APPLICATION OF THE BOUNDARY ELEMENT METHOD USING DISCRETIZATION IN TIME FOR NUMERICAL SOLUTION OF HYPERBOLIC EQUATION

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Abstract. The hyperbolic equation (1D problem) supplemented by adequate boundary and initial conditions is considered. This equation is solved using the combined variant of the boundary element method. The problem is also solved in analytical way. The comparison of the results obtained by means of these two methods confirms the effectiveness and accuracy of the BEM.

1. Formulation of the problem

The following equation is considered

$$\frac{\partial U(x,t)}{\partial t}+\frac{\partial^2 U(x,t)}{\partial t^2}=\frac{\partial^2 U(x,t)}{\partial x^2} \tag{1}$$

where $U(x, t)$ is an unknown function, x is the spatial co-ordinate and t is the time. The equation (1) is supplemented by the boundary conditions

$$\begin{aligned} &t>0, \quad x=0: \quad U(0,t)=0 \\ &t>0, \quad x=1: \quad U(1,\ t)=0 \end{aligned} \tag{2}$$

and the initial ones

$$\begin{aligned} &0<x<1, \quad t=0: \quad U(x,0)=U_0>0 \\ &0<x<1, \quad t=0: \quad \frac{\partial U(x,t)}{\partial t}\Big|_{t=0}=0 \end{aligned} \tag{3}$$

This type of boundary and initial conditions allows to solve the problem analytically and in this way the results obtained by means of the boundary element method using discretization in time can be compared with the analytical solution.

2. Boundary element method

To solve the equation (1), the BEM using discretization in time is applied [1, 2]. So, the time grid

$$0 = t^0 < t^1 < \ldots < t^{f-2} < t^{f-1} < t^f \ldots < t^F < \infty \tag{4}$$

with constant step $\Delta t = t^f - t^{f-1}$ is introduced.

For the time interval $\left[t^{f-2}, t^f\right]$ the following approximations of time derivative can be taken into account

$$\left.\frac{\partial U(x,t)}{\partial t}\right|_{t=t^f} = \frac{U(x,t^f) - U(x,t^{f-1})}{\Delta t} \tag{5}$$

and

$$\left.\frac{\partial U(x,t)}{\partial t}\right|_{t=t^f} = \frac{U(x,t^f) - U(x,t^{f-2})}{2\,\Delta t} \tag{6}$$

or

$$\left.\frac{\partial U(x,t)}{\partial t}\right|_{t=t^f} = \frac{3U(x,t^f) - 4U(x,t^{f-1}) + U(x,t^{f-2})}{2\,\Delta t} \tag{7}$$

The second time derivative is approximated in following way

$$\frac{\partial^2 U(x,\ t)}{\partial t^2} = \frac{U(x,t^f) - 2U(x,\ t^{f-1}) + U(x,\ t^{t-2})}{(\Delta t)^2} \tag{8}$$

Let $\beta = 1/\Delta t$ and $U^f = U(x,\ f\Delta t)$. At the f-th time step $t = f\Delta t \quad (f \geq 2)$ the equation (1) can be approximately rewritten as

$$\frac{\partial^2 U^f}{\partial x^2} - AU^f + BU^{f-1} - CU^{f-2} = 0 \tag{9}$$

where for the first variant (equation (5))

$$A = \beta^2 + \beta, \quad B = 2\beta^2 + \beta, \quad C = \beta^2 \tag{10}$$

for the second variant (equation (6))

$$A = \beta^2 + \frac{1}{2}\beta,\ \ B = 2\beta^2,\ \ C = \beta^2 - \frac{1}{2}\beta \tag{11}$$

and for the third variant (equation (7))

$$A=\beta^2+\frac{3}{2}\beta,\ \ B=2(\beta^2+\beta),\ \ C=\beta^2+\frac{1}{2}\beta \tag{12}$$

