ABOUT THE EQUIVALENCE OF THE TANGENCY RELATION OF ARCS

Tadeusz Konik

Institute of Mathematics, Czestochowa University of Technology, Poland, konik@imi.pcz.pl

Abstract. In this paper the problem of the equivalence of the tangency relation $T_l(a, b, k, p)$ of the rectifiable arcs in the generalized metric spaces is considered. Some sufficient conditions for the equivalence of this relation of the rectifiable arcs have been given here.

Introduction

Let E be an arbitrary non-empty set, and E_0 the family of all non-empty subsets of the set E. Let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$, and let l_0 be the function of the form:

$$l_0(x, y) = l(\{x\}, \{y\}) \quad \text{for} \quad x, y \in E \tag{1}$$

If we put some conditions on the function l, then the function l_0 defined by the formula (1) will be the metric of the set E. For this reason the pair (E, l) can be treated as a certain generalization of the metric space and we will call it (see [1]) the generalized metric space.

Using (1) we may define in the space (E, l), similarly as in a metric space, the following notions: the sphere $S_l(p, r)$ and the ball $K_l(p, r)$ with centre at the point p and radius r.

Let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0^+]{} 0 \tag{2}$$

By $S_l(p, r)_u$ (see [1, 2]) we will denote the so-called u-neighbourhood of the sphere $S_l(p, r)$ in the space (E, l) defined by the formula:

$$S_l(p, r)_u = \begin{cases} \bigcup\limits_{q \in S_l(p,r)} K_l(q, u) & \text{for} \quad u > 0 \\ S_l(p, r) & \text{for} \quad u = 0 \end{cases} \tag{3}$$

We say that the pair (A, B) of sets $A, B \in E_0$ is (a, b)-clustered at the point p of the space (E, l), if 0 is the cluster point of the set of all real numbers $r > 0$ such that $A \cap S_l(p, r)_{a(r)} \neq \emptyset$ and $B \cap S_l(p, r)_{a(r)} \neq \emptyset$.

Let (see [3, 4])

$$T_l(a, b, k, p) = \{(A, B):\ A, B \in E_0, \text{ the pair } (A, B) \text{ is } (a, b)\text{-clustered at the point } p \text{ of the space } (E, l) \text{ and } \frac{1}{r^k} l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0\} \tag{4}$$

If $(A, B) \in T_l(a, b, k, p)$, then we say that the set $A \in E_0$ is (a, b)-tangent of order $k > 0$ to the set $B \in E_0$ at the point p of the space (E, l).

The set $T_l(a, b, k, p)$ defined by (4) we will call the (a, b)-tangency relation of order k of sets at the point p in the generalized metric space (E, l).

We say that the tangency relation $T_l(a, b, k, p)$ is the equivalence in the set E, if is reflexive, symmetric and transitive relation in this set.

Let ρ be a metric of the set E and let A, B be arbitrary sets of the family E_0. Let us denote

$$\rho(A, B) = \inf\{\rho(x, y) : x \in A,\ y \in B\},\ \ d_\rho A = \sup\{\rho(x, y) : x, y \in A\} \tag{5}$$

By $\mathfrak{F}_\rho$ we shall denote the class of all functions l fulfilling the conditions:

1^0 $l : E_0 \times E_0 \longrightarrow [0, \infty)$,

2^0 $\rho(A, B) \leq l(A, B) \leq d_\rho(A \cup B)$ for $A, B \in E_0$.

From (1) and from the condition 2^0 we get the equality:

$$l(\{x\}, \{y\}) = l_0(x, y) = \rho(x, y) \text{ for } l \in \mathfrak{F}_\rho \text{ and } x, y \in E \tag{6}$$

From the above equality it follows that every function $l \in \mathfrak{F}_\rho$ generates on the set E the metric ρ.

In this paper the problem of the equivalence of the tangency relation $T_l(a, b, k, p)$ of the rectifiable arcs in the spaces (E, l), for the functions l belonging to the class $\mathfrak{F}_\rho$ is considered.

1. The equivalence of the tangency relation of the rectifiable arcs

Let ρ be a metric of the set E, and let A be any set of the family E_0. By A' we shall denote the set of all cluster points of the set A.

By $\widetilde{A}_p$ we will denote the class of sets of the form (see [5, 6]):

$\widetilde{A}_p = \{A \in E_0$: A is rectifiable arc with the origin at the point $p \in E$ and

$$\lim_{A\, x \to p} \frac{\ell(\widetilde{px})}{\rho(p,x)} = g < \infty\} \tag{7}$$

where $\ell(\widetilde{px})$ denotes the lenght of the arc $\widetilde{px}$ with the ends p and x.

