2	0	0	3	0	1	0	11	$\frac{11}{3}$
s_3	0	1	1	0	0	1	5	5

the actual basic feasible solution is $\mathbf{x}_1^T = [0 \ 0 \ 10 \ 11 \ 5]$.

Step 3. Determine if the basic feasible solution is optimal (from Table 2.1, sign of the coefficients in row 0).

Step 4. The current solution is not optimal. There are negative coefficients in row 0. Since y has the most negative coefficient in row 0, and s_2 has the lowest ratio, the entering and leaving variables are y and s_2 , respectively.

Step 5. Use elementary row operations to solve for the new basic feasible solution. Return to step 3. The new basic feasible solution is shown in Table 2.2.

Table 2.2. The tableau for the new basic feasible solution in the first iteration

Basic	z	x	y	s_1	s_2	s_3	RHS	ratio
z	1	-120	0	0	$160/3$	0	$1760/3$	
s_1	0	2	0	1	0	0	10	5
y	0	0	1	0	$1/3$	0	$11/3$	none
s_3	0	1	0	0	$-1/3$	1	$4/3$	$4/3$

The actual basic feasible solution is $\mathbf{x}_2^T = \left[0 \quad 11/3 \quad 10 \quad 0 \quad 4/3 \right]$, with the objective function value $1760/3$.

The new basic feasible solution is still not optimal. There is a negative coefficient in row 0. The objective function value can be increased by increasing the value of x . Since x has the most negative coefficient in row 0 and s_3 has the lowest ratio, the entering and the leaving variables are x and s_3 , respectively.

Step 5. Use elementary row operations to solve for the new basic feasible solution. Return to step 3. The new basic feasible solution is shown in Table 2.3.

Step 3. Since there are no negative coefficients in row 0, we have reached the optimal solution where the objective function value is $2240/3$, and **the optimal solution** is $\mathbf{x}_{opt}^T = \left[4/3 \quad 11/3 \quad 22/3 \quad 0 \quad 0 \right]$.

Table 2.3. The tableau for the new basic feasible solution in the second iteration

Basic	z	x	y	s_1	s_2	s_3	RHS	ratio
z	1	0	0	0	$40/3$	120	$2240/3$	
s_1	0	0	0	1	$2/3$	-2	$22/3$	
y	0	0	1	0	$1/3$	0	$11/3$	
x	0	1	0	0	$-1/3$	1	$4/3$	

Finally, the carpentry company should build $4/3$ type 1 tables and $11/3$ type 2 tables to maximize the revenue to \$746,67.

To summarize, the simplex algorithm can be described by the following steps:

1. Convert the linear program to its standard form.
2. Obtain a basic feasible solution from the standard form (an initial basic feasible solution can be found by setting all decision variables equal to zero).
3. Determine whether the basic feasible solution is optimal by using the **optimality condition**.
4. If the current basic feasible solution is not optimal, select a non-basic variable that should become a basic variable (it is the so-called **entering variable**), and basic variable which should become a non-basic variable (called the **leaving variable**) to determine a new basic feasible solution with an improved objective function value.
5. Use elementary row operations to solve for the new basic feasible solution. Return to step 3.

To solve the minimization problems, the entering variable condition and optimality condition should be modified. Instead of choosing the entering variable on the basis of the most negative coefficient in row 0 of the tableau, select the entering variable on the basis of the **most positive coefficient**. Then terminate the algorithm when entries in row 0 of the tableau corresponding to non-basic variables are all non-positive.

The second way to solve the minimization problem is to transform the problem as a maximization problem by just multiplying the objective function by (-1) and keeping each constraint unchanged. Now solve the LP using the simplex algorithm as we did previously.

Let us consider the following examples:

Example 14:

$$z = 2x - 3y \rightarrow \min$$

$$\text{s.t. } \begin{cases} x + y \leq 4 \\ x - y \leq 6 \end{cases}$$

$$x, y \geq 0$$

Solution

First transform to the maximization problem

$$w = -z = -2x + 3y \rightarrow \max$$

$$\text{s.t. } \begin{cases} x + y \leq 4 \\ x - y \leq 6 \end{cases}$$

$$x, y \geq 0$$

next, the standard form of LP

$$w = -z = -2x + 3y + 0s_1 + 0s_2 \rightarrow \max$$

$$\text{s.t.} \quad \begin{cases} x + y + s_1 = 4 \\ x - y + s_2 = 6 \end{cases}$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The initial simplex tableau is:

Basic	z	x	y	s_1	s_2	RHS	ratio
w	1	2	-3	0	0	0	
s_1	0	1	1	1	0	4	4
s_2	0	1	-1	0	1	6	-

the basic variables are $s_1 = 4$, $s_2 = 6$, non-basic variable are x , y . Since the coefficient of variable y is the most negative non-basic variable coefficient in the top row, y is the entering variable. The first constraint row yields the smallest ratio of 4, so s_1 becomes a non-basic variable and is replaced by s_1 . The new basic feasible solution after first iteration is shown in the following table:

Basic	z	x	y	s_1	s_2	RHS	ratio
w	1	5	0	3	0	12	
y	0	1	1	1	0	4	
s_2	0	2	0	1	1	10	-

the basic variables are $y = 4$, $s_2 = 10$, non-basic variables are x , s_1 , and $w = 12$. Since all non-basic variable coefficients in the row 0 of the tableau are positive, this solution $\mathbf{x}_{opt}^T = [0 \ 4 \ 0 \ 10]$ is optimal with an optimal value of the objective function $w = -z = -12$.

Example 15: Solve the given minimization problem by using the modified simplex algorithm:

$$z = -5x_1 + 2x_2 \rightarrow \min$$

$$\text{s.t. } \begin{cases} x_1 \leq 2 \\ 3x_1 + 2x_2 \leq 6 \\ 8x_1 + 3x_2 \leq 12 \end{cases}$$

$$x_1, x_2 \geq 0$$

Solution

the standard form of LP is the following

$$z = -5x_1 + 2x_2 \rightarrow \min$$

$$\text{s.t. } \begin{cases} x_1 + s_1 = 2 \\ 3x_1 + 2x_2 + s_2 = 6 \\ 8x_1 + 3x_2 + s_3 = 12 \end{cases}$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

and the initial tableau:

Basic	z	x_1	x_2	s_1	s_2	s_3	RHS	ratio
z	1	5	-2	0	0	0	0	
s_1	0	1	0	1	0	0	2	2
s_2	0	3	2	0	1	0	6	2
s_3	0	8	3	0	0	1	12	$\frac{3}{2}$

the basic variables are $s_1=2$, $s_2=6$, $s_3=12$, non-basic variables are x_1, x_2 . Since 5 is the most positive non-basic variable coefficient in the top row, x_1 is the entering variable. The third constraint row yields the smallest ratio of $\frac{3}{2}$, so s_3 becomes non-basic variable and is replaced by x_1 . The new basic feasible solution after first iteration is shown in the following table:

Basic	z	x_1	x_2	s_1	s_2	s_3	RHS	ratio
z	1	0	$-\frac{31}{8}$	0	0	$-\frac{5}{8}$	$-\frac{15}{2}$	
s_1	0	0	$-\frac{3}{8}$	1	0	$-\frac{1}{8}$	$\frac{1}{2}$	
s_2	0	0	$\frac{7}{8}$	0	1	$-\frac{3}{8}$	$\frac{3}{2}$	
x_1	0	1	$\frac{3}{8}$	0	0	$\frac{1}{8}$	$\frac{3}{2}$	

the basic variables are $x_1 = 3/2$, $s_1 = 1/2$, $s_2 = 3/2$, non-basic variables are x_2 , s_3 and $z = -15/2$. Since all non-basic variable coefficients in row 0 of the tableau are non-positive, this solution $\mathbf{x}_{opt}^T = [3/2 \ 0 \ 1/2 \ 3/2 \ 0]$ is optimal with the value of the objective function $z = -15/2$.

2.4.3. The Big M -method

The more general form of a linear programming problem is the problem with a different kind of constraint: “ \leq ”, “ \geq ” or “ $=$ ”. Some modifications of the simplex algorithm referred as “Big M -method” may be used to solve a problem like this.

The Big M -method steps are following:

1. Modify the constraints so that the right – hand side of each constraint is nonnegative. After modification, identify each constraint as a “ \leq ”, “ \geq ” or “ $=$ ” constrain.
2. Convert each inequality constraint to standard form – if constraint i is a “ \leq ” constraint, we add a slack variable s_i ; if constraint i is a “ \geq ” constraint, we subtract an excess (also called surplus variable) variable e_i (add slack and excess variable to the objective function with coefficients equal zero).
3. Add an “**artificial variable**” a_i to the constraints identified as “ \geq ” or “ $=$ ” constraints at the end of step 1. Also add the sign restriction $a_i \geq 0$.
4. Let M denote a very large positive number. If the LP is a minimization problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a maximization problem, add (for each artificial variable) $-Ma_i$ to the objective function.
5. Since each artificial variable will be in starting basis, all artificial variables must be eliminated from row “0” before beginning the simplex by using an elementary row operation. Now solve the transformed problem by the simplex method.

If the non-negative conditions are not met, for example, one of the decision variables: $x_k \in \mathbb{R}$ is unrestricted in sign, and we replace this variable with the difference of two non-negative variables in accordance with the following formula:

$$x_k = x_k^* - x_k^{**}, \quad x_k^* \geq 0, \quad x_k^{**} \geq 0.$$

The number M is chosen to be a very large number. While there is no definitive rule as how large M should be, it is important that M be significantly larger than any of the coefficients in the objective function, large enough so that any non-zero artificial variable value in the final solution results in a significantly larger z value. Thus, we call this approach for solving the LP the “**Big M -Method**”.

All artificial variables should be equal to zero in the final solution to the original linear programming problem. If any artificial variables are not equal to zero in the optimal solution, the original problem is **infeasible**.

Example 16: An example of such an LP is given by:

$$\begin{aligned} z = 3x_1 + x_2 &\rightarrow \min \\ \text{s.t. } &\begin{cases} x_1 + x_2 \leq 6 \\ x_1 \geq 1 \\ 2x_1 + x_2 = 10 \\ x_1, x_2 \geq 0 \end{cases} \end{aligned}$$

Solution

First, convert this problem to appropriate form. Introduce a slack variable s_1 , in the first constraint, in the second constraint subtract a nonnegative surplus variable e_2 and artificial variable a_2 . In the third constraint, introduce artificial variable a_3 . So, the standard form of the problem is:

$$\begin{aligned} z = 3x_1 + x_2 + 0s_1 + 0e_2 + Ma_2 + Ma_3 &\rightarrow \min \\ \text{s.t. } &\begin{cases} x_1 + x_2 + s_1 = 6 \\ x_1 - e_2 + a_2 = 1 \\ 2x_1 + x_2 + a_3 = 10 \\ x_1, x_2, s_1, e_2, a_2, a_3 \geq 0 \end{cases} \end{aligned}$$

This yields a basic feasible solution with basic variables: $s_1 = 6$, $a_2 = 1$, $a_3 = 10$. The introduction of artificial variables therefore eliminates the difficulty of finding an initial basic feasible solution. The initial tableau for $M = 20$ is:

Basic	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	ratio
z	1	-3	-1	0	0	-20	-20	0	
s_1	0	1	1	1	0	0	0	6	
a_2	0	1	0	0	-1	1	0	1	
a_3	0	2	1	0	0	0	1	10	

Since a_2, a_3 are basic variables in the basic feasible solution, and have non-zero coefficients in row 0, elementary row operations are necessary to obtain appropriate form of the initial simplex tableau (pivot on the coefficient in row “ i ” and column “ j ” to convert column “ j ” into “ $1j$ ” by multiply row “ i ” by a constant, and add multiples of row “ i ” to other rows, in this example: $20 \cdot R2 + R0 \Rightarrow \text{new } R0$; $20 \cdot R3 + R0 \Rightarrow \text{new } R0$).

Performing these two pivots yields the following initial simplex tableau:

Basic	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	ratio
z	1	57	19	0	-20	0	0	220	
s_1	0	1	1	1	0	0	0	6	$\frac{6}{1} = 6$
a_2	0	1	0	0	-1	1	0	1	$\frac{1}{1} = 1$
a_3	0	2	1	0	0	0	1	10	$\frac{10}{2} = 5$

Now, the simplex algorithm can be started. For the minimization problem, we focus on the column corresponding to the non-basic variable whose coefficient in row 0 of the tableau, is most positive. In this example, the variable with the most positive coefficient in the top row is x_1 , it will be the entering variable, the ratio test determines a_2 as the leaving variable. Performing elementary row operations yields the new basic feasible solution in the next simplex tableau:

Basic	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	ratio
z	1	0	19	0	37	-57	0	163	
s_1	0	0	1	1	1	-1	0	5	$\frac{5}{1} = 5$
x_1	0	1	0	0	-1	1	0	1	none
a_3	0	0	1	0	2	-2	1	8	$\frac{8}{2} = 4$

The actual basic feasible solution is (non-basic variables equal to zero, basic variables from tableau) $\mathbf{x}_1^T = [1 \ 0 \ 5 \ 0 \ 0 \ 0 \ 8]$, and the value of the objective function is equal to 163. This solution is not optimal. There are positive coefficients in row 0. Since e_2 has the most positive coefficient in row 0, and a_3 has the lowest value in ratio test, the entering and the leaving variables are e_2 and a_3 , respectively.

Two more iterations of the simplex algorithm are then necessary to achieve the optimal solution.

Basic	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	ratio
z	1	0	$\frac{1}{2}$	0	0	-20	$-\frac{37}{2}$	15	
s_1	0	0	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	1	$\frac{1}{\frac{1}{2}}=2$
x_1	0	1	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	5	$\frac{5}{\frac{1}{2}}=10$
e_2	0	0	$\frac{1}{2}$	0	1	-1	$\frac{1}{2}$	4	$\frac{4}{\frac{1}{2}}=8$

Basic	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	ratio
z	1	0	0	-1	0	-20	-18	14	
x_2	0	0	1	2	0	0	-1	2	
x_1	0	1	0	-1	0	0	1	4	
e_2	0	0	0	-1	1	-1	1	3	

Recall that because this is a minimization problem, we terminate the algorithm when entries in the top row of the tableau corresponding to non-basic variables are all non-positive. Thus, the solution is optimal with $x_1=4$, $x_2=2$ and the objective function value of $z=14$. It is very significant to notice that in this solution, both artificial variables are non-basic, which means that they take the value of zero.

2.5. Duality theory

Duality occurs when two related parts constitute the whole of something. In the context of linear programming, duality refers to the idea that associated with any linear problem is another linear problem, so-called dual, the solution of which gives the information about the solution of original linear problem. The dependencies between the original and the one that is dual are very useful in a variety of ways.