For equation (9) the weighted residual criterion is applied [1]

$$\int_0^1\left(\frac{\partial^2 U^f}{\partial x^2}-AU^f+BU^{f-1}-CU^{f-2}\right)U^*(\xi,x)\,dx=0 \tag{13}$$

where $\xi\in(0, 1)$ is the observation point, $U^*(\xi, x)$ is the fundamental solution and this function should fulfil the equation

$$\frac{\partial^2 U^*(\xi,x)}{\partial x^2}-AU^*(\xi,x)=-\delta(\xi,x) \tag{14}$$

where $\delta(\xi, x)$ is the Dirac function. It can be check that the following function

$$U^*(\xi,x)=\frac{1}{2\sqrt{A}}\exp(-|x-\xi|\sqrt{A}) \tag{15}$$

fulfills the equation (14).

Additionally, the function $q^*(\xi, x)$ resulting from fundamental solution is defined

$$q^*(\xi,x)=-\frac{\partial U^*(\xi,x)}{\partial x} \tag{16}$$

and it can be calculated analytically

$$q^*(\xi,x)=\frac{\text{sgn}(x-\xi)}{2}\exp(-|x-\xi|\sqrt{A}) \tag{17}$$

where sgn(·) is the sign function.

Integrating twice by parts the first component of equation (13) and taking into account the property (14) of fundamental solution one obtains

$$\left[U^*(\xi,x)\frac{\partial U^f}{\partial x}\right]_{x=0}^{x=1}-\left[U^f\,\frac{\partial U^*(\xi,x)}{\partial x}\right]_{x=0}^{x=1}-U(\xi,t^f)+ \\ +\int_0^1(BU^{f-1}-CU^{f-2})\,U^*(\xi,x)\,dx=0 \tag{18}$$

or

$$U(\xi,\ t^f) + U^*(\xi,1)\, q(1,t^f) - U^*(\xi,0)\, q(0,t^f) = \\ = q^*(\xi,1)\, U(1,t^f) - q^*(\xi,0)\, U(0,t^f) + P(\xi,t^f) \tag{19}$$

where $q(x,t^f) = -\partial U(x,t^f)/\partial x$ and

$$P(\xi,t^f) = \int_0^1 (B\,U^{f-1} - C\,U^{f-2})\, U^*(\xi,x)\, dx \tag{20}$$

For $\xi \to 0^+$ one obtains

$$U(0,\ t^f) + U^*(0,1)\, q(1,t^f) - U^*(0,\ 0)\, q(0,t^f) = \\ = q^*(0^+,1)\, U(1,t^f) - q^*(0^+,0)\, U(0,t^f) + P(0,t^f) \tag{21}$$

and for $\xi \to 1^-$ one has

$$U(1,\ t^f) + U^*(1,1)\, q(1,t^f) - U^*(1,\ 0)\, q(0,t^f) = \\ = q^*(1^-,1)\, U(1,t^f) - q^*(1^-,0)\, U(0,t^f) + P(1,t^f) \tag{22}$$

The system of equations (21), (22) can be written in the matrix form

$$\begin{bmatrix} -U^*(0,0) & U^*(0,1) \\ -U^*(1,0) & U^*(1,1) \end{bmatrix} \begin{bmatrix} q(0,t^f) \\ q(1,t^f) \end{bmatrix} = \\ \begin{bmatrix} -q^*(0^+,0)-1 & q^*(0^+,1) \\ -q^*(1^-,0) & q^*(1^-,1)-1 \end{bmatrix} \begin{bmatrix} U(0,t^f) \\ U(1,t^f) \end{bmatrix} + \begin{bmatrix} P(0,t^f) \\ P(1,t^f) \end{bmatrix} \tag{23}$$

or

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} q(0,t^f) \\ q(1,t^f) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} U(0,t^f) \\ U(1,t^f) \end{bmatrix} + \begin{bmatrix} P(0,t^f) \\ P(1,t^f) \end{bmatrix} \tag{24}$$