From the considerations of the paper [4] and from Lemma 2.1 of the paper [7] follows the following corollary:

Corollary 1. *If the function a fulfils the condition*

$$\frac{a(r)}{r} \xrightarrow[r \to 0^+]{} 0 \tag{8}$$

then for an arbitrary arc $A \in \widetilde{A}_p$

$$\frac{1}{r} d_\rho(A \cap S_\rho(p,r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{9}$$

We say that the tangency relation $T_l(a,b,k,p)$ is reflexive in the set E, if

$$(A,A) \in T_l(a,b,k,p) \quad \text{for} \quad A \in E_0 \tag{10}$$

Using Corollary 1 we shall prove the following theorem:

Theorem 1. *If* $l \in \mathfrak{F}_\rho$, *functions* a, b *fulfil the condition*

$$\frac{a(r)}{r} \xrightarrow[r \to 0^+]{} 0 \quad \textit{and} \quad \frac{b(r)}{r} \xrightarrow[r \to 0^+]{} 0 \tag{11}$$

then the tangency relation $T_l(a,b,1,p)$ *is reflexive in the class* $\widetilde{A}_p$ *of the rectifiable arcs.*

Proof. From the inequality

$$d_\rho(A \cup B) \le d_\rho A + d_\rho B + \rho(A,B) \quad \text{for} \quad A, B \in E_0 \tag{12}$$

and from the fact that

$$\rho(A \cap S_\rho(p,r)_{a(r)}, A \cap S_\rho(p,r)_{b(r)}) = 0 \quad \text{for} \quad A \in E_0 \tag{13}$$

we get

$$\begin{aligned} 0 &\le l(A \cap S_\rho(p,r)_{a(r)}, A \cap S_\rho(p,r)_{b(r)}) \\ &\le d_\rho((A \cap S_\rho(p,r)_{a(r)}) \cup (A \cap S_\rho(p,r)_{b(r)})) \end{aligned}$$

$$\le d_\rho(A\cap S_\rho(p,r)_{a(r)}) + d_\rho(A\cap S_\rho(p,r)_{b(r)}))$$
$$+ \rho(A\cap S_\rho(p,r)_{a(r)}, A\cap S_\rho(p,r)_{b(r)})$$
$$= d_\rho(A\cap S_\rho(p,r)_{a(r)}) + d_\rho(A\cap S_\rho(p,r)_{b(r)}) \tag{14}$$

From the assumption (8) and from Corollary 1 it follows that

$$\frac{1}{r} d_\rho(A\cap S_\rho(p,r)_{a(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{15}$$

and

$$\frac{1}{r} d_\rho(A\cap S_\rho(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{16}$$

From (15), (16) and from the inequality (14) we get

$$\frac{1}{r} l(A\cap S_\rho(p,r)_{a(r)},\ A\cap S_\rho(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{17}$$

Hence and from the fact that the pair of arcs (A,A) is (a,b)-clustered at the point p of the space (E,l) it follows that $(A,A)\in T_l(a,b,1,p)$, what means that the tangency relation $T_l(a,b,1,p)$ is reflexive in the class $\widetilde{A}_p$.

We call the tangency relation $T_l(a,b,k,p)$ symmetric in the set E, iff

$$(A,B)\in T_l(a,b,k,p) \Rightarrow (B,A)\in T_l(a,b,k,p) \quad \text{for} \quad A,B\in E_0. \tag{18}$$

Theorem 2. *If functions a,b fulfil the condition (11) and $l\in\mathfrak{F}_\rho$, then for arbitrary arcs of the class $\widetilde{A}_p$ the tangency relation $T_l(a,b,1,p)$ is symmetric.*

Proof. We assume that $(A,B)\in T_l(a,b,1,p)$ for $A,B\in\widetilde{A}_p$ and $l\in\mathfrak{F}_\rho$. From here and from the Theorem 2 of the paper [4] on the compatibility of the tangency relation of arcs it follows that $(A,B)\in T_l(b,a,1,p)$.
Therefore

$$\frac{1}{r} l(A\cap S_\rho(p,r)_{b(r)}, B\cap S_\rho(p,r)_{a(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{19}$$

From the inequality (12) and from the assumption that $l\in\mathfrak{F}_\rho$, we get

$$0\le l(B\cap S_\rho(p,r)_{a(r)}, A\cap S_\rho(p,r)_{b(r)})$$
$$\le d_\rho((B\cap S_\rho(p,r)_{a(r)})\cup(A\cap S_\rho(p,r)_{b(r)}))$$
$$\le d_\rho(A\cap S_\rho(p,r)_{b(r)}) + d_\rho(B\cap S_{l_0}(p,r)_{a(r)})$$

$$+ \rho(A \cap S_\rho(p,r)_{b(r)}, B \cap S_\rho(p,r)_{a(r)})$$
$$\leq d_\rho(A \cap S_\rho(p,r)_{b(r)}) + d_\rho(B \cap S_\rho(p,r)_{a(r)})$$
$$+ l(A \cap S_\rho(p,r)_{b(r)}, B \cap S_\rho(p,r)_{a(r)}).$$

Hence, from (19) and from Corollary 1 of this paper it follows that

$$\frac{1}{r} l(B \cap S_\rho(p,r)_{a(r)}, A \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{20}$$

Hence and from the fact that the pair of arcs (B, A) is (a, b)-clustered at the point p of the space (E, l) it follows that $(B, A) \in T_l(a, b, 1, p)$. This means that the tangency relation $T_l(a, b, 1, p)$ is symmetric in the class of arcs $\widetilde{A}_p$.