2.5.1. Formulating the dual LP

Let us consider the maximization problem having n decision variables and m constraints, written in matrix notation as:

$$z = \mathbf{c} \cdot \mathbf{x} \rightarrow \mathbf{max}$$

$$\text{s.t. } \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

where \mathbf{x} is a column vector that belongs to \mathbb{R}^n , \mathbf{c} is a row vector in \mathbb{R}^n , \mathbf{b} belongs to \mathbb{R}^m , and \mathbf{A} is an m -by- n matrix.

For example: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{c} = [4 \ 3]$, $\mathbf{b} = \begin{bmatrix} 8 \\ 28 \\ 32 \end{bmatrix}$, and $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 2 \end{bmatrix}$.

The expanded form of this LP is given by:

$$z = 4x_1 + 3x_2 \rightarrow \mathbf{max}$$

$$\text{s.t. } \begin{cases} x_1 \leq 8 \\ 2x_1 + 2x_2 \leq 28 \\ 3x_1 + 2x_2 \leq 32 \end{cases}$$

$$x_1, x_2 \geq 0$$

Let \mathbf{y} denote a row vector in \mathbb{R}^m .

Definition 1. The **dual problem** of given linear problem is written in matrix form as:

$$w = \mathbf{y} \cdot \mathbf{b} \rightarrow \mathbf{min}$$

$$\mathbf{y} \cdot \mathbf{A} \geq \mathbf{c}$$

$$\mathbf{y} \geq \mathbf{0}$$

The original linear programming problem is called the **primal problem**.

The general features of the dual linear programming problem:

1. Its goal is minimization if the primal is a maximization problem.
2. Its objective function coefficients are determined from the right-hand sides of the original LP's constraints.

3. The right-hand side of the primal are the objective coefficients of the dual problem.
4. The column coefficients in a constraint of a primal variables will be the left-hand side coefficients of the dual constraints.

In the example model $w = [y_1 \ y_2 \ y_3]$ is a row vector in \mathbb{R}^3 , and the dual LP's

$$\text{objective function is given by } w = [y_1 \ y_2 \ y_3] \begin{bmatrix} 8 \\ 28 \\ 32 \end{bmatrix} = 8y_1 + 28y_2 + 32y_3.$$

So, the expanded form of the dual is:

$$\begin{aligned} w &= 8y_1 + 28y_2 + 32y_3 \rightarrow \min \\ \text{s.t. } &\begin{cases} y_1 + 2y_2 + 3y_3 \geq 4 \\ 2y_2 + 2y_3 \geq 3 \end{cases} \\ &y_1, y_2, y_3 \geq 0 \end{aligned}$$

A comparison of the primal problem with its dual yields:

- the primal LP involves three constraints in two decision variables,
- the dual LP involves two constraints in three decision variables.

The duality is a very useful feature in practice: we can obtain the solution of the LP by solving its dual. Thus, one has the freedom to solve either of the LP's, depending upon which is computationally easier.

2.5.2. The dual theorems

The path to establishing relationships between an LP and its corresponding dual begins with the following theorem:

Theorem 1 (Weak Duality Theorem). Suppose \mathbf{x}_0 is a primal-feasible solution, meaning it satisfies the constraints and sign conditions in primal LP, and \mathbf{y}_0 is a dual-feasible solution, meaning it satisfies the constraints and sign restrictions in dual LP. Then

$$\mathbf{c} \cdot \mathbf{x}_0 \leq \mathbf{y}_0 \cdot \mathbf{b}$$

In other words, any feasible solution to the dual can be used to develop a bound on the optimal value of the primal objective function.

Theorem 2 (Strong Duality Theorem). If the linear programming problem has an optimal solution \mathbf{x}_0 , then its dual also has an optimal solution, \mathbf{y}_0 , and their corresponding objective function values, z_0 and w_0 , are equal.

Important Fact: the dual of a linear problem's dual is the original linear problem.

Combining **Theorem 1**, **Theorem 2** and the important fact, the following major result can be formulated, which summarizes the three possible outcomes for a linear problem and its dual:

Theorem 3. For a linear problem and its dual, one of the three possible outcomes must occur:

1. If the primal LP has an optimal solution, then so does the dual LP and the conclusions of both the Weak and Strong Duality Theorems hold.
2. If the primal LP is unbounded, then the dual is infeasible.
3. If the primal LP is infeasible, then the dual LP is either infeasible or unbounded.

There are two methods of obtaining the dual linear problem solution. One way is to solve it directly. The second way of reaching the dual solution is to do so indirectly from the solution of the primal problem. This can be done using the following theorem.

Theorem 4 (the Complementary Slackness Property). Assume that \mathbf{x}_0 is a feasible solution to LP and \mathbf{y}_0 is a feasible solution to dual LP. Then a necessary and sufficient condition for both to be simultaneously optimal solutions of their respective LP's is that

$$[\mathbf{y}_0 \mathbf{A} - \mathbf{c}]_i [\mathbf{x}_0]_i = 0, \quad i = 1, 2, \dots, n \quad \text{and} \quad [\mathbf{y}_0]_j [\mathbf{b} - \mathbf{A} \mathbf{x}_0]_j = 0, \quad j = 1, 2, \dots, m$$

In other words, if a constraint in either the primal or dual is a strict inequality, then the corresponding (complementary) variable in the other problem must be equal to zero.

The table below provides a summary of the general formulation for an LP having m constraints in n variables. In this table, "u. r. s." denotes a decision variable that is unrestricted in sign.

Primal	Dual
maximize $\mathbf{c} \cdot \mathbf{x}$	minimize $\mathbf{y} \cdot \mathbf{b}$
Constraint sign	Variable sign
$\sum_{i=1}^n a_{ij} x_i \leq b_j$ for some $1 \leq j \leq m$	$y_j \geq 0$ for this same j
$\sum_{i=1}^n a_{ij} x_i = b_j$ for some $1 \leq j \leq m$	y_j u.r.s. for this same j
Variable sign	Constraint sign
$x_i \geq 0$ for some $1 \leq i \leq n$	$\sum_{i=1}^n y_j a_{ij} \geq c_i$ for this same i
x_i u.r.s. for some $1 \leq i \leq n$	$\sum_{i=1}^n y_j a_{ij} = c_i$ for this same i

Example 17: A furniture factory makes desks, tables, and chairs. Each product needs the limited resources of R1, R2 and R3. The limits of resources are described in the table.

Let us suppose that a businessman has been found who is willing to buy all resources from this factory. Determine the resources price that will encourage the furniture factory to sell the resources.

Resources	Desk	Table	Chairs	Max available
R1	8	6	1	48
R2	4	2	1.5	20
R3	2	1.5	0.5	8
Price [\$]	60	30	20	

Solution. The problem can be described by the dual problem. The decision variables in dual are the resources price – price paid for one unit of resource R1, R2, and R3. The total cost of purchasing the resources (dual problem) should be equal to the total profit (primal problem).

The primal and the dual problems are

$$z = 60x_1 + 30x_2 + 20x_3 \rightarrow \max$$

$$\begin{aligned} \text{s.t. } & \begin{cases} 8x_1 + 6x_2 + x_3 \leq 48 \\ 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\ 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \end{cases} \\ & x_1, x_2, x_3 \geq 0 \\ w = & 48y_1 + 20y_2 + 8y_3 \rightarrow \min \end{aligned}$$

$$\begin{aligned} \text{s.t. } & \begin{cases} 8y_1 + 4y_2 + 2y_3 \geq 60 \\ 6y_1 + 2y_2 + 1.5y_3 \geq 30 \\ y_1 + 1.5y_2 + 0.5y_3 \geq 20 \end{cases} \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

The solution of the dual problem can be obtained in two ways, directly with the simplex method or indirectly from the solution of the primal problem. So, using the second way for this example, the optimal solution of the primal LP obtained with the simplex method is the following $\mathbf{x}_{opt} = [2 \ 0 \ 8]$, and $z_{opt} = 280$. Next, the solution to the dual problem will be found using the duality theorems. From the strong duality property, the so-called complementary slackness conditions, the following dependencies arise: if the optimal variable is positive, its corresponding (complementary) dual constraint holds with equality, and if a primal constraint holds with strict inequality, then the corresponding dual variable must be zero.

$$\begin{cases} 8 \cdot 2 + 6 \cdot 0 + 8 = 24 < 48 \Rightarrow y_1 = 0 \\ 4 \cdot 2 + 2 \cdot 0 + 1.5 \cdot 8 = 20 \\ 2 \cdot 2 + 1.5 \cdot 0 + 0.5 \cdot 8 = 8 \end{cases}$$

additionally, since the first and third optimal variable are positive, the corresponding dual constraints are fulfilled with equality:

$$\begin{cases} 8 \cdot 0 + 4y_2 + 2y_3 = 60 \\ 0 + 1.5y_2 + 0.5y_3 = 20 \end{cases}$$

The solution of this system of equations is the optimal solution of the dual problem: $\mathbf{y}_{opt} = [0 \ 10 \ 10]$,

$$\text{and } w = 48y_1 + 20y_2 + 8y_3 = 48 \cdot 0 + 20 \cdot 10 + 8 \cdot 10 = 280$$

Interpretation of the results: an entrepreneur can offer the following resource price, \$10 for resource R2 and \$10 for resource R3, he or she should not buy resource R1, the minimum purchase cost will then be \$280.

2.5.3. Economic interpretation of dual variables

When the primal is a standard max problem, the dual variables are related to the value of resources available to the decision-maker. For this reason, dual variables are often referred to as **resource shadow prices**.

Definition 1. The shadow price for the i -th constraint of an LP is the amount by which the optimal z value is “improved” (increased in a maximization problem and decreased in a minimization problem) if the right-hand side of the i -th constraint is increased by 1.

This definition applies only if the change in RHS of the constraint leaves the current basis optimal.

A “ \geq ” constraint will always have a nonpositive shadow price; a “ \leq ” constraint will always have a non-negative shadow price.

Example 18: For the given linear problem

$$z = 15x_1 + 25x_2 \rightarrow \max$$

$$\text{s.t.} \quad \begin{cases} 3x_1 + 4x_2 \leq 100 \\ 2x_1 + 3x_2 \leq 70 \\ x_1 + 2x_2 \leq 30 \\ x_2 \geq 3 \end{cases}$$

$$x_1, x_2 \geq 0$$

the optimal solution to the problem is $\mathbf{x}_{opt} = [24 \ 3]$, and $z_{opt} = 435$. Find the shadow price of each constraint.

Solution. The shadow price of each constraint is the optimal value of the corresponding dual variable of each constraint. Let us start with dual problem

$$w = 100y_1 + 70y_2 + 30y_3 + 3y_4 \rightarrow \min$$

$$\text{s.t.} \quad \begin{cases} 3y_1 + 2y_2 + y_3 \geq 15 \\ 4y_1 + 3y_2 + 2y_3 + y_4 \geq 25 \end{cases}$$

$$y_1, y_2, y_3 \geq 0, y_4 \leq 0$$

So, using the duality theorems, the shadow prices (optimal solution of dual) are:

$$\mathbf{y}_{opt} = [0 \ 0 \ 15 \ -5], \text{ and } w_{opt} = 435.$$

2.6. Sensitivity analysis

The second important topic in linear programming, next to duality, is sensitivity analysis. This chapter is about sensitivity analysis that takes place after a linear programming problem has been solved. Sensitivity analysis is precisely the study of how the optimal solution to a linear programming problem depends on model parameters (the technological coefficients a_{ij} , right-hand side b_i , and objective function coefficients c_j). In other words: what happens to the optimal solution value if one number in the data is changed. Typically, greater attention is given to performing sensitivity analysis on the b_i , and c_j than on the a_{ij} parameters. On real problems with hundreds or thousands of constraints and variables, the effect of changing one a_{ij} value is usually negligible, but changing one b_i , or c_j value can have a real impact.

2.6.1. Sensitivity to an objective function coefficient

Two cases of objective coefficient value change should be analyzed:

- 1) If the objective function coefficient corresponds to a non-basic variable in the solution of the original linear problem, then the range of values δ for which the optimal solution values remain unchanged is straightforward to determine. We can subtract δ from the coefficient of the corresponding variable in the top row of the original final tableau.
- 2) If the coefficient corresponds to a basic variable, we should pivot to update the value of the objective function and to determine whether another iteration of the simplex algorithm is necessary. For values of δ not leading to an additional iteration, the decision variable value in the optimal solution is the same as it is in the original LP. However, the objective function value will now depend upon δ , in a manner determined by the pivots.

Example 19: Consider the following problem:

$$z = 4x_1 + 3x_2 + 6x_3 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 3x_1 + x_2 + 3x_3 \leq 30 \\ 2x_1 + 2x_2 + 3x_3 \leq 40 \end{cases}$$

$$x_1, x_2, x_3 \geq 0$$

The final simplex tableau corresponding to this problem is:

Basic	z	x_1	x_2	x_3	s_1	s_2	RHS
z	1	1	0	0	1	1	70
x_3	0	$\frac{4}{3}$	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{20}{3}$
x_2	0	-1	1	0	-1	1	10

The basic variables at the final iteration are given by $x_2 = 10$ and $x_3 = \frac{20}{3}$. Suppose that the objective coefficient c_1 increases from 4 to $4 + \delta$. In this case, no additional pivots are necessary in order to update the objective value. This is because the coefficient corresponds to a decision variable that was non-basic in the optimal solution of the original LP. This is the first case described above, so we get the following table:

Basic	z	x_1	x_2	x_3	s_1	s_2	RHS
z	1	$1 - \delta$	0	0	1	1	70
x_3	0	$\frac{4}{3}$	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{20}{3}$
x_2	0	-1	1	0	-1	1	10

The current solution remains optimal when the optimality condition is met $1 - \delta \geq 0 \Rightarrow \delta < 1$.

Example 20: Consider the following problem

$$z = 2x_1 + 3x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 + 2x_2 \leq 14 \\ x_1 + 2x_2 \leq 8 \\ 4x_1 \leq 16 \\ x_1, x_2 \geq 0 \end{cases}$$

with the final simplex tableau:

Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0	$\frac{3}{2}$	$\frac{1}{8}$	14
s_1	0	0	0	1	-1	$-\frac{1}{4}$	2
x_2	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{8}$	2
x_1	0	1	0	0	0	$\frac{1}{4}$	4

The basic variables at the final iteration were given by x_1, x_2, s_1 . Suppose that we want to change coefficient c_1 , which corresponds to the basic variable (the second case):

Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	$0 - \delta = -\delta$	0	0	$\frac{3}{2}$	$\frac{1}{8}$	14
s_1	0	0	0	1	-1	$-\frac{1}{4}$	2
x_2	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{8}$	2
x_1	0	1	0	0	0	$\frac{1}{4}$	4

to update the value of x_1 and the objective, an additional pivot is necessary:

Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0	$\frac{3}{2}$	$\frac{1}{8} + \frac{1}{4} \cdot \delta$	$14 + 4 \cdot \delta$
s_1	0	0	0	1	-1	$-\frac{1}{4}$	2
x_2	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{8}$	2
x_1	0	1	0	0	0	$\frac{1}{4}$	4

From this tableau, we see that if $\frac{1}{8} + \frac{1}{4} \cdot \delta \geq 0$, $\delta \geq -\frac{1}{2}$ then the optimal value of the decision variable remains at the same point $(x_1, x_2) = (4, 2)$. However, the corresponding objective function value changes to $z = 14 + 4\delta$.