where $G_{11} = -G_{22} = -1/(2\sqrt{A})$, $G_{12} = -G_{21} = -1/(2\sqrt{A})\exp(-\sqrt{A})$,

$$H_{11} = H_{22} = -1/2\,,\quad H_{12} = H_{21} = 1/2\exp(-\sqrt{A})$$

This system of equations allows to find the boundary values $U(0,\ t^f)$, $U(1,\ t^f)$. Next, the values of U at the internal nodes $\xi \in (0,\ 1)$ are calculated using the formula (c.f. equation (19))

$$U(\xi, t^f) = q^*(\xi,1)\,U(1,t^f) - q^*(\xi,0)\,U(0,t^f) + \\ -U^*(\xi,1)\,q(1,t^f) + U^*(\xi,0)\,q(0,t^f) + P(\xi,t^f) \tag{25}$$

this means (c.f. formulas (15) and (17))

$$U(\xi,t^f) = \frac{1}{2}\exp(-\xi\sqrt{A})\,U(0,t^f) + \frac{1}{2}\exp[-(1-\xi)\sqrt{A}]\,U(1,t^f) + \\ + \frac{1}{2\sqrt{A}}\exp(-\xi\sqrt{A})\,q(0,t^f) - \frac{1}{2\sqrt{A}}\exp[-(1-\xi)\sqrt{A}]\,q(1,t^f) + P(\xi,t^f) \tag{26}$$

3. Analytical solution

To solve the problem (1), (2), (3) analytically, the Fourier method [3, 4] is applied. So, one assumes that

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) \tag{27}$$

where

$$U_n(x,t) = X_n(x)\;T_n(t) \tag{28}$$

and then

$$\frac{\partial U_n(x,t)}{\partial t} = X_n(x)\,T_n^{'}(t), \quad \frac{\partial^2 U_n(x,t)}{\partial t^2} = X_n(x)\,T_n^{''}(t) \\ \frac{\partial U_n(x,t)}{\partial x} = X_n^{'}(x)\,T_n(t), \quad \frac{\partial^2 U_n(x,t)}{\partial x^2} = X_n^{''}(x)\,T_n(t) \tag{29}$$

Putting (29) into (1) one obtains

$$X_n(x)\,T_n^{'}(t) + X_n(x)\,T_n^{''}(t) = X_n^{''}(x)\,T_n(t) \tag{30}$$

this means

$$\frac{T_n^{'}(t)}{T_n(t)} + \frac{T_n^{''}(t)}{T_n(t)} = \frac{X_n^{''}(x)}{X_n(x)} \tag{31}$$

It is assumed that

$$\frac{T_n^{'}(t)}{T_n(t)} + \frac{T_n^{''}(t)}{T_n(t)} = \frac{X_n^{''}(x)}{X_n(x)} = -\lambda_n^2 \tag{32}$$

where $\lambda_n \neq 0$ are the constants.

If the functions $X_n(x)$, $T_n(t)$ will be the solutions of equations

$$\frac{T_n^{'}(t)}{T_n(t)} + \frac{T_n^{''}(t)}{T_n(t)} = -\lambda_n^2, \quad \frac{X_n^{''}(x)}{X_n(x)} = -\lambda_n^2 \tag{33}$$

then these functions will fulfill the equation (31). The equations (33) can be written in the form

$$T_n^{''}(t) + T_n^{'}(t) + \lambda_n^2 T_n(t) = 0 \tag{34}$$

and

$$X_n^{''}(x) + \lambda_n^2 X_n(x) = 0 \tag{35}$$

The solution of equation (35) is following

$$X_n(x) = A_n \cos \lambda_n x + B_n \sin \lambda_n x \tag{36}$$

Taking into account the boundary conditions (2) one has

$$X_n(0) = A_n = 0, \quad X_n(1) = B_n \sin \lambda_n = 0 \rightarrow \lambda_n = n\pi \tag{37}$$

