ANALYTICAL SOLUTION OF CATTANEO EQUATION

Maria Lupa

Institute of Mathematics and Computer Science, Czestochowa University of Technology, Poland
email: lupa@imi.pcz.pl

Abstract. The Cattaneo equation supplemented by adequate boundary and initial conditions is considered. The solution of this equation is found in analytic way. In the final part of the paper the computations basing on the solution presented are shown.

1. Cattaneo equation

The following equation is considered

$$\frac{\partial u}{\partial t}+\tau\frac{\partial^2 u}{\partial t^2}=a\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $u=u(x, t)$ is an unknown function, a, τ are the positive constants. The equation (1) is supplemented by the following boundary conditions

$$\begin{aligned} t>0, \quad x=0: \quad u\left(0, t\right)=0 \\ t>0, \quad x=L: \quad u\left(L, t\right)=0 \end{aligned} \tag{2}$$

and initial ones

$$\begin{aligned} 0<x<L, \quad t=0: \quad u\left(x, 0\right)=u_0>0 \\ 0<x<L, \quad t=0: \quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0 \end{aligned} \tag{3}$$

2. Analytical solution

The equation (1) can be written in the form

$$\frac{\partial^2 u}{\partial x^2}-\frac{\tau}{a}\frac{\partial^2 u}{\partial t^2}-\frac{1}{a}\frac{\partial u}{\partial t}=0 \tag{4}$$

To solve the problem (4), (2), (3), the Fourier method [1] is applied. So, one assumes that

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{5}$$

where

$$u_n(x,t)=X_n(x)T_n(t) \tag{6}$$

and then

$$\begin{aligned}\frac{\partial u_n(x,t)}{\partial t}&=X_n(x)T_n^{/}(t), \qquad \frac{\partial^2 u_n(x,t)}{\partial t^2}=X_n(x)T_n^{//}(t)\\ \frac{\partial u_n(x,t)}{\partial x}&=X_n^{/}(x)T_n(t), \qquad \frac{\partial^2 u_n(x,t)}{\partial x^2}=X_n^{//}(x)T_n(t)\end{aligned} \tag{7}$$

Putting (7) into (4) one obtains

$$X_n^{//}(x)T_n(t)-\frac{\tau}{a}X_n(x)T_n^{//}(t)-\frac{1}{a}X_n(x)T_n^{/}(t)=0 \tag{8}$$

this means

$$\frac{X_n^{//}(x)}{X_n(x)}-\frac{\tau}{a}\frac{T_n^{//}(t)}{T_n(t)}-\frac{1}{a}\frac{T_n^{/}(t)}{T_n(t)}=0 \tag{9}$$

or

$$\frac{X_n^{//}(x)}{X_n(x)}=\frac{\frac{\tau}{a}T_n^{//}(t)+\frac{1}{a}T_n^{/}(t)}{T_n(t)} \tag{10}$$

It is assumed that

$$\frac{X_n^{//}(x)}{X_n(x)}=\frac{\frac{\tau}{a}T_n^{//}(t)+\frac{1}{a}T_n^{/}(t)}{T_n(t)}=-\lambda_n^2 \tag{11}$$

where $\lambda_n \neq 0$ are the constants.

If the function $X_n(x)$, $T_n(t)$ will be the solutions of equations

$$\frac{X_n^{//}(x)}{X_n(x)}=-\lambda_n^2, \quad \frac{\frac{\tau}{a}T_n^{//}(t)+\frac{1}{a}T_n^{/}(t)}{T_n(t)}=-\lambda_n^2 \tag{12}$$

then these functions will fulfil the equation (9). The equations (12) can be written in the form

$$X_n^{//}(x) + \lambda_n^2 X_n(x) = 0 \tag{13}$$

and

$$\tau\, T_n^{//}(t) + T_n^{/}(t) + a\,\lambda_n^2 T_n(t) = 0 \tag{14}$$

The solution of equation (13) is following

$$X_n(x) = A_n \cos\lambda_n x + B_n \sin\lambda_n x \tag{15}$$

Taking into account the boundary conditions (2) one has

$$X_n(0) = A_n = 0, \quad X_n(L) = B_n \sin\lambda_n L = 0 \;\rightarrow \lambda_n = \frac{n\pi}{L} \tag{16}$$

this means

$$X_n(x) = B_n \sin\frac{n\pi x}{L} \tag{17}$$

The equation (14) can be written in the form

$$\tau\, T_n^{//}(t) + T_n^{/}(t) + \frac{n^2\pi^2 a}{L^2}\, T_n(t) = 0 \tag{18}$$

Under the assumption that

$$L^2 - 4\, n^2\pi^2 a\,\tau \;>\; 0 \tag{19}$$

one obtains the following solution of equation (18)

$$T_n(t) = \exp\left(-\frac{t}{2\tau}\right)\left[C_n \exp(-\alpha_n t) \;+\; D_n \exp(\alpha_n t)\right] \tag{20}$$

where

$$\alpha_n = \frac{\sqrt{L^2 - 4\, n^2\pi^2 a\,\tau}}{2L\tau} \tag{21}$$

Finally, the function (5) has a form

$$u(x, t) = \exp\left(-\frac{t}{2\tau}\right)\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left[E_n \exp(-\alpha_n t) \;+\; F_n \exp(\alpha_n t)\right] \tag{22}$$

where $E_n = B_n C_n$, $F_n = B_n D_n$ are the constants.