2.6.2. Sensitivity to constraint bounds

Now we consider the effect of changing one or more constants that bound the right-hand sides of linear problem constraints. In matrix form we can write this as follows: $\mathbf{b}' = \mathbf{b} + \delta \cdot \mathbf{y}$, where: \mathbf{b}' it is vector of coefficients of right-hand side after changes, \mathbf{b} it is vector of coefficients of right-hand side of the optimal solution of an original linear problem, and \mathbf{y} represents the column vector of slack or artificial variables corresponding to the constraint that right-hand side being changed. Suppose we wish to increase or decrease the right-hand side entries by δ . If we want to determine this value, we have to solve the following inequalities: $\mathbf{b} + \delta \cdot \mathbf{y} \geq \mathbf{0}$. We can do it using the final tableau corresponding to the optimal solution of original linear problem.

Example 21: Consider the given LP (**Example 20**) formed by changing the bound on the second constraint:

$$\begin{aligned} z &= 2x_1 + 3x_2 \rightarrow \max \\ \text{s.t. } &\begin{cases} 2x_1 + 2x_2 \leq 14 \\ x_1 + 2x_2 \leq 8 \\ 4x_1 \leq 16 \end{cases} \\ &x_1, x_2 \geq 0 \end{aligned}$$

Let us consider the following system of inequalities: $\mathbf{b} + \delta \cdot \mathbf{y} \geq \mathbf{0}$.

We can do it using the final tableau corresponding to the optimal solution of original LP

where $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$,

is the right-hand side in the final tableau

and $\mathbf{y} = \begin{bmatrix} -1 \\ 1/2 \\ 0 \end{bmatrix}$,

\mathbf{y} – represents the column vector of slack variables s_2 corresponding to the second constraint in final tableau.

From the given system of inequalities, we obtain

$$\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + \delta \cdot \begin{bmatrix} -1 \\ 1/2 \\ 0 \end{bmatrix} \geq \mathbf{0} \Rightarrow \begin{cases} 2 - \delta_2 \geq 0 \\ 2 + \frac{1}{2}\delta_2 \geq 0 \\ 4 + 0 \cdot \delta_2 \geq 0 \end{cases} \Rightarrow \begin{cases} \delta_2 \leq 2 \\ \delta_2 \geq -4 \end{cases}$$

Therefore, since $b'_2 = 4 + \delta_2$, the solution remains feasible only if $-4 \leq \delta_2 \leq 2$, that is, $0 \leq b_2 \leq 6$. Note that the value of optimal basic variables and optimal value of the objective function can change. For example, for $\delta_2 = 1$, the new optimal solution is:

$$\Rightarrow \begin{bmatrix} 2-1 \\ 2+\frac{1}{2} \cdot 1 \\ 4+0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5}{2} \\ 4 \end{bmatrix} \Rightarrow x_1 = 4, x_2 = \frac{5}{2} \quad \text{and} \quad z = \frac{31}{2}.$$

2.7. Special cases of linear programming problems

The simplex algorithm has a very useful feature, as it is possible to determine from the tableau whether an LP has alternative optimal solutions or is unbounded or the original problem is infeasible.

The original problem is **infeasible** – if any artificial variables are not equal to zero in the final tableau (optimal solution).

The LP problem has an **alternative optimal solution** – if non-basic variable in the top row of LP has a coefficient equal to zero in the optimal solution, this non-basic variable entering into the basis does not change the objective function value, so the new solution is also optimal.

The LP is **unbounded** – if the variable in the top row of an LP maximum problem has a negative coefficient and all the elements in the column of this variable are negative or equal to zero. This non-basic variable is the entering variable and can improve the value of the objective function without limit.

Degeneracy – if the LP has at least one basic feasible solution in which a basic variable is equal to zero, then it is a degenerate solution. This means that the extreme point represented by a basic solution is formed by the intersection of more constraints than are needed. Therefore, the same extreme point can be expressed algebraically by several basic solutions.

Exercises

4. The farm raising dairy cows determined that cows need at least minimum amounts of four nutrients to grow healthy. The manager is considering two types of feed for cows. The table shows the number of units of each nutrient in each kilogram of food, the minimum daily requirements of each nutrient for each cow, and the cost of the food. Formulate a mathematical model of the problem that will help the manager raise cows at minimum cost.

	food 1	food 2	min daily requirement [units]
nutrient 1	20	30	110
nutrient 2	10	10	18
nutrient 3	50	30	90
nutrient 4	6	2.5	14
cost [PLN/kg]	41	36	

5. Graphically solve the following linear problems:

a) $z = 3x_1 + x_2 \rightarrow \max$

$$\text{s.t. } \begin{cases} 3x_1 - 2x_2 \leq 0 \\ x_1 + x_2 \leq 6 \\ 9x_1 + 5x_2 \leq 45 \end{cases} \quad x_1, x_2 \geq 0$$

b) $z = -x_1 \rightarrow \min$

$$\text{s.t. } \begin{cases} -6x_1 + 4x_2 \leq 12 \\ x_1 + x_2 \leq 4 \\ 2x_1 - 4x_2 = 4 \end{cases} \quad x_1, x_2 \geq 0$$

6. The linear programming problem has the feasible area determined by the given system of linear inequalities:

$$\begin{cases} x + 2y \leq 10 \\ 3x + y \leq 15 \end{cases} \quad x, y \geq 0$$

Graph this feasible area. The corner points for this problem are the following: $O(0,0)$; $A(0,5)$; $B(4,3)$; and $C(5,0)$. For the given objective function $z = ax + by$, determine conditions on parameters a and b that will ensure that the maximum value of the objective function z occurs:

- a) only at A,
- b) only at B,
- c) only at C,
- d) at both A and B,
- e) at both B and C.

7. Convert the given linear programming problem to the standard form:

$$z = 2x_1 + 3x_2 + 5x_3 \rightarrow \max$$

$$\text{s.t. } \begin{cases} x_1 + x_2 - x_3 \geq -5 \\ -6x_1 + 7x_2 - 9x_3 \leq 4 \\ x_1 + x_2 + 4x_3 = 10 \end{cases} \quad x_1, x_2 \geq 0, x_3 \text{ u.r.s.}$$

Express the problem in matrix form.

8. For a given linear programming problem, find all possible bases and all basic feasible solutions. Determine the optimal solution by inspection of the objective function value for an obtained basic feasible solution. Solve the problem graphically and verify the obtained optimal solution.

$$z = x_1 + 3x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} x_1 + x_2 \leq 2 \\ -2x_1 + x_2 \leq 4 \end{cases} \quad x_1, x_2 \geq 0$$

9. Solve the problems using the simplex method step-by-step:

a) $z = 2x_1 + 3x_2 \rightarrow \max$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \leq 4 \\ x_1 + 2x_2 \leq 5 \end{cases} \quad x_1, x_2 \geq 0$$

b) $z = 4x_1 - x_2 \rightarrow \min$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \leq 8 \\ x_2 \leq 5 \\ x_1 - x_2 \leq 4 \end{cases} \quad x_1, x_2 \geq 0$$

c) $z = 4x_1 + x_2 \rightarrow \min$

$$\text{s.t. } \begin{cases} 3x_1 + x_2 = 3 \\ 4x_1 + 3x_2 \geq 6 \\ x_1 + 2x_2 \leq 4 \end{cases} \quad x_1, x_2 \geq 0$$

10. Find the dual of the given linear problems:

a) $z = 5x_1 + 12x_2 + 4x_3 \rightarrow \max$

$$\text{s.t. } \begin{cases} x_1 + 2x_2 + x_3 \leq 10 \\ 2x_1 - x_2 + x_3 = 8 \end{cases} \quad x_1, x_2, x_3 \geq 0$$

b) $z = 15x_1 + 12x_2 \rightarrow \min$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \geq 3 \\ x_1 - 4x_2 \leq 7 \end{cases} \quad x_1, x_2 \geq 0$$

11. Consider the given linear problem

a) $z = 5x_1 + 3x_2 + x_3 \rightarrow \max$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 + x_3 \leq 6 \\ x_1 + 2x_2 + x_3 \leq 7 \end{cases} \quad x_1, x_2, x_3 \geq 0$$

the optimal solution to the problem is $\mathbf{x}_{opt} = \left[\frac{5}{3} \quad \frac{8}{3} \quad 0 \right]$, and $z_{opt} = \frac{49}{3}$,

b) $z = 3x_1 + 4x_2 + x_3 + 5x_4 \rightarrow \max$

$$\text{s.t. } \begin{cases} x_1 + 2x_2 + x_3 + 2x_4 \leq 5 \\ 2x_1 + 3x_2 + x_3 + 3x_4 \leq 8 \end{cases} \quad x_1, x_2, x_3, x_4 \geq 0$$

the optimal solution to the problem is $\mathbf{x}_{opt} = [1 \quad 0 \quad 0 \quad 2]$, and $z_{opt} = 13$.

Use the complementary slackness theorem to find the optimal dual solution of these problems.

CHAPTER 3

TRANSPORTATION PROBLEM

The transportation problem is one of the original applications of linear programming problems which is usually used to minimize the transportation cost for the industries with a number of sources and a number of destinations while satisfying the supply limit and demand requirements. Transportation models play an important role in logistics and supply-chain management for reducing cost and improving service. It is now accepted as one of the most important analytical and planning tools in business and industry. Many practical problems which are focused on cost-effective ways to transport goods and services of one type or another have the same form as transportation problems, for example: the assignment problem, the transshipment problem.

A short history: the transportation model was used during the Second World War to determine how to move troops (located, for example, at training basis in different parts of the United States) to battlegrounds in Europe and Asia. It was first studied by F.L. Hitchcock in 1941, then separately by T.C. Koopmans in 1947, and finally placed in the framework of linear programming and solved by the simplex method by G.B. Dantzig in 1951.

3.1. Transportation problem – mathematical model formulating

The transportation problem is an optimization problem with a linear objective function and linear constraints. Generally, it can be described as follows: a company produces articles at m different supply centers, $i = 1, 2, \dots, m$. The supply produced at supply center i is s_i . The demand for the article is spread out at n different demand centers, $j = 1, 2, \dots, n$. The demand at the j -th demand center is d_j . The problem of the company is to get articles from supply centers to demand centers at minimum cost. Assuming that the cost of shipping one unit from supply center i to the demand center j is c_{ij} and that shipping cost is a linear function then if you shipped x_{ij} units from supply center i to demand j , the cost would be $c_{ij} x_{ij}$.

The problem is to identify the minimum cost shipping schedule. The constraints are that you must meet demand at each demand center and not exceed supply at each supply center.

The relevant data can be formulated in a *transportation tableau*:

	Demand point 1	Demand point 2	Demand point n	Supply
Supply point 1	c_{11}	c_{12}		c_{1n}	s_1
Supply point 2	c_{21}	c_{22}		c_{2n}	s_2
.....				
Supply point m	c_{m1}	c_{m2}		c_{mn}	s_m
Demand	d_1	d_2		d_n	

Summing up, the mathematical model for the transportation problem is:

$$z = \sum_i \sum_j c_{ij} x_{ij} \rightarrow \min$$

$$\text{s.t. } \begin{cases} \sum_{i=1}^m x_{ij} = s_i & \text{supply constraints} \\ \sum_{j=1}^n x_{ij} = d_j & \text{demand constraints} \end{cases} \quad x_{ij} \geq 0$$

where x_{ij} – decision variables, it is the number of units shipped from supply point i to demand point j , and the cost of transportation on this route is $c_{ij} x_{ij}$.

Example 22: Consider a power company that has three electric power plants to meet the needs of four cities. Each power plant can supply the following kWh of electricity: plant 1 – 35 million, plant 2 – 50 million, and plant 3 – 40 million. The peak power demands in these cities are as follows (in kWh): city 1 – 45 million, city 2 – 20 million, city 3 – 30 million, city 4 – 30 million. The costs (in dollars) of sending 1 million kWh of electricity from the plant to city are given in the table below. To minimize the cost of meeting each city’s peak power demand, formulate a transportation problem in a transportation tableau and represent the problem as an LP model.

From	To			
	City 1	City 2	City 3	City 4
Plant 1	\$8	\$6	\$10	\$9
Plant 2	\$9	\$12	\$13	\$7
Plant 3	\$14	\$9	\$16	\$5

Solution

The transportation tableau for the problem is the following:

	city 1	city 2	city 3	city 4	Supply
plant 1	8	6	10	9	35
plant 2	9	12	13	7	50
plant 3	14	9	16	5	40
Demand	45	20	30	30	125

Total supply and total demand both equal 125 – it is the so-called balanced transportation problem, and its mathematical model is:

$$z = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} \rightarrow \min$$

$$\text{s.t. } \begin{cases} x_{11} + x_{12} + x_{13} + x_{14} = 35 \\ x_{21} + x_{22} + x_{23} + x_{24} = 50 \\ x_{31} + x_{32} + x_{33} + x_{34} = 40 & \text{supply constraints} \\ x_{11} + x_{21} + x_{31} = 45 \\ x_{12} + x_{22} + x_{32} = 20 \\ x_{13} + x_{23} + x_{33} = 30 \\ x_{14} + x_{24} + x_{34} = 30 & \text{demand constraints} \end{cases}$$

$$x_{ij} \geq 0; \quad i = 1, 2, 3; \quad j = 1, 2, 3, 4$$

3.1.1. Basic definitions

Definition 1. The balanced transportation problem – a transportation problem is said to be balanced if the total supply from all sources equals the total demand in the destinations. Thus, for a balanced problem we have

$$\sum_{i=1}^m s_i = \sum_{j=1}^n d_j$$

Note that if a solution has $(m + n - 1)$ positive allocations for a balanced transportation problem, then the **basic feasible solution** is obtained. This solution has $(m + n - 1)$ basic variables.

Definition 2. Degenerate the basic feasible solution – if the number of positive allocations in the basic feasible solution is less than $(m + n - 1)$.

Definition 3. An ordered sequence of at least four different cells in the transportation tableau is called a **loop** if:

- any two consecutive cells lie in either the same row or same column,
- no three consecutive cells lie in the same row or column,
- the last cell in the sequence has a row or column in common with the first cell in the sequence.