this means

$$X_n(x) = B_n \sin n\pi x \tag{38}$$

The equation (34) can be written in the form

$$T_n^{''}(t) + T_n^{'}(t) + n^2\pi^2 T_n(t) = 0 \tag{39}$$

Because

$$1 - 4n^2\pi^2 < 0 \tag{40}$$

so the solution of equation (39) is the following

$$T_n(t) = \exp(-\frac{t}{2})\,(C_n \cos \alpha_n t + D_n \sin \alpha_n t) \tag{41}$$

where

$$\alpha_n = \frac{\sqrt{4n^2\pi^2 - 1}}{2} \tag{42}$$

Finally, the function (27) has the form

$$U(x,t) = \exp(-\frac{t}{2}) \sum_{n=1}^{\infty} \sin n\pi x \, (E_n \cos \alpha_n t + F_n \sin \alpha_n t) \tag{43}$$

where: $E_n = B_n\, C_n, \quad F_n = B_n\, D_n$ are the constants.

Now, the initial conditions (3) should be taken into account, this means

$$U(x,0) = \sum_{n=1}^{\infty} E_n \sin n\pi x = U_0 \tag{44}$$

and

$$\left.\frac{\partial U(x,t)}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} (-\frac{1}{2} E_n + \alpha_n F_n) \sin n\pi x = 0 \tag{45}$$

For the arguments $x \in [-1, 1]$ function $U(x, 0)$ can be extended on the uneven function

$$\overline{U}(x,0) = \begin{cases} 0, & x = -1 \\ -U_0, & x \in (-1, 0) \\ 0, & x = 0 \\ U_0, & x \in (0,1) \\ 0, & x = 1 \end{cases} \tag{46}$$

Taking into account the expansion of this function into a Fourier series one obtains

$$E_n = 2U_0 \int_0^1 \sin n\pi x \, dx = \frac{2U_0}{n\pi}\left[1 - (-1)^n\right] \tag{47}$$

From condition (45) results that the zero function is expanded into the Fourier series. In such case

$$-\frac{1}{2} E_n + \alpha_n F_n = 0 \tag{48}$$

this means

$$F_n = \frac{U_0}{n\pi\alpha_n}\left[1 - (-1)^n\right] \tag{49}$$

Finally, one obtains

$$U(x,t)=\frac{U_0}{\pi}\exp(-\frac{t}{2})\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin n\pi x\,(2\cos\alpha_n t+\frac{1}{\alpha_n}\sin\alpha_n t) \qquad (50)$$

4. Results of computations

Application of the boundary element method using discretization in time requires a proper assumption of time step Δt. Additionally, the integral (20) should be determined with sufficient accuracy. To calculate this integral the domain [0, 1] is divided into equal M sub-domains and six-point Gauss integral formula is used. All computations have been done under the assumption that $U_0 = 1$.

In Figure 1 the curves at the point $x = 0.5$ for different approximations of time derivative (c.f. formulas (5), (6), (7)) are shown. It is visible that the results are practically the same. The calculations have been done for $\Delta t = 0.02$, $M = 100$ and for these values very good agreement with analytical solution has been observed. Figure 2 illustrates the distribution of function U for times 1, 2, 3, 4 and 5 s.