Because the equation (4) is linear, so the function (22) fulfils the equation (4) and boundary conditions (2).

Now, the initial conditions (3) should be taken into account, this means

$$u(x,0)=\sum_{n=1}^{\infty}(E_n+F_n)\sin\frac{n\pi x}{L}=u_0 \tag{23}$$

and

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=\sum_{n=1}^{\infty}\left[\left(-\frac{1}{2\tau}-\alpha_n\right)E_n+\left(-\frac{1}{2\tau}+\alpha_n\right)F_n\right]\sin\frac{n\pi x}{L}=0 \tag{24}$$

For the arguments $x\in[-L,L]$ function $u(x,0)$ can be extended on the uneven function

$$\bar{u}(x,0)=\begin{cases}0, & x=-L\\ -u_0, & x\in(-L,0)\\ 0, & x=0\\ u_0, & x\in(0,L)\\ 0, & x=L\end{cases} \tag{25}$$

Taking into account the expansion of this function into a Fourier series one obtains

$$E_n+F_n=\frac{2u_0}{L}\int_0^L\sin\frac{n\pi x}{L}\mathrm{d}x=\frac{2u_0}{n\pi}\left[1-(-1)^n\right] \tag{26}$$

From condition (24) results that the zero function is expanded into the Fourier series. In such case

$$\left(-\frac{1}{2\tau}-\alpha_n\right)E_n+\left(-\frac{1}{2\tau}+\alpha_n\right)F_n=0 \tag{27}$$

The solution of the system of equations (26), (27) is following

$$E_n=\frac{u_0}{n\pi}\left[1-(-1)^n\right]\frac{-1+2\alpha_n\tau}{2\alpha_n\tau} \tag{28}$$

and

$$F_n=\frac{u_0}{n\pi}\left[1-(-1)^n\right]\frac{1+2\alpha_n\tau}{2\alpha_n\tau} \tag{29}$$

Finally, one obtains

$$u(x,t) = \frac{u_0}{2\pi\tau}\exp\left(-\frac{t}{2\tau}\right)\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi x}{L}$$
$$\left[\frac{-1+2\alpha_n\tau}{\alpha_n}\exp(-\alpha_n t) + \frac{1+2\alpha_n\tau}{\alpha_n}\exp(\alpha_n t)\right] \tag{30}$$

or

$$u(x,t) = \frac{u_0}{\pi\tau}\exp\left(-\frac{t}{2\tau}\right)\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\frac{(2k-1)\pi x}{L}$$
$$\left[\frac{-1+2\alpha_{2k-1}\tau}{\alpha_{2k-1}}\exp(-\alpha_{2k-1} t) + \frac{1+2\alpha_{2k-1}\tau}{\alpha_{2k-1}}\exp(\alpha_{2k-1} t)\right] \tag{31}$$

3. Example of computations

The heat transfer in biological tissue [2] of thickness L = 0.01 m is considered. The initial temperature of tissue equals $u(x, 0) = 37^{o}$C. On the surfaces $x = 0$ and $x = L$ the Dirichlet condition $u(0, t) = u(L, t) = 0$ is accepted. The following values of parameters are assumed: thermal diffusivity $a = 2.67\cdot 10^{-7}$ m^2/s and relaxation time $\tau = 0.05$ s. The values of L, a, τ fulfil the assumed condition (19).

The temperature field in the domain analyzed is described by equation (1), the boundary conditions (2) and the initial conditions (3). The solution of the problem formulated is given by equation (31).

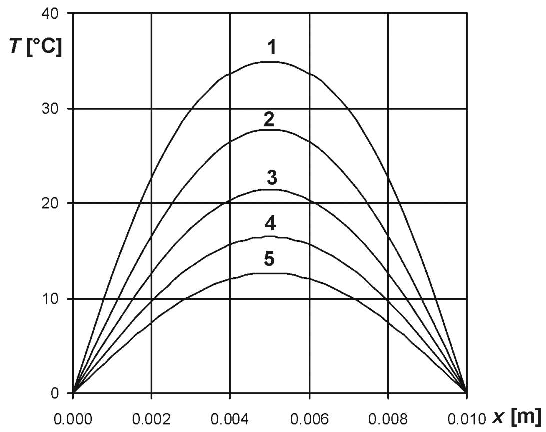

Fig. 1. Temperature distribution for times 10, 20, 30, 40 and 50 seconds