Problems that are not balanced are called an **unbalanced transportation problem** and

$$\sum_{i=1}^m s_i \neq \sum_{j=1}^n d_j$$

There are two possible cases:

1. **Excess supply** $\sum_{i=1}^m s_i > \sum_{j=1}^n d_j$

If total supply exceeds total demand, we can balance a transportation problem by creating a **dummy demand point** to consume the excess supply. Since shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. These shipments indicate unused supply capacity:

$$d_{n+1} = \sum_{i=1}^m s_i - \sum_{j=1}^n d_j.$$

$$z = \sum_{i=1}^m \sum_{j=1}^{n+1} c_{ij} x_{ij} \rightarrow \min$$

$$\text{s.t.} \quad \begin{cases} \sum_{j=1}^{n+1} x_{ij} = s_i \\ \sum_{i=1}^m x_{ij} = d_j \end{cases}$$

$$x_{ij} \geq 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n + 1$$

2. **Unmet demand** $\sum_{j=1}^n d_j > \sum_{i=1}^m s_i$

If the total demand exceeds the total supply, the demand of some of the demand points will not be satisfied. To balance this problem, a **dummy supply point** should be introduced, and the demand of this dummy supply point is equal:

$$s_{m+1} = \sum_{j=1}^n d_j - \sum_{i=1}^m s_i$$

$$z = \sum_{i=1}^{m+1} \sum_{j=1}^n c_{ij} x_{ij} \rightarrow \min$$

$$\text{s.t.} \quad \begin{cases} \sum_{j=1}^n x_{ij} = s_i \\ \sum_{i=1}^{m+1} x_{ij} = d_j \end{cases}$$

$$x_{ij} \geq 0, \quad i = 1, 2, \dots, m + 1, \quad j = 1, 2, \dots, n$$

3.2. Initial basic feasible solution

The transportation problem is a special case of a linear programming problem, so theoretically it can be solved by the simplex method. However, due to the special form of the constraints, other much more convenient methods of solving these problems have been developed. The methods begin by determining the basic feasible solution, and then determine whether it is the optimal solution. If not, then another basic feasible solution is determined for which the value of objective is smaller. The procedure is repeated until a solution is found for which the value of the objective function is minimal.

There are three methods that can be used to find a basic feasible solution for a balanced transportation problem:

1. Northwest corner method
2. Minimum cost method
3. Vogel's approximation method

3.2.1. Northwest corner method

The method starts from the upper left corner of the transportation tableau and sets x_{11} as large as possible: $\min\{s_1, d_1\}$ (clearly, x_{11} can be no larger than of s_1 and d_1).

- If $x_{11} = s_1$, cross out the first row of the tableau and change d_1 to $d_1 - s_1$.
- If $x_{11} = d_1$, cross out the first column of the tableau and change s_1 to $s_1 - d_1$.
- If $x_{11} = s_1 = d_1$, cross out either row 1 or column 1 (but not both!)
 - If you cross out a row, change d_1 to 0
 - If you cross out a column, change s_1 to 0

Continue applying this procedure to the most northwest cell in the tableau that does not lie in a cross-out row or column.

Finally, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cell's row or column. An initial basic feasible solution has now been obtained.

3.2.2. Minimum cost method

To begin the minimum cost method, find the variable with the smallest shipping cost (call it x_{ij}), then assign x_{ij} its largest possible value, $\min\{s_i, d_j\}$.

As in the NWC method, cross out row i or column j and reduce the supply or demand of the non-cross-out of row or column by the value of x_{ij} . Continue like the NWC method (instead of assigning upper left corner, the cell with the minimum cost is assigned).

3.2.3. Vogel's approximation method

Begin by computing for each row and column a penalty equal to the difference between the two smallest costs in the row and column. Next, find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest cost. As described in the NWC method, make this variable as large as possible, cross out a row or column, and change the supply or demand associated with the basic variable. Now recompute new penalties (using only cells that do not lie in a crossed-out row or column) and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with the variable and cross out the variable's row and column.

Example 23: The company has manufacturing plants in cities A, B, C, and distribution centers in cities D, E, F. The production ability of the plants is 50, 70 and 30 units, respectively, while the demand forecasts at the centers are 20, 40, 90 units, respectively. Unit transportation costs are shown in the table below. Find the initial basic feasible solution using the Northwest corner method, Minimum cost method, and Vogel’s approximation method.

	city D	city E	city F	SUPPLY
plant A	3	5	7	50
plant B	12	10	9	70
plant C	13	3	9	30
DEMAND	20	40	90	150

Solution: The mathematical model is

$$z = 3x_{11} + 5x_{12} + 7x_{13} + 12x_{21} + 10x_{22} + 9x_{23} + 13x_{31} + 3x_{32} + 9x_{33} \rightarrow \min$$

$$\text{s.t.} \begin{cases} x_{11} + x_{12} + x_{13} = 50 \\ x_{21} + x_{22} + x_{23} = 70 \\ x_{31} + x_{32} + x_{33} = 30 & \text{supply constraints} \\ x_{11} + x_{21} + x_{31} = 20 \\ x_{12} + x_{22} + x_{32} = 40 \\ x_{13} + x_{23} + x_{33} = 90 & \text{demand constraints} \\ x_{ij} \geq 0; i = 1, 2, 3; j = 1, 2, 3 \end{cases}$$

The problem is balanced (total supply equals total demand). Let us start with the northwest corner method. In this method the unit costs can be omitted, as they are not needed to find initial basic feasible solution. The procedure starts at the upper left corner and is assigned to cell (1,1) the value $x_{11} = \min\{20, 50\} = 20$. Because $x_{11} = d_1$, the first column is crossed out and correct $s_1 = s_1 - d_1 = 50 - 20 = 30$. Next, applying this procedure to the most northwest cell that is (1, 2), and $x_{12} = \min\{30, 40\} = 30$, and the first row is crossed out, and correction to $d_1 = d_2 - s_1 = 40 - 30 = 10$. Continue by following this procedure until there is only one cell that can be assigned a value of $x_{33} = 30$.

20			30
0			70
0			30
0	40	90	

20	30	0	0
0			70
0			30
20	10	90	

20	30	0	0
0	10		60
0	0		30
0	0	90	

20	30	0	0
0	10	60	0
0	0		30
0	0	30	

20	30	0	0
0	10	60	0
0	0	30	30
0	0	90	

When the NWC method was applied to this problem, the initial basic feasible solution in the last table is obtained (check: there exist $m + n - 1$ basic variables):

$$x_{11} = 20, x_{12} = 30, x_{22} = 10, x_{23} = 60, x_{33} = 30 \text{ – basic variables,}$$

$$x_{13} = x_{21} = x_{31} = x_{32} = 0 \text{ – non-basic variables.}$$

The minimum cost method

Let us start with the smallest shipping cost cell – x_{32} , assign $x_{32} = \min\{30, 40\} = 30$, cross out row 3, and reduce $d_2 = d_2 - s_3 = 40 - 30 = 10$:

	3		5		7	50
	12		10		9	70
0	13	30*	3	0	9	0
20		10		90		

20*	3		5		7	30
0	12		10		9	70
0	13	30*	3	0	9	0
0	10	90				

20*	3	10*	5		7	20
0	12	0	10		9	70
0	13	30*	3	0	9	0
20	0	90				

20*	3	10*	5	20*	7	0
0	12	0	10		9	70
0	13	30*	3	0	9	0
0	10	70				

20*	3	10*	5	20*	7	0
0	12	0	10	70*	9	0
0	13	30*	3	0	9	0
0	0	0				

When the minimum cost method was applied to this problem, the initial basic feasible solution in the last table is obtained:

$$x_{11} = 20, x_{12} = 10, x_{13} = 20, x_{23} = 70, x_{32} = 30 \text{ – basic variables}$$

$$x_{21} = x_{22} = x_{31} = x_{33} = 0 \text{ – non-basic variables.}$$

The Vogel’s approximation method

20*	3		5		7	30
0	12		10		9	70
0	13		3		9	30
0	40	90				

20*	3		5		7	30
0	12		10		9	70
0	13	30*	3	0	9	0
0	10	90				

20*	3	10*	5	7	20
0	12	0	10	9	70
0	13	30*	3	0	9
20	0	90			

20*	3	10*	5	20*	7	0
0	12	0	10	9	70	
0	13	30*	3	0	9	0
0	10	70				

20*	3	10*	5	20*	7	0
0	12	0	10	70*	9	0
0	13	30*	3	0	9	0
0	0	0				

When the Vogel's approximation method was applied to this problem, the initial basic feasible solution in the last table is obtained:

$$x_{11} = 20, x_{12} = 10, x_{13} = 20, x_{23} = 70, x_{32} = 30 \text{ – basic variables}$$

$$x_{21} = x_{22} = x_{31} = x_{33} = 0 \text{ – non-basic variables.}$$

3.3. Optimal solution for the transportation problem

To find the optimal solution of the transportation problem, the modified simplex method can be applied. This modification is called the transportation simplex method.

Steps of the transportation Simplex method

For the optimality test, we have to calculate the following number $c_{ij} = u_i + v_j$

u_i – is the dual variables associated with the supply constraint

v_j – is the dual variables associated with the demand constraint

Step 1. For an unbalanced problem, balance it.

Step 2. Use one of the methods (NWC, MC or Vogel's) to find an initial basic feasible solution for the problem.

Step 3. Use the fact that $c_{ij} = u_i + v_j$ and assume $u_1 = 0$ for all basic variables to find the u 's and v 's for the current basic feasible solution.

Definition 4. Optimality test: If $u_i + v_j - c_{ij} \leq 0$ for all non-basic variables, then the current basic feasible solution is optimal.

Step 4. If this is not the case, we enter the variable with the most positive $u_i + v_j - c_{ij}$ (pick this variable as the entering variable) into the basis using the pivoting procedure. This yields a new basic feasible solution. Return to step 3.

For the maximization problem, proceed as stated, but replace step 4 by the following step:

Step 5. If optimality indexes are nonnegative ($u_i + v_j - c_{ij} \geq 0$) for all non-basic variables, then the current basic feasible solution is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure. This yields a new basic feasible solution. Return to step 3.

Pivoting procedure

1. Find the loop involving the entering variable (determined at step 4 of the transportation simplex method) and some or all of the basic variables.
2. Counting only cells in the loop, label those that are an even number (0, 2, 4, and so on) of cells away from the entering variable as **even cells**. Also label those that are an odd number of cells away from the entering variable as **odd cells**.
3. Find the odd cell whose variable assumes the smallest value. Call this value Ω . The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by Ω and increase the value of each even cell by Ω . The values of variables not in the loop remain unchanged. The pivot is now complete. If $\Omega = 0$, the entering variable will equal 0, and the odd variable that has a current value of 0 will leave the basis.

Example 24: Apply the transportation simplex method to solve the problem described in **Example 23**, using the initial basic feasible solution obtained by the northwest corner method.

Solution

The first and second steps of the method are done, so the next is step 3, which checks whether this initial solution is optimal by applying the optimality test for the transportation simplex method.

Step 3. The initial basic feasible solution obtained by the NWC method is:

20*	3	30*	5	0	7
0	12	10*	10	60*	9
0	13	0	3	30*	9

We give each equation that corresponds to a basic variable in our initial basic feasible solution:

for basic variable x_{11} : $c_{11} = u_1 + v_1 = 3$

for basic variable x_{12} : $c_{12} = u_1 + v_2 = 5$

for basic variable x_{22} : $c_{22} = u_2 + v_2 = 10$

for basic variable x_{23} : $c_{23} = u_2 + v_3 = 9$

for basic variable x_{33} : $c_{33} = u_3 + v_3 = 9$

Assuming $u_1 = 0$, and solving the given system of equations yields:

$$u_1 = 0, v_1 = 3, u_2 = 5, v_2 = 5, u_3 = 5, v_3 = 4.$$

Note that this derivation of the u_i and v_j values depends on which x_{ij} variables are basic variables in the current basic feasible solution, so this derivation will need to be repeated each time a new basic feasible solution is obtained.

Now, perform a computation for each non-basic variable $\hat{c}_{ij} = u_i + v_j - c_{ij}$

for non-basic variables

$$x_{13}: \hat{c}_{13} = u_1 + v_3 - c_{13} = 0 + 4 - 7 = -3$$

$$x_{21}: \hat{c}_{21} = u_2 + v_1 - c_{21} = 5 + 3 - 12 = -4$$

$$x_{31}: \hat{c}_{31} = u_3 + v_1 - c_{31} = 5 + 3 - 13 = -5$$

$$x_{32}: \hat{c}_{32} = u_3 + v_2 - c_{32} = 5 + 5 - 3 = 7$$

Step 4. From the optimality test – the current basic feasible solution is not optimal (there exists the $\hat{c}_{ij} > 0$ for non-basic variables). Therefore, the transportation simplex method must go to the iteration to find a better basic feasible solution.

20*	0	30*	0	0	-3	$u_1 = 0$
0	-4	10*	0	60*	0	$u_2 = 5$
0	-5	0	7	30*	0	$u_3 = 5$
$v_1 = 3$	$v_2 = 5$	$v_3 = 4$				

Since \hat{c}_{32} is the most positive one, we would next enter x_{32} into the basis using the pivoting procedure. Each unit of x_{32} that is entered into the basis will decrease the total cost by \$7. The loop involving x_{32} is: $(3, 2) - (2, 2) - (2, 3) - (3, 3)$, $\Omega = 10$, x_{33} would leave the basis (leaving variable).

20*	0	30*	0	0	-3	$u_1 = 0$
0	-4	$10 - \Omega$	0	$60 + \Omega$	0	$u_2 = 5$
0	-5	Ω	7	$30 - \Omega$	0	$u_3 = 5$
$v_1 = 3$	$v_2 = 5$	$v_3 = 4$				

To find a new basic feasible solution, we change values of variables in the loop only, and the values of variables not in the loop remain unchanged. The pivot is now complete, and a new basic feasible solution is obtained.

The new basic feasible solution is shown in the following table:

20*	0	30*	0	0	-3
0	-4	0	0	70*	0
0	-5	10*	7	20*	0

$$\text{where } \mathbf{x}_2 = \begin{bmatrix} x_{11} = 20 & x_{12} = 30 & x_{13} = 0 \\ x_{21} = 0 & x_{22} = 0 & x_{23} = 70 \\ x_{31} = 0 & x_{32} = 10 & x_{33} = 20 \end{bmatrix},$$

and the objective function value is $z = 1050$. Return to step 3.