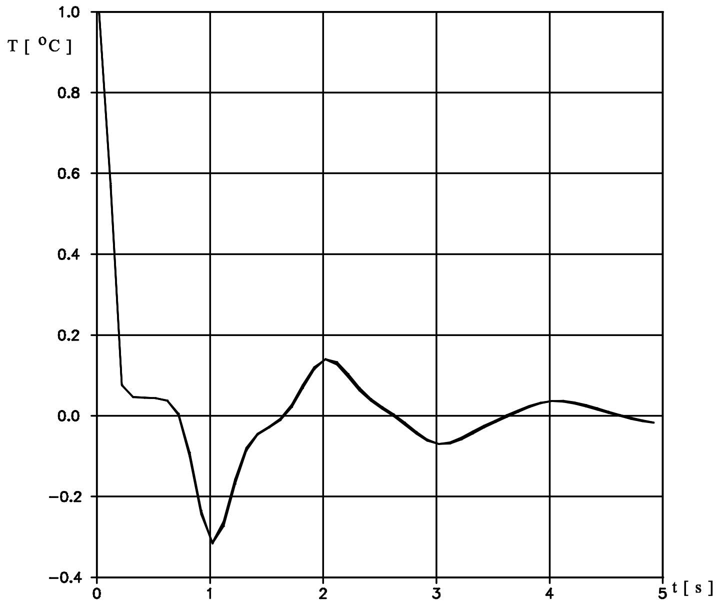

Fig. 1. Course of function U at $x = 0.5$

It should be pointed out that the influence of time step Δt on the results of computations is big. In Figure 3 the curves at the point $x = 0.25$ and for $\Delta t = 0.01,\ 0.02,\ 0.05$ ($M = 100$) are shown.

Summing up, the BEM using discretization in time constitutes the effective numerical method of hyperbolic equation solution but it requires a proper choice of time step Δt and number of internal cells *M*.

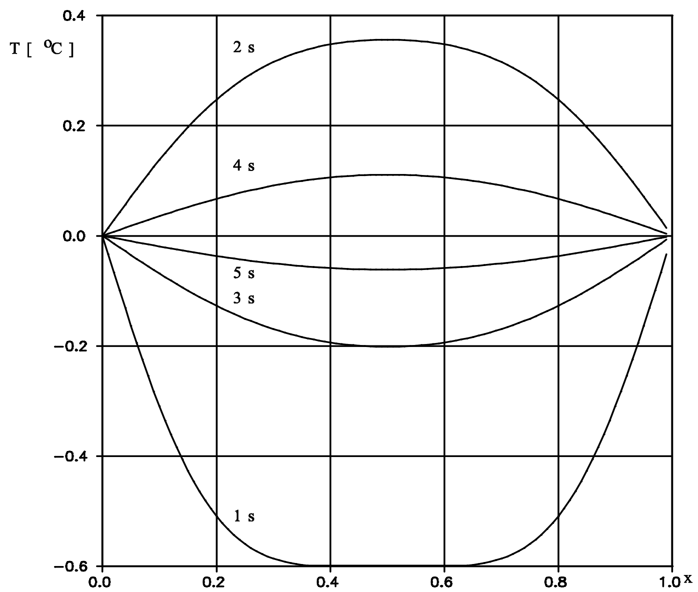

Fig. 2. Distribution of function *U*

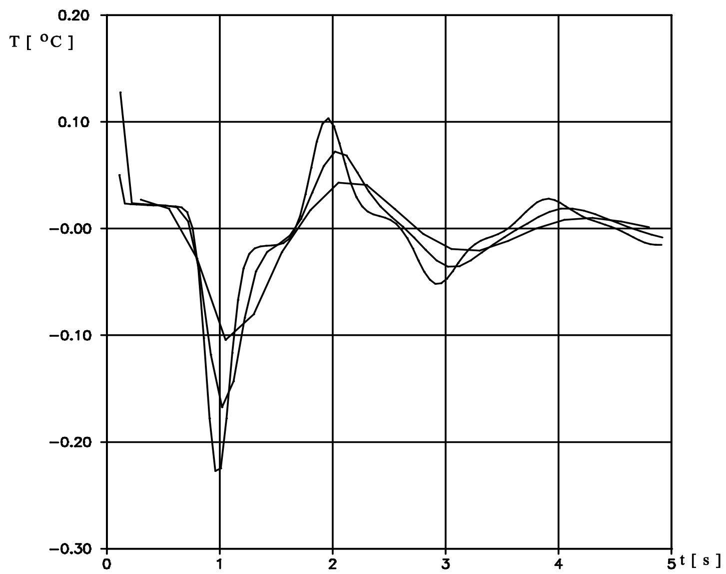

Fig. 3. Curves for different Δt at the point $x = 0.25$

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