We say that the tangency relation $T_l(a, b, k, p)$ is transitive in the set E, if for $A, B, C \in E_0$

$$[(A, B) \in T_l(a, b, k, p) \ \wedge \ (B, C) \in T_l(a, b, k, p)] \Rightarrow (A, C) \in T_l(a, b, k, p).$$

Theorem 3. *If functions a, b fulfil the condition* (11) *and $l \in \mathfrak{F}_\rho$, then for arbitrary arcs of the class $\widetilde{A}_p$ the tangency relation $T_l(a, b, 1, p)$ is transitive relation.*

Proof. We assume that $(A, B) \in T_l(a, b, 1, p)$ and $(B, C) \in T_l(a, b, 1, p)$ for arbitrary arcs $A, B, C \in \widetilde{A}_p$ and the function $l \in \mathfrak{F}_\rho$.
From here it follows that

$$\frac{1}{r} l(A \cap S_\rho(p,r)_{a(r)}, B \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{21}$$

and

$$\frac{1}{r} l(B \cap S_\rho(p,r)_{a(r)}, C \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{22}$$

From (22) and from the Theorem 2 of the paper [4] on the compatibility of the tangency relation of arcs it results

$$\frac{1}{r} l(B \cap S_\rho(p,r)_{b(r)}, C \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{23}$$

From the conditions (12), (13) and from the fact that $l \in \mathfrak{F}_\rho$, we get

$$0 \leq l(A \cap S_\rho(p,r)_{a(r)}, C \cap S_\rho(p,r)_{b(r)})$$
$$\leq d_\rho((A \cap S_\rho(p,r)_{a(r)}) \cup (C \cap S_\rho(p,r)_{b(r)}))$$

$$
\begin{aligned}
&\le d_\rho(((A\cap S_\rho(p,r)_{a(r)})\cup(B\cap S_\rho(p,r)_{b(r)}))\\
&\cup((B\cap S_\rho(p,r)_{b(r)})\cup(C\cap S_\rho(p,r)_{b(r)})))\\
&\le d_\rho((A\cap S_\rho(p,r)_{a(r)})\cup(B\cap S_\rho(p,r)_{b(r)}))\\
&+d_\rho((B\cap S_\rho(p,r)_{b(r)})\cup(C\cap S_\rho(p,r)_{b(r)}))\\
&\le d_\rho(A\cap S_\rho(p,r)_{a(r)})+d_\rho(B\cap S_\rho(p,r)_{b(r)})\\
&+\rho(A\cap S_\rho(p,r)_{a(r)},B\cap S_\rho(p,r)_{b(r)})\\
&+d_\rho(B\cap S_\rho(p,r)_{b(r)})+d_\rho(C\cap S_\rho(p,r)_{b(r)})\\
&+\rho(B\cap S_\rho(p,r)_{b(r)},C\cap S_\rho(p,r)_{b(r)})\\
&\le d_\rho(A\cap S_\rho(p,r)_{a(r)})+2d_\rho(B\cap S_\rho(p,r)_{b(r)})+d_\rho(C\cap S_\rho(p,r)_{b(r)})\\
&+l(A\cap S_\rho(p,r)_{a(r)},B\cap S_\rho(p,r)_{b(r)})+l(B\cap S_\rho(p,r)_{b(r)},C\cap S_\rho(p,r)_{b(r)})
\end{aligned}
$$

From the above inequality, from the assumptions of this theorem, from Corollary 1 of this paper and from the conditions (21) and (23) it follows that

$$\frac{1}{r}l(A\cap S_\rho(p,r)_{a(r)},C\cap S_\rho(p,r)_{b(r)})\xrightarrow[r\to 0^+]{}0 \tag{24}$$

Because the pair (A,C) of arcs of the class $\widetilde{A}_p$ is (a,b)-clustered at the point p of the space (E,l), then from here and from the condition (24) it follows that $(A,C)\in T_l(a,b,1,p)$, what means that the tangency relation $T_l(a,b,1,p)$ is transitive relation for arbitrary arcs belonging to the class $\widetilde{A}_p$ and the function funkcji $l\in\mathfrak{F}_\rho$.

From the Theorems 1-3 of this paper we get the following corollary:

Corollary 2. *If $l\in\mathfrak{F}_\rho$ and the functions a,b fulfil the condition (11), then the tangency relation $T_l(a,b,1,p)$ is the equivalence in the class $\widetilde{A}_p$ of rectifiable arcs.*

If

$$\lim_{A\ni x\to p}\frac{\ell(\widetilde{px})}{\rho(p,x)}=1 \tag{25}$$

then we say say that the rectifiable arc $A\in E_0$ with the origin at the point $p\in E$ has the Archimedean property at the point p of the metric space (E,ρ).

The class of all arcs having the Archimedean property at the point $p\in E$ we denote by A_p. Obvious is following inclusion: $A_p\subset\widetilde{A}_p$.

From here it follows that all results presented in this paper are true for the rectifiable arcs of the class A_p.

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