Step 3. Find the u_i and v_j for the current BF solution and compute new \hat{c}_{ij} for each non-basic variable:

20*	0	30*	0	0	4	$u_1 = 0$
0	-11	0	-7	70*	0	$u_2 = -2$
0	-12	10*	0	20*	0	$u_3 = -2$
$v_1 = 3$		$v_2 = 5$		$v_3 = 11$		

$$\hat{c}_{13} = 4, \hat{c}_{21} = -11, \hat{c}_{22} = -7, \hat{c}_{31} = -12.$$

Step 4. The current basic feasible solution is not optimal (there exists the $\hat{c}_{ij} > 0$). Since \hat{c}_{13} is the most positive one, we would next enter x_{13} into the basis (x_{13} – entering variable). The loop involving x_{13} is $(1, 3) - (3, 3) - (3, 2) - (1, 2)$, $\Omega = 20$ (see table), x_{33} would leave the basis.

20*	0	30 - Ω	0	Ω	4	$u_1 = 0$
0	-11	0	-7	70*	0	$u_2 = 5$
0	-12	10 + Ω	0	20 - Ω	0	$u_3 = 5$
$v_1 = 3$		$v_2 = 5$		$v_3 = 4$		

The new BFS is the following:

20*	0	10*	0	20*	0
0	-7	0	-3	70*	0
0	-12	30*	0	0	-4

$$\mathbf{x}_3 = \begin{bmatrix} x_{11} = 20 & x_{12} = 10 & x_{13} = 20 \\ x_{21} = 0 & x_{22} = 0 & x_{23} = 70 \\ x_{31} = 0 & x_{32} = 30 & x_{33} = 0 \end{bmatrix},$$

where the objective function value is $z = 970$.

$$\hat{c}_{21} = -7, \hat{c}_{22} = -3, \hat{c}_{31} = -12, \hat{c}_{33} = -4$$

Since all \hat{c}_{ij} 's are negative, an optimal solution has been obtained.

3.4. Standard application of the transportation problem – Transshipment problems

One of the variants of transportation problems is the problem with intermediate points between the supply centers (called sources) and receiving centers (called destinations). These points are referred to as the **transshipment points** in which goods can be transshipped on their journey from sources to destinations. The transportation problem with this characteristic is a **transshipment problem**.

The optimal solution to a transshipment problem can be found by converting this transshipment problem to a transportation problem and then solving this transportation problem.

Remark: as stated in the “Formulating Transportation Problem”, we define a **supply point** as a point that can send goods to another point but cannot receive goods from any other point.

Similarly, a **demand point** is a point that can receive goods from other points but cannot send goods to any other points.

Steps of converting process

1. If the problem is unbalanced, balance it

Let s be a total available supply (or demand) for balanced problem

2. Construct a transportation tableau as follows:

A row in the tableau will be needed for each supply point and transshipment point

A column will be needed for each demand point and transshipment point

Each supply point will have a supply equal to its original supply

Each demand point will have a demand equal to its original demand

Each transshipment point will have a supply equal to “that point’s original supply + s ”

Each transshipment point will have a demand equal to “that point’s original demand + s ”

3. Solve the transportation problem.

Example 25: A company manufactures TVs at two factories, one in Gdansk and one in Szczecin. The Gdansk factory can produce up to 150 TVs per day, and the Szczecin factory can produce up to 200 TVs per day. TVs are shipped by air to customers in London and Paris. The customer in each city requires 130 TVs per day. Because of the deregulation of air fares, the company believes that it may be cheaper to first fly some TVs to Brussels or Hamburg and then fly to their final destinations. The costs of flying a TV are shown in the table below. The company wants to minimize the total cost of shipping the required TVs to its customers.

From [euros]	To					
	Gdansk	Szczecin	Brussels	Hamburg	London	Paris
Gdansk	0	–	8	13	25	28
Szczecin	–	0	15	12	26	25
Brussels	–	–	0	6	16	17
Hamburg	–	–	–	0	14	16
London	–	–	–	–	0	–
Paris	–	–	–	–	–	0

Solution: In this problem, Brussels and Hamburg are transshipment points.

Step 1. Total supply = $150 + 200 = 350$, and total demand = $130 + 130 = 260$

This problem is unbalanced, so we have to balance it:

dummy's demand = $350 - 260 = 90$, $s = 350$ total available supply for the balanced problem

Step 2. Constructing the transportation tableau:

transshipment point's demand = its original demand + $s = 0 + 350 = 350$

transshipment point's supply = its original supply + $s = 0 + 350 = 350$

	Brussels	Hamburg	London	Parish	Dummy	SUPPLY
Gdansk	8	13	25	28	0	150
Szczecin	15	12	26	25	0	200
Brussels	0	6	16	17	0	350
Hamburg	6	0	14	16	0	350
DEMAND	350	350	130	130	90	

Step 3. Mathematical model

$$z = 8x_{11} + 13x_{12} + 25x_{13} + 28x_{14} + 0x_{15} + 15x_{21} + 12x_{22} + 26x_{23} + 25x_{24} + 0x_{25} + 0x_{31} + 6x_{32} + 16x_{33} + 17x_{34} + 0x_{35} + 6x_{41} + 0x_{42} + 14x_{43} + 16x_{44} + 0x_{45} \rightarrow \min$$

$$\text{s.t. } \begin{cases} x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 150 \\ x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 200 \\ x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 350 \\ x_{41} + x_{42} + x_{43} + x_{44} + x_{45} = 350 \\ x_{11} + x_{21} + x_{31} + x_{41} = 350 \\ x_{12} + x_{22} + x_{32} + x_{42} = 350 \\ x_{13} + x_{23} + x_{33} + x_{43} = 130 \\ x_{14} + x_{24} + x_{34} + x_{44} = 130 \\ x_{15} + x_{25} + x_{35} + x_{45} = 90 \\ x_{ij} \geq 0; \quad i = 1, 2, 3, 4; \quad j = 1, 2, 3, 4, 5 \end{cases}$$

solving this transportation problem, we obtained

	Brussels	Hamburg	London	Parish	Dummy	SUPPLY
Gdansk	130	0	0	0	20	150
Szczecin	0	0	0	130	70	200
Brussels	220	0	130	0	0	350
Hamburg	0	350	0	0	0	350
DEMAND	350	350	130	130	90	1050

Result: The company should produce 130 TVs at Gdansk, ship them to Brussels, and transship them from Brussels to London. The 130 TVs produced at Szczecin should be shipped directly to Paris. The total shipment is 6370 Euros.

Exercises

12. A particleboard manufacturing company has production plants in three cities P1, P2, P3, and three warehouses in cities W1, W2, W3. The production capacity of the plants is 5,00, 4,00, and 2,00 units per month, respectively, and the cost of producing one plate at these plants is 3, 6, and 1 PLN. The first warehouse needs 2500 plates, the second 4500, and the third needs 3000 plates per month. Unit transportation costs (in PLN) are shown in the table. Determine which cities to produce particleboard in and how to distribute them to warehouses so that the total production and transportation costs are minimized.

From	To			Supply
	W1	W2	W3	
P1	2	7	6	5000
P2	7	3	1	4000
P3	6	5	9	2000
Demand	2500	4500	3000	

13. The 3 Parties supply the following quantity of coal $P1 = 14t$, $P2 = 12t$, $P3 = 5t$ to the 3 consumers whose coal requirements are as follows $C1 = 6t$, $C2 = 10t$, $C3 = 15t$. The cost matrix is given below. Formulate a transportation problem which minimizes the total cost in a transportation tableau and represents the problem as an LP model. Find the initial basic feasible solution using the northwest corner method, minimum cost method, and Vogel's method. For these solutions find the corresponding values of the objective function.

From	To		
	C1	C2	C3
P1	6	8	4
P2	4	9	3
P3	1	2	6

14. Consider the following transportation problem which involves the supply system of 4 breweries, supplying the needs of 4 hotels for beer. The transportation cost for a barrel of beer from each brewery to each hotel is given in table. The production capacities of breweries 1, 2, 3 and 4 are 20, 10, 10 and 5 barrels per day, respectively. The demands of hotels A, B, C and D are 5, 20, 10 and 10 barrels per day, respectively.

Breweries	Hotels			
	A	B	C	D
1	8	14	12	17
2	11	9	15	13
3	12	19	10	6
4	12	5	13	8

Formulate a linear programming model for this problem. Find an initial basis using the: northwest corner method, least cost method, Vogel approximation method. For each initial BFS check the optimality condition.

15. Solve the given transportation problem starting with an initial solution obtained by the Vogel's approximation method.

	1	2	3	4	Supply
1	10	5	6	7	25
2	8	2	7	6	25
3	9	3	4	8	50
Demand	15	20	30	35	

CHAPTER 4

INTEGER LINEAR PROGRAMMING

For many linear programming applications, the solution in which decision variables are integer-valued is sought. Within this chapter a general technique for solving linear programming problems that require integer-valued solutions is discovered.

4.1. Formulating the integer linear problem

Definition 1. The **integer linear programming** problem (abbreviated by ILP) is a linear problem in which additional conditions called integer conditions, must be satisfied. These conditions require all or only some decision variables take integer values in the optimal solution.

An ILP in which all variables are required to be integers is called a **pure ILP problem**. If some variables are restricted to be integers and some are not, then the problem is a **mixed ILP problem**. The third case of ILP is a problem where the integer variables are additionally restricted to be 0 or 1. Such problems are called **pure or mixed binary integer linear programming problems**.

4.2. The branch and bound method

The most effective algorithm for obtaining the optimal solution in integer programming problems is a method called the “branch and bound method”. This method was first proposed in the early 1960’s by Ailsa Land and Alison Doig, the professors of Operational research in the Management Department at the London School of Economics.

The branch and bound method is a technique that is based on solving a carefully chosen sequence of closely related linear programming problems. The first step of the method requires solving the corresponding problem called linear relaxation. To any integer linear problem, a linear problem that corresponds is called the **relaxation of the ILP**.

Definition 2. The **linear relaxation** is a problem formed by using the ILP, eliminating the restriction that decision variables be integer-valued.

In general, the optimal objective function value of an integer linear problem involving maximization is no larger than that of the corresponding relaxation LP. For the minimization problem, the optimal objective value for linear relaxation is less than or equal to that of IP. If linear relaxation is infeasible, then so is IP. Consequently, if the solution of the relaxation is integer-valued, then this solution is also a solution of the integer linear programming problem. If not, the next step of the branch and bound method must be applied. In this step, the original feasible region is partitioned into two non-overlapping regions (sub-problems), neither of which contains the solution of linear relaxation.

Suppose that in a given sub-region (sub-problem, call it subr. I), assumes a fractional value $x_i = \frac{p}{q}$, $p > q$. Then the two newly generated sub-problems are:

$$\text{subr. II} = \text{subr. I} + \text{constraint } x_i \leq \mathbf{E}\left(\frac{p}{q}\right),$$

$$\text{subr. III} = \text{subr. I} + \text{constraint } x_i \geq \mathbf{E}\left(\frac{p}{q} + 1\right),$$

where \mathbf{E} is a **floor and ceiling function**.

Each of these two regions initiate a new linear problem, which is simply the original LP together with an additional constraint. Then, these two new LPs are solved. A process will be referred to as **branching**. If an integer-valued solution results from one of these new linear problems, then it becomes a candidate solution for the original integer linear problem. Otherwise, the corresponding region is again partitioned, and the process repeats itself. In the end, after exhausting all cases, we get the solution to the integer programming problem.

Example 26. Let us consider a production problem in the toy company, which manufactures plush toy dogs and cats. Both toys are comprised of two grades of materials, labeled grades A and B. Each toy dog requires 2 pounds of each grade material and sells for 3 PLN. The plush toy cat requires 1 pound of grade A and 3 pounds of grade B and sells for 4 PLN. Suppose the daily availability of materials is given by 6 pounds for grade A and 9 pounds for grade B. Under the assumption the company sells all the toys that it produces, determine how many toys of each type should be produced, subject to the given constraints, so as to maximize daily revenue.

Solution. Let us start with the mathematical model. Let x_1 and x_2 denote the numbers of produced toy dogs and toy cats, respectively, the mathematical model of the problem can be formulated as:

$$z = 3x_1 + 4x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \end{cases}; \quad x_1, x_2 \geq 0, \quad x_1, x_2 \in Z$$

where Z is a set of integer numbers, and objective function z – represents the profit. Note that, in particular, we require that x_1 and x_2 be integer-valued in the solution, so it is an integer linear programming problem. The feasible region of the problem with integer solutions marked is shown in Figure 4.1.

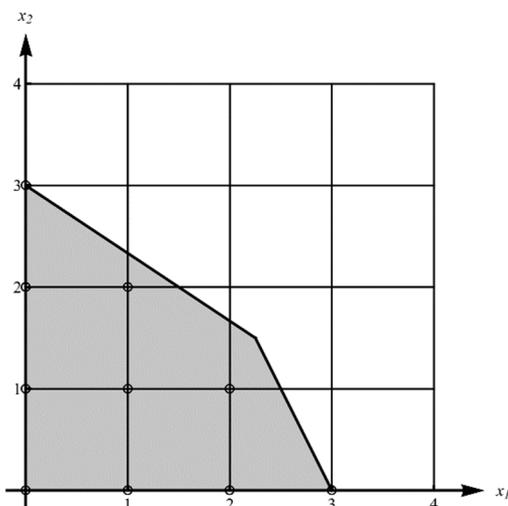


Figure 4.1. Feasible region and integer solution for ILP

These points are candidates for the optimal solution of the original integer problem. These kinds of candidates are called feasible “**lattice points**”.

For a problem with two decision variables, one of the ways to solve the integer linear problem is to evaluate the objective function at all feasible lattice points and to determine which produces the largest objective value. If the constraint’s inequalities produced a bounded feasible region, then only a finite number of candidates exist from which to choose. The graphical inspection of feasible region (Fig. 4.1) yields the following lattice points for **Example 26**:

$$(0,0) \rightarrow z = 0; \quad (1,0) \rightarrow z = 3; \quad (2,0) \rightarrow z = 6; \quad (3,0) \rightarrow z = 9;$$

$$(0,1) \rightarrow z = 4; \quad (0,2) \rightarrow z = 8; \quad (0,3) \rightarrow z = 12; \quad (1,1) \rightarrow z = 7;$$

$$(1,2) \rightarrow z = 11; \quad (2,1) \rightarrow z = 10;$$

So, the optimal solution is the point with the largest objective value: $(0,3) \rightarrow z = 12$.

For small scale problems, this approach is satisfactory, but for larger scale problems, it proves extremely inefficient, and the other method of solution is needed, which is the branch and bound method. The first step of this method is the linear relaxation of the ILP, for **Example 26**, it is given by the following mathematical model:

$$\begin{aligned} z &= 3x_1 + 4x_2 \rightarrow \max \\ \text{s.t. } &\begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \end{cases} \quad x_1, x_2 \geq 0, \quad x_1, x_2 \in Z \end{aligned}$$

The relaxation solution is straightforward to compute (for example by graphical method) and is given by:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} \\ 3 \\ \frac{3}{2} \end{bmatrix} \Rightarrow z = \frac{51}{4} = 12.75.$$

The linear relaxation gives a solution in which decision variables are not-integer valued. A naive approach to solving the ILP is rounding off the relaxation solution to the nearest lattice point. Unfortunately, this approach can yield the wrong results: the ILP solution determined by graphical inspection of feasible region is given by $x_1 = 0, x_2 = 3 \Rightarrow z = 12$. Rounding down the relaxation solution gives: $x_1 = 2, x_2 = 1 \Rightarrow z = 10$, which is the feasible solution but not optimal. On the other hand, rounding up $x_1 = 3, x_2 = 2 \Rightarrow z = 17$ is not a feasible solution.

Coming back to the branch and bound method, the partition of the feasible region will be considered. Neither of the decision variables in the relaxation solution is integer-valued, so to partition the feasible region into two disjoint regions, any decision variables can be chosen to branch. Let us select branch variable x_1 . Two new linear problems are obtained, the first one with additional constraint $x_1 \leq \mathbf{E}\left(\frac{9}{4}\right) = 2$ and the second one with new constraint $x_1 \geq \mathbf{E}\left(\frac{9}{4} + 1\right) = 3$:

$$\text{I. } z = 3x_1 + 4x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \\ x_1 \leq 2 \end{cases} \quad x_1, x_2 \geq 0$$

$$\text{II. } z = 3x_1 + 4x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \\ x_1 \geq 3 \end{cases} \quad x_1, x_2 \geq 0$$

and the following solutions are obtained:

$$\text{I. } x_1 = 2, x_2 = 1.6 \Rightarrow z = 12.6$$

$$\text{II. } x_1 = 3, x_2 = 0 \Rightarrow z = 9$$

The second solution is integer-valued and a candidate for the optimal solution of original LP.

So, now we branch the problem I, the solution of I produces a value of x_2 that is not integer-valued, so we branch on x_2 . The new sub-problems are:

$$\text{III. } z = 3x_1 + 4x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \\ x_1 \leq 2 \\ x_2 \leq 1 \end{cases} \quad x_1, x_2 \geq 0$$

$$\text{IV. } z = 3x_1 + 4x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \\ x_1 \leq 2 \\ x_2 \geq 2 \end{cases} \quad x_1, x_2 \geq 0$$

and the respective solutions are as follows:

$$\text{III. } x_1 = 2, x_2 = 1 \Rightarrow z = 10$$

$$\text{IV. } x_1 = 1.5, x_2 = 2 \Rightarrow z = 12.5$$

These results indicate that $(x_1 = 2, x_2 = 1)$ is also a candidate solution. Because its objective value is larger than that produced by the first candidate solution, $(x_1 = 3, x_2 = 0 \Rightarrow z = 9)$, we eliminate the first candidate from further consideration.

The solution of IV produces a value of x_1 that is not integer-valued, so we branch on x_1 , creating two new LPs:

$$\text{V. } z = 3x_1 + 4x_2 \rightarrow \max \quad \text{s.t.} \quad \begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \\ x_1 \leq 2 \\ x_2 \geq 2 \\ x_1 \leq 1 \end{cases} \quad x_1, x_2 \geq 0$$

$$\text{VI. } z = 3x_1 + 4x_2 \rightarrow \max \quad \text{s.t.} \quad \begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \\ x_1 \leq 2 \\ x_2 \geq 2 \\ x_1 \geq 2 \end{cases} \quad x_1, x_2 \geq 0$$

The respective solutions are as follows:

$$\text{V. } x_1 = 1, x_2 = 2.3 \Rightarrow z = 12.3$$

VI. infeasible.

Branching V on x_2 gives two more LPs:

$$\text{VII. } z = 3x_1 + 4x_2 \rightarrow \max$$

$$\text{s.t.} \quad \begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \\ x_1 \leq 2 \\ x_2 \geq 2 \\ x_1 \leq 1 \\ x_2 \leq 2 \end{cases} \quad x_1, x_2 \geq 0$$

$$\text{VIII. } z = 3x_1 + 4x_2 \rightarrow \max$$

$$\text{s.t.} \quad \begin{cases} 2x_1 + x_2 \leq 6 \\ 2x_1 + 3x_2 \leq 9 \\ x_1 \leq 2 \\ x_2 \geq 2 \\ x_1 \leq 1 \\ x_2 \geq 3 \end{cases} \quad x_1, x_2 \geq 0$$

Linear programming problems VII and VIII produce solutions of $\mathbf{x} = [1 \ 2]^T$ and $\mathbf{x} = [0 \ 3]^T$ respectively, with corresponding objective values $z = 11$ and $z = 12$. Both are candidate solutions. At this stage, the branching process is complete, and four candidate solutions are determined:

$\mathbf{x} = [0 \ 3]^T$ has already been eliminated from consideration, and

$\mathbf{x} = [1 \ 2]^T$, $\mathbf{x} = [2 \ 1]^T$, $\mathbf{x} = [0 \ 3]^T$,

the last yields the largest objective value of $z = 12$. Thus, the original ILP has its solution given by $\mathbf{x} = [0 \ 3]^T$. So, the company should produce 0 toy dogs and 3 toy cats to maximize daily revenue.

4.3. Binary and mixed integer linear problem

A special case of integer programming that appears often in real-world applications is mixed integer linear programming problems, and binary integer linear problems, in which some, but not all decision variables must be integer-valued in the solution or must take the value 0 or 1. Solving these problems is easily accomplished using the branch and bound method, with branching done only on these variables required to take the particular values – integer, 0, or 1.

Example 27: (Binary ILP) There are four possible projects which run for three years and have the following characteristics:

project	return	capital requirements		
		year 1	year 2	year 3
1	0.2	0.5	0.3	0.2
2	0.3	1	0.5	0.2
3	0.5	1.5	1.5	0.3
4	0.1	0.1	0.4	0.1
available capital		3.1	2.5	0.4

A businessman wants to know which project to choose to maximize the total profit. Use the branch and bound method for solving this problem.

Solution. Let us assume

$$x_j = \begin{cases} 1, & \text{project } j \text{ is select} \\ 0, & \text{project } j \text{ is not select} \end{cases}$$

this is the binary linear programming problem with following mathematical model

$$\begin{aligned}
 \text{I.} \quad & z = 0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4 \rightarrow \max \\
 \text{s.t.} \quad & \begin{cases} 0.5x_1 + x_2 + 1.5x_3 + 0.1x_4 \leq 3.1 \\ 0.3x_1 + 0.8x_2 + 1.5x_3 + 0.4x_4 \leq 2.5 \\ 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 \leq 0.4 \\ x_j \geq 0, x_j \in \{0,1\}, j=1,2,3,4 \end{cases}
 \end{aligned}$$

The first step in the solution is to replace $x_j \in \{0,1\}$ by $0 \leq x_j \leq 1$, which give the linear relaxation problem with the following solution:

$$x_1 = x_4 = 0, x_2 = 0.5, x_3 = 1 \Rightarrow z = 0.65.$$

The variable $x_2 = 0.5$ does not meet the binary condition, as it is fractional. Branching on x_2 generates two new problems:

$$\begin{aligned}
 \text{I.} \quad & z = 0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4 \rightarrow \max \\
 \text{s.t.} \quad & \begin{cases} 0.5x_1 + x_2 + 1.5x_3 + 0.1x_4 \leq 3.1 \\ 0.3x_1 + 0.8x_2 + 1.5x_3 + 0.4x_4 \leq 2.5 \\ 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 \leq 0.4 \\ x_2 = 0 \\ x_j \geq 0, 0 \leq x_j \leq 1, j=1,2,3,4 \end{cases}
 \end{aligned}$$

with solution: $x_2 = x_4 = 0, x_1 = 0.5, x_3 = 1 \Rightarrow z = 0.6$, and

$$\begin{aligned}
 \text{II.} \quad & z = 0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4 \rightarrow \max \\
 \text{s.t.} \quad & \begin{cases} 0.5x_1 + x_2 + 1.5x_3 + 0.1x_4 \leq 3.1 \\ 0.3x_1 + 0.8x_2 + 1.5x_3 + 0.4x_4 \leq 2.5 \\ 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 \leq 0.4 \\ x_2 = 1 \\ x_j \geq 0, 0 \leq x_j \leq 1, j=1,2,3,4 \end{cases}
 \end{aligned}$$

with solution: $x_1 = x_4 = 0, x_2 = 1, x_3 = 0.67 \Rightarrow z = 0.63$.

Choosing problem II (the larger value of the objective function) and branching on x_3 (is fractional), yields

$$\begin{array}{l}
 \text{III.} \quad z = 0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4 \rightarrow \max \\
 \text{s.t.} \quad \begin{cases} 0.5x_1 + x_2 + 1.5x_3 + 0.1x_4 \leq 3.1 \\ 0.3x_1 + 0.8x_2 + 1.5x_3 + 0.4x_4 \leq 2.5 \\ 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 \leq 0.4 \\ x_2 = 1 \\ x_3 = 0 \end{cases} \\
 x_j \geq 0, 0 \leq x_j \leq 1, \quad j=1,2,3,4
 \end{array}$$

with solution: $x_1 = x_2 = 1, x_3 = x_4 = 0, \Rightarrow z = 0.5$, and

$$\begin{array}{l}
 \text{IV.} \quad z = 0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4 \rightarrow \max \\
 \text{s.t.} \quad \begin{cases} 0.5x_1 + x_2 + 1.5x_3 + 0.1x_4 \leq 3.1 \\ 0.3x_1 + 0.8x_2 + 1.5x_3 + 0.4x_4 \leq 2.5 \\ 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 \leq 0.4 \\ x_2 = 1 \\ x_3 = 1 \end{cases} \\
 x_j \geq 0, 0 \leq x_j \leq 1, \quad j=1,2,3,4
 \end{array}$$

with solution: infeasible.

The solution of problem III is feasible for the original ILP and is a candidate solution with $z = 0.5$.

Choosing the only remaining problem I, and branching on x_1 ,

$$\begin{array}{l}
 \text{V.} \quad z = 0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4 \rightarrow \max \\
 \text{s.t.} \quad \begin{cases} 0.5x_1 + x_2 + 1.5x_3 + 0.1x_4 \leq 3.1 \\ 0.3x_1 + 0.8x_2 + 1.5x_3 + 0.4x_4 \leq 2.5 \\ 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 \leq 0.4 \\ x_2 = 0 \\ x_1 = 0 \end{cases} \\
 x_j \geq 0, 0 \leq x_j \leq 1, \quad j=1,2,3,4
 \end{array}$$

with solution: $x_3 = x_4 = 1, x_1 = x_2 = 0, \Rightarrow z = 0.6$, and

$$\begin{aligned}
 \text{VI.} \quad & z = 0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4 \rightarrow \max \\
 \text{s.t.} \quad & \begin{cases} 0.5x_1 + x_2 + 1.5x_3 + 0.1x_4 \leq 3.1 \\ 0.3x_1 + 0.8x_2 + 1.5x_3 + 0.4x_4 \leq 2.5 \\ 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 \leq 0.4 \\ x_2 = 0 \\ x_1 = 1 \\ x_j \geq 0, 0 \leq x_j \leq 1, j=1,2,3,4 \end{cases}
 \end{aligned}$$

with solution: $x_2 = x_4 = 0, x_1 = 1, x_3 = 0.67 \Rightarrow z = 0.53$.

Problem V is new candidate solution with $z = 0.6$.

Problem VI does not meet the binary condition (the variable $x_3 = 0.67$ is fractional), and cannot be branched, value of the objective function is less than the current z – value and cannot yield the optimal solution to the original binary problem.

So, the optimal solution to the binary integer problem is

$x_3 = x_4 = 1, x_1 = x_2 = 0, \Rightarrow z = 0.6$. This means that the businessman should choose project number 3 and 4 to maximize the return.

Exercises

16. Find the optimal solution to the mixed ILP using the branch and bound method.

$$\begin{aligned}
 & z = x_1 + 3x_2 + 3x_3 \rightarrow \max \\
 \text{s.t.} \quad & \begin{cases} x_1 + 3x_2 + 2x_3 \leq 7 \\ 2x_1 + 2x_2 + x_3 \leq 11 \\ x_1, x_2, x_3 \geq 0, x_1, x_3 \in Z \end{cases}
 \end{aligned}$$

17. A woman is organizing a housewarming party. Her friends decided to buy the presents together, and they have 3000 PLN available. She gave them a list of things that she needed with preference on a scale of 2 to 5. The thing that is needed the least is worth 2 points, and the thing that is most needed is worth 5 points.

No.	Presents	Price [PLN]	Preference
1.	robot Vacuum Cleaner, 3 In 1	2000	5
2.	set of cups	150	2
3.	champagne glasses	150	5
4.	cognac glasses	200	4
5.	potted flowers	250	3

What things should friends buy to meet the person's needs and not exceed the amount available?

CHAPTER 5

COMPUTER IMPLEMENTATION

The purpose of this chapter is to provide a quick introduction to the computer algebra system Maple and, more importantly, to demonstrate these aspects of Maple that can be useful in modeling and solving a wide range of mathematical programming problems.

Maple is one of the mathematical types of software known as Computer Algebra System (CAS), it is a mathematical problem-solving environment for advanced mathematics that gives us the ability to work with mathematical expressions in a way similar to the traditional computations. It combines symbolic, numerical and graphical methods used in different areas of mathematics, for example, algebra, calculus, discrete mathematics, and many other areas.

5.1. An introduction to using Maple package

A large amount of information about the Maple package can be found in the “Maple User Manual” or the “Maple Fundamentals Guide” that are available at Maple’s web site: www.maplesoft.com. On the other hand, Maple has an extensive Help System. To obtain help about a command, it is enough to type the question mark “?”, and then the name of command.

One of the advanced features of Maple is its document interface where we can perform certain calculations in a “what you see is what you get” style. Computation in this style is convenient and useful.

There are a few rules that are important for Maple’s users:

- Maple takes case-sensitivity into account,
- every command line entered into Maple must end with either a semicolon (i.e., ;) or a colon (i.e., :). If a semicolon is at the end of a statement, then the output will be displayed on the screen, while if a colon is used, then the output is suppressed - although the statement is still executed. To execute a command, press the Enter key, while the blinking cursor is anywhere in the statement. The cursor does not need to be at the end of the statement in order to execute it,

- the assignment operation ($:=$) allows the user to name or value Maple objects – variables and other types of structures,
- Maple remembers the order in which commands are executed in the worksheet. To clear all values in the worksheet, write “restart” at the first command line and execute the worksheet again,
- to add the line below the current cursor line type key combination: “CTRL – J”, to insert additional line above, type “CTRL – K” instead,
- to add any notes in the command line, first type the “#” sign.

In this chapter, the solutions of specific problems of linear programming with the Maple package will be presented. Every Maple worksheet displayed in this chapter has been tested with Maple 15.

5.2. Linear algebra with Maple

In this section of Chapter 5, the Maple packages including “*LinearAlgebra*”, and “*VectorCalculus*” are used. To find out something about a package’s concrete command, just type the “question sign(?) and command name”.

The “*Linear Algebra*” package has the ability to perform a wide variety of operations on matrices, vectors, and the linear system of equations. To load this package, give the following command:

```
restart;
with(LinearAlgebra):.
```

If you load any package using the “with” command by ending with a semicolon, Maple shows the names of all commands defined by the package. If you end a command with a colon these definitions will not displayed.

Maple has different possibilities to construct the matrices,

- matrix command: “Matrix(m, n, L)” where m – denotes the number of rows, and n – number of columns, L – is a list of elements, reading across the rows,

```
restart;
with(LinearAlgebra) :
A := Matrix(2, 3, [1, 2, 3, 4, 5, 6])
```

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- column vector notation:
we can use the angle bracket shortcut, $\langle \rangle$ to describe a column vector, separate entries with commas, and separate columns with vertical bars, $|$ in matrices

restart;

with(LinearAlgebra) :

$k1 := \langle 1, 4 \rangle;$

$$k1 := \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$k2 := \langle 2, 5 \rangle;$

$$k2 := \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$k3 := \langle 3, 6 \rangle;$

$$k3 := \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$A := \langle k1|k2|k3 \rangle;$

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- use the matrix palette to interactively create a matrix without command. In the matrix palette, the specification of the matrix size and properties is available. To insert a matrix, just click the insert Matrix button

$$\begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix}$$

enter the values of the entries; to move to the next entry placeholder, press **Tab**. After specifying all entries, press **Enter**.

Below, several “LinearAlgebra” package commands are shown in examples:

Example 28: Compute the determinant of matrix $\mathbf{X} = \begin{bmatrix} 1 & 3 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 4 \\ -1 & 2 & 4 & 5 \end{bmatrix}$:

Solution

restart;

with(LinearAlgebra) :

$X := \text{Matrix}(4, 4, [1, 3, 2, 3, 1, 3, 2, 0, 2, 1, 4, 4, -1, 2, -2, 5])$

$$\begin{bmatrix} 1 & 3 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 4 \\ -1 & 2 & -2 & 5 \end{bmatrix}$$

$\text{Determinant}(X)$

0.

Example 29: Solve the matrix equation $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 7 \\ 1 \\ 10 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution

restart;

with(LinearAlgebra) :

$A := \text{Matrix}(3, 3, [1, -2, 3, 2, 5, 2, 3, 1, 5]);$

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$b := \langle 7, 1, 10 \rangle;$

$$\begin{bmatrix} 7 \\ 1 \\ 10 \end{bmatrix}$$

$M := \langle A|b \rangle;$

$$\begin{bmatrix} 1 & -2 & 3 & 7 \\ 2 & 5 & 2 & 1 \\ 3 & 1 & 5 & 10 \end{bmatrix}$$

$\text{GaussianElimination}(M)$

$$\begin{bmatrix} 1 & -2 & 3 & 7 \\ 0 & 9 & -4 & -13 \\ 0 & 0 & -\frac{8}{9} & -\frac{8}{9} \end{bmatrix}$$

ReducedRowEchelonForm($\langle A|b \rangle$);

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The solution is: $x = \begin{bmatrix} x_1 = 2 \\ x_2 = -1 \\ x_3 = 1 \end{bmatrix}$.

5.3. A step-by-step Maple implementation to solve the linear programming problem

5.3.1. The graphical solution of LP

To start with graphical solution, the plot of the feasible region for a linear problem is necessary. In Maple, the plot commands are contained in the *plots* package, so first type “*with(plots)*” in the current worksheet. The region in the plane that consist of points satisfying the given inequalities, or constraints of the linear problem, can be graphed using the command “*inequal(inequalities, options)*”, the inequalities must be linear and enclosed within brackets and separated by commas. The options impose the plot region, the color of the area satisfying the inequalities, (*optionsfeasible*), the color of the boundary of the area, (*optionsclosed*), and the color of the points not satisfying at least one inequality, (*optionsexcluded*).

Example 30: Use the graphical method to solve the problem:

$$z = f(x_1, x_2) = 2x_1 + x_2 \rightarrow \max$$

$$\text{s.t.} \begin{cases} 2x_1 \leq 10 \\ 2x_1 + 5x_2 \leq 60 \\ x_1 + x_2 \leq 18 \\ 3x_1 + x_2 \leq 44 \\ x_1, x_2 \geq 0 \end{cases}$$

Solution: the following Maple commands were used to obtain the solution:

restart;

with(plots) :

$f := (x1, x2) \rightarrow 2 \cdot x1 + x2;$

$CS := [x2 \leq 10, 2 \cdot x1 + 5 \cdot x2 \leq 60, x1 + x2 \leq 18, 3 \cdot x1 + x2 \leq 44, x1 \geq 0, x2 \geq 0];$

$FR := \text{inequal}(CS, x1 = 0 \dots 15, x2 = 0 \dots 15, \text{optionsfeasible} = (\text{color} = \text{grey}), \text{optionsexcluded} = (\text{color} = \text{white}), \text{optionsclosed} = (\text{color} = \text{black}), \text{thickness} = 2) :$

$PL := \text{contourplot}(f(x1, x2), x1 = 0 \dots 20, x2 = 0 \dots 20, \text{contours} = [10, 15, 20, 31, 35], \text{thickness} = 2, \text{linestyle} = 4) :$

display(FR, PL);

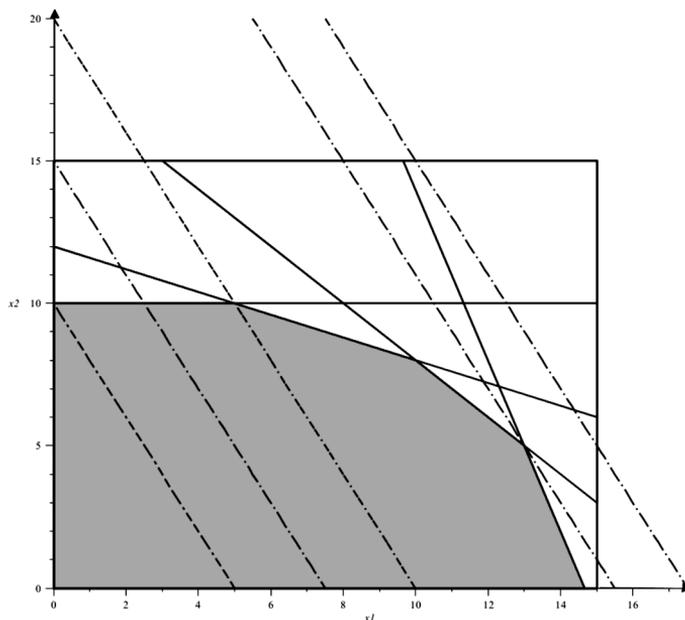


Figure 5.1. Feasible region with izoprofit lines for $z = 10, 15, 20, 31, 35$

$sol := \text{solve}(\{3 \cdot x1 + x2 = 44, x1 + x2 = 18, \})$

$\{x1 = 13, x2 = 5\}$

$Point := \text{pointplot}([\text{rhs}(sol[1]), \text{rhs}(sol[2])], \text{color} = [\text{green}], \text{thickness} = 6, \text{symbolsize} = 20, \text{symbol} = \text{solidcircle}) :$

display(FR, PL, Point);

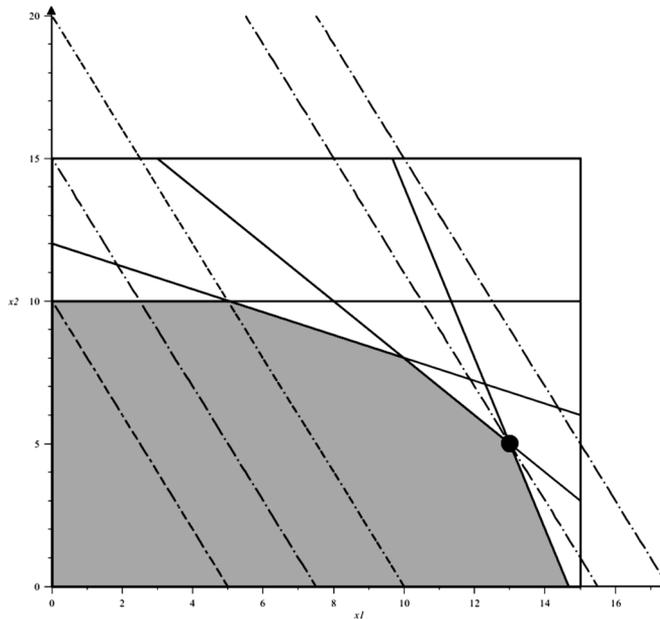


Figure 5.2. Feasible region with optimal solution point

$$f(x_1 = 13, x_2 = 5)$$

$$2x_1 + x_2 = 31.$$

where:

CS – describe the set of constraints,

FR – feasible region,

PL – isoprofit lines for $z = 10, 15, 20, 31, 35$

Point – optimal solution, intersection point for the third and fourth constraints.

5.3.2. The Simplex method

To solve any linear programming problem, Maple has a package called “simplex”. To load this package, just type “*with(simplex)*”, then define the objective function, constraints, and then the option: maximize or minimize.

Example 31: $z = f(x_1, x_2) = 4x_1 + 3x_2 \rightarrow \max$

$$\text{s.t. } \begin{cases} x_1 \leq 8 \\ 2x_1 + 2x_2 \leq 28 \\ 3x_1 + 3x_2 \leq 32 \\ x_1, x_2 \geq 0 \end{cases}$$

Solution

restart, with(simplex) :

obj := 4·x1 + 3·x2;

*cons := {x1 ≤ 8, 2·x1 + 2·x2 ≤ 28, 3·x1 + 3·x2 ≤ 32, x1 ≥ 0,
x2 ≥ 0};*

maximize(obj, cons)

$$\left\{ x1 = 8, x2 = \frac{8}{3} \right\}$$

Sol := maximize(obj, cons, NONNEGATIVE);

$$Sol := \left\{ x1 = 8, x2 = \frac{8}{3} \right\};$$

to find the optimal value of the objective function, write:

subs(cornerpoint, obj);

40

or

assign(Sol); obj;

40.

Example 32:

$$z = f(x_1, x_2) = x_1 + x_2 \rightarrow \min$$

$$\text{s.t. } \begin{cases} 4x_1 + 3x_2 \leq 5 \\ 3x_1 + 4x_2 \leq 4 \end{cases} \quad x_1, x_2 \geq 0$$

Solution:

restart, with(simplex) :

obj := x1 + x2;

$$obj := x1 + x2$$

*Sol := minimize(obj, {4·x1 + 3·x2 ≤ 5, 3·x1 + 4·x2 ≤ 4},
NONNEGATIVE);*

$$Sol := \{x1 = 0, x2 = 0\}$$

```
subs(Sol, obj);
```

0

The integer programming problem can be solved with Maple directly by using package “Optimization”. To load this package, give the following command:

```
„restart; with(Optimization) : ”.
```

To solve an ILP having no sign restrictions, use Maple’s “LPSolve” command, which is available in optimization package, and simply add the option “assume=integer”. For an ILP requiring nonnegative decision variable values, add “assume=nonnegint” instead. For a mixed ILP, the option “integervariables=[list]” specifies which particular variables are required to take on integer variables. For a binary LP, adding “assume=binary” requires all decision variables equal to 0 or 1 in the solution, whereas “binaryvariables=[list]” specifies which subset of the decision variables have binary values. You can find a sample worksheet illustrating the use of these command options in the following examples:

Example 33: Solve the given linear programming problems directly with Maple:

a) the LP problem is given in matrix form

$$\begin{aligned} z &= \mathbf{c} \cdot \mathbf{x} \rightarrow \mathbf{min} \\ \text{s.t. } \mathbf{A} \cdot \mathbf{x} &\geq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}, \text{ and integer} \end{aligned}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{c} = [50 \ 100], \quad \mathbf{b} = \begin{bmatrix} 28 \\ 24 \end{bmatrix}, \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 7 & 2 \\ 2 & 12 \end{bmatrix}.$$

Solution.

```
restart;
```

```
with(Optimization) :
```

```
with(LinearAlgebra) :
```

```
c := Vector[row]([50, 100]);
```

```
A := Matrix(2, 2, [7, 2, 2, 12]);
```

```
b := <28, 24>;
```

$x := \langle x1, x2 \rangle;$

$ConstraintMatrix := A.x - b;$

$constraints := seq(ConstraintMatrix[i] \geq 0, i = 1 ..2);$

$LPSolve(c.x, [constraints], assume$
 $= nonnegint);$

$[400, [x1 = 4, x2 = 2]]$

b) $z = 0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4 \rightarrow \max$

$$\text{s.t. } \begin{cases} 0.5x_1 + x_2 + 1.5x_3 + 0.1x_4 \leq 3.1 \\ 0.3x_1 + 0.8x_2 + 1.5x_3 + 0.4x_4 \leq 2.5 \\ 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 \leq 0.4 \end{cases}$$

$$x_i = 0 \text{ or } 1, \quad i = 1, 2, 3, 4$$

Solution.

restart;

with(Optimization) :

with(LinearAlgebra) :

$c := Vector[row]([0.2, 0.3, 0.5, 0.1]);$

$A := Matrix(3, 4, [0.5, 1, 1.5, 0.1, 0.3, 0.8, 1.5, 0.4,$
 $0.2, 0.2, 0.3, 0.1]);$

$b := \langle 3.1, 2.5, 0.4 \rangle;$

$x := \langle x1, x2, x3, x4 \rangle;$

$ConstraintMatrix := A.x - b;$

$constraints := seq(ConstraintMatrix[j] \leq 0, j = 1$
 $..3)$

$LPSolve(c.x, [constraints], assume = 'binary', 'maximize');$

$[0.6, [x1 = 0, x2 = 0, x3 = 1, x4 = 1]] .$

5.3.3. Transportation problem

To solve the transportation problem with Maple, the “Simplex” package can be used. The example shows the Maple commands that can be used to obtain the transportation problem solution.

Example 34: There are 3 Parties who supply the following quantity of coal P1 = 14t, P2 = 12t, P3 = 5t. There are 3 consumers who require the coal as follows C1 = 6t, C2 = 10t, C3 = 15t. The cost matrix is given below. Find the schedule of transportation policy which minimizes the cost.

From	To		
	C1	C2	C3
P1	6	8	4
P2	4	9	3
P3	1	2	6

Solution. The mathematical model of the problem is:

$$z = 6x_{11} + 8x_{12} + 4x_{13} + 4x_{21} + 9x_{22} + 3x_{23} + x_{31} + 2x_{32} + 6x_{33} \rightarrow \min$$

$$\text{s.t.} \begin{cases} x_{11} + x_{12} + x_{13} = 14 \\ x_{21} + x_{22} + x_{23} = 12 \\ x_{31} + x_{32} + x_{33} = 5 \\ x_{11} + x_{21} + x_{31} = 6 \\ x_{12} + x_{22} + x_{32} = 10 \\ x_{13} + x_{23} + x_{33} = 15 \\ x_{ij} \geq 0 \end{cases}$$

and the solution with Maple:

restart;

with(LinearAlgebra) : with(simplex) :

Cost := Matrix(3, 3, [6, 8, 4, 4, 9, 3, 1, 2, 6]);

$$Cost := \begin{bmatrix} 6 & 8 & 4 \\ 4 & 9 & 3 \\ 1 & 2 & 6 \end{bmatrix}$$

$x := \text{array}(1..3, 1..3) :$

$TotalCost := \text{add}(\text{add}(Cost[i, j] \cdot x[i, j], i = 1..3), j = 1..3);$

$$\begin{aligned} TotalCost := & 6x_{1,1} + 4x_{2,1} + x_{3,1} + 8x_{1,2} \\ & + 9x_{2,2} + 2x_{3,2} + 4x_{1,3} + 3x_{2,3} \\ & + 6x_{3,3} \end{aligned}$$

$Demand := \text{Vector}([6, 10, 15]);$

$$Demand := \begin{bmatrix} 6 \\ 10 \\ 15 \end{bmatrix}$$

$Supply := \text{Vector}([14, 12, 5]);$

$$Supply := \begin{bmatrix} 14 \\ 12 \\ 5 \end{bmatrix}$$

$CSD := \text{seq}(\text{add}(x[i, j], i = 1..3) = Demand[j], j = 1..3);$

$$\begin{aligned} CSD := & x_{1,1} + x_{2,1} + x_{3,1} = 6, x_{1,2} + x_{2,2} \\ & + x_{3,2} = 10, x_{1,3} + x_{2,3} + x_{3,3} = 15 \end{aligned}$$

$CSS := \text{seq}(\text{add}(x[i, j], j = 1..3) = Supply[i], i = 1..3);$

$$\begin{aligned} CSS := & x_{1,1} + x_{1,2} + x_{1,3} = 14, x_{2,1} + x_{2,2} + x_{2,3} = 12, x_{3,1} \\ & + x_{3,2} + x_{3,3} = 5 \end{aligned}$$

$CS := \{CSD, CSS\};$

$sol := \text{minimize}(TotalCost, CS, \text{NONNEGATIVE});$

$$\begin{aligned} sol := & \{x_{1,1} = 0, x_{1,2} = 5, x_{1,3} = 9, x_{2,1} = 6, x_{2,2} = 0, x_{2,3} = 6, x_{3,1} \\ & = 0, x_{3,2} = 5, x_{3,3} = 0\} \end{aligned}$$

$\text{assign}(sol); TotalCost;$

Exercises

18. Solve the following problems graphically with Maple software.

a. $z = f(x_1, x_2) = 4x_1 + x_2 \rightarrow \min$

$$\text{s.t. } \begin{cases} 3x_1 + x_2 \geq 6 \\ 4x_1 + x_2 \geq 12 \\ x_1 \geq 2 \\ x_1, x_2 \geq 0 \end{cases}$$

b. $z = f(x_1, x_2) = -4x_1 + x_2 \rightarrow \max$

$$\text{s.t. } \begin{cases} 3x_1 + 2x_2 \leq 8 \\ x_1 + x_2 \leq 12 \\ x_1 \geq 0, x_2 \text{ u.r.s.} \end{cases}$$

c. $z = f(x_1, x_2) = 5x_1 - x_2 \rightarrow \max$

$$\text{s.t. } \begin{cases} 2x_1 + 3x_2 \geq 12 \\ x_1 - 3x_2 \geq 0 \\ x_1, x_2 \geq 0 \end{cases}$$

d. $z = f(x_1, x_2) = x_1 + 5x_2 \rightarrow \max$

$$\text{s.t. } \begin{cases} x_1 - 3x_2 \geq 0 \\ x_1 + x_2 \leq 8 \\ x_1, x_2 \geq 0 \end{cases}$$

e. $z = f(x_1, x_2) = 5x_1 + x_2 \rightarrow \min$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \geq 6 \\ x_1 + x_2 \geq 4 \\ x_1 + 5x_2 \geq 10 \\ x_1, x_2 \geq 0 \end{cases}$$

f. $z = f(x_1, x_2) = 4x_1 + x_2 \rightarrow \min$

$$\text{s.t. } \begin{cases} 3x_1 + x_2 \geq 10 \\ x_1 + x_2 \geq 5 \\ x_1 \geq 3 \end{cases}$$

$$x_1, x_2 \geq 0$$

19. Use the simplex algorithm to solve the following problems.

a) Marc's physics teacher has given his class three long lists of problems with the instruction to submit no more than 100 of them for credit (only problems correctly solved). The problems in the first set are worth 5 points each, those in the second set are worth 4 points each, and the problems in the last set are worth 6-points each. Mark knows that he requires on average 3 minutes to solve a 5-point problem, 2 minutes to solve a 4-point problem, and 4 minutes to solve a 6-point one. He cannot afford to devote more than 3.5 hours altogether on this physics assignment. Moreover, the first two sets of problems involve numerical calculations, and he knows that he cannot stand more than 2.5 hours of work on this type of problem. How many problems in each of the three categories shall he solve in order to get the maximum possible credit for his effort?

b) $z = f(x_1, x_2) = 2x_1 + 3x_2 \rightarrow \max$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \leq 4 \\ x_1 + 2x_2 \leq 5 \end{cases}$$

$$x_1, x_2 \geq 0$$

c) $z = f(x_1, x_2) = 2x_1 - 3x_2 \rightarrow \min$

$$\text{s.t. } \begin{cases} x_1 + x_2 \leq 4 \\ x_1 - x_2 \leq 6 \end{cases}$$

$$x_1, x_2 \geq 0$$

d) $z = f(x_1, x_2) = 5x_1 + 4x_2 + 6x_3 - 8x_4 \rightarrow \min$

$$\text{s.t. } \begin{cases} x_1 + 2x_2 + 2x_3 + 4x_4 \leq 40 \\ 2x_1 - 2x_2 + x_3 + 2x_4 \leq 8 \\ 4x_1 - 2x_2 + x_3 - x_4 \leq 10 \end{cases}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$e) z = f(x_1, x_2) = 3x_1 + x_2 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 2x_1 + x_2 \leq 4 \\ x_1 + x_2 = 3 \\ x_1, x_2 \geq 0 \end{cases}$$

$$f) z = f(x_1, x_2) = 4x_1 + 4x_2 + x_3 \rightarrow \min$$

$$\text{s.t. } \begin{cases} x_1 + x_2 + x_3 \leq 2 \\ 2x_1 + x_2 \leq 3 \\ 2x_1 + x_2 + 3x_3 \geq 3 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

20. Consider the transportation problem with n supply points and m demand points. The dates are shown in the table. Find the schedule of transportation which minimizes the total cost.

$$a) n = 3, m = 4$$

From	To				Supply
	D1	D2	D3	D4	
S1	4	1	3	4	60
S2	2	3	2	3	35
S3	3	5	2	4	45
Demand	22	15	18	35	

$$b) n = 3, m = 4$$

From	To				Supply
	D1	D2	D3	D4	
S1	5	7	3	8	500
S2	24	6	9	5	400
S3	32	6	4	54	200
Demand	250	450	300	400	

Supply centers are also production points, and unit production cost is: 4, 3, 5, respectively.

ANSWERS TO SELECTED PROBLEMS**2.1.** 5.**2.2.** 4.**2.3.** 6×4 .

3a.
$$\begin{cases} x = 1 \\ y = 2 \end{cases}$$

3b.
$$\begin{cases} x = 1 \\ y = 3 \end{cases}$$

3c.
$$\begin{cases} x = 2 \\ y = 2 \\ z = 2 \end{cases}$$

3d.
$$\begin{cases} z = p \\ y = 5 - 2p, \quad p \in \mathbb{R}. \\ x = 1 - p \end{cases}$$

4. $z = 41x_1 + 36x_2 \rightarrow \min$

$$\text{s.t.} \begin{cases} 20x_1 + 30x_2 \geq 110 \\ 10x_1 + 10x_2 \geq 18 \\ 50x_1 + 30x_2 \geq 90 \\ 6x_1 + 2.5x_2 \geq 14 \end{cases}, \quad x_1, x_2 \geq 0$$

5a. $x_1 = \frac{12}{5}, \quad x_2 = \frac{18}{5}, \quad z = \frac{54}{5},$

5b. $x_1 = \frac{10}{3}, \quad x_2 = \frac{2}{3}, \quad z = \frac{-10}{3},$

6a. $2a < b$ **6b.** $\frac{a}{3} < b < 2a$ **6c.** $b < \frac{a}{3}$

6d. $b = 2a$ **6e.** $b = \frac{a}{3}$

8. $x_1 = 0, \quad x_2 = 2, \quad z = 6$ **9a.** The linear program in standard form is:

$$z = 2x_1 + 3x_2 \rightarrow \max$$

$$\text{s.t.} \begin{cases} 2x_1 + x_2 + s_1 = 4 \\ x_1 + 2x_2 + s_2 = 5 \\ x_1, x_2, s_1, s_2 \geq 0 \end{cases}$$

The initial tableau – I iteration

Basic	z	x_1	x_2	s_1	s_2	RHS	ratio
z	1	-2	-3	0	0	0	
s_1	0	2	1	1	0	4	4
s_2	0	1	2	0	1	5	$\frac{5}{2}$

The actual BFS is $\mathbf{x}_1^T = [0 \ 0 \ 4 \ 5]$, $z(\mathbf{x}_1) = 0$.

III iteration – the optimal simplex tableau:

Basic	z	x_1	x_2	s_1	s_2	RHS	ratio
z	1	0	0	$\frac{1}{3}$	$\frac{4}{3}$	8	
x_1	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	1	
x_2	0	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	2	

Since there are no negative coefficients in row 0, we have reached the optimal solution

$$\mathbf{x}_{opt}^T = [1 \ 2], \quad z(\mathbf{x}_{opt}) = 8$$

9b. $\mathbf{x}_{opt}^T = [0 \ 5], \quad z(\mathbf{x}_{opt}) = -5$

9c. $\mathbf{x}_{opt}^T = \left[\frac{2}{5} \ \frac{9}{5} \right], \quad z(\mathbf{x}_{opt}) = \frac{17}{5}$

10a. $w = 10y_1 + 8y_2 \rightarrow \min$

$$\text{s.t.} \quad \begin{cases} y_1 + 2y_2 \geq 5 \\ 2y_1 - y_2 \geq 12 \\ y_1 + y_2 \geq 4 \end{cases} \quad y_1 \geq 0; y_2 \text{ u.r.s.}$$

10b. $w = 3y_1 + 7y_2 \rightarrow \max$

$$\text{s.t.} \quad \begin{cases} 2y_1 + y_2 \geq 15 \\ y_1 - 4y_2 \geq 12 \end{cases} \quad y_1 \leq 0; y_2 \geq 0.$$

$$11a. w = 6y_1 + 7y_2 \rightarrow \min$$

$$\text{s.t. } \begin{cases} 2y_1 + y_2 \geq 5 \\ y_1 + 2y_2 \geq 3 \\ y_1 + y_2 \geq 1 \end{cases} \quad y_1, y_2 \geq 0; \mathbf{y}_{opt}^T = \begin{bmatrix} 7 & 1 \\ 3 & 3 \end{bmatrix}$$

$$11b. w = 5y_1 + 8y_2 \rightarrow \min$$

$$\text{s.t. } \begin{cases} y_1 + 2y_2 \geq 3 \\ 2y_1 + 3y_2 \geq 4 \\ y_1 + y_2 \geq 1 \\ 2y_1 + 3y_2 \geq 5 \end{cases} \quad y_1, y_2 \geq 0; \mathbf{y}_{opt}^T = [1 \quad 1]$$

$$12. \mathbf{x}_{opt} = \begin{bmatrix} x_{11} = 2500 & x_{12} = 1500 & x_{13} = 0 & x_{14} = 1000 \\ x_{21} = 0 & x_{22} = 1000 & x_{23} = 3000 & x_{24} = 0 \\ x_{31} = 0 & x_{32} = 2000 & x_{33} = 0 & x_{34} = 0 \end{bmatrix}, z(\mathbf{x}_{opt}) = 69500$$

Hint. This is unbalanced problem, total supply (11000) exceeds total demand (10000), create a dummy demand point.

$$15. \mathbf{x}_{opt} = \begin{bmatrix} x_{11} = 0 & x_{12} = 0 & x_{13} = 0 & x_{14} = 25 \\ x_{21} = 0 & x_{22} = 15 & x_{23} = 0 & x_{24} = 10 \\ x_{31} = 15 & x_{32} = 5 & x_{33} = 30 & x_{34} = 0 \end{bmatrix}, z(\mathbf{x}_{opt}) = 535$$

$$16. \mathbf{x}_{opt}^T = \begin{bmatrix} 0 & \frac{1}{3} & 3 \end{bmatrix}, z(\mathbf{x}_{opt}) = 10$$

17. The mathematical model:

$$z = 5x_1 + 2x_2 + 5x_3 + 4x_4 + 3x_5 \rightarrow \max$$

$$\text{s.t. } 2000x_1 + 150x_2 + 150x_3 + 200x_4 + 250x_5 \leq 3000$$

$$x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}$$

$$18a. z = 12; x_1 = 3, x_2 = 0$$

18b. unbounded

$$18c. z = 6; x_1 = 0, x_2 = 6$$

$$18e. z = 16; x_1 = 6, x_2 = 2$$

18f. $z = 14; x_1 = 3, x_2 = 2$

19a. The mathematical model:

$$z = 5x_1 + 4x_2 + 6x_3 \rightarrow \max$$

$$\text{s.t. } \begin{cases} 3x_1 + 2x_2 + 4x_3 \leq 210 \\ 3x_1 + 2x_2 \leq 150 \end{cases} \quad x_1, x_2, x_3 \geq 0$$

19b. $z = 8; x_1 = 1, x_2 = 2$

19d. $z = -48; x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 8$

19f. $z = 1; x_1 = 0, x_2 = 0, x_3 = 1$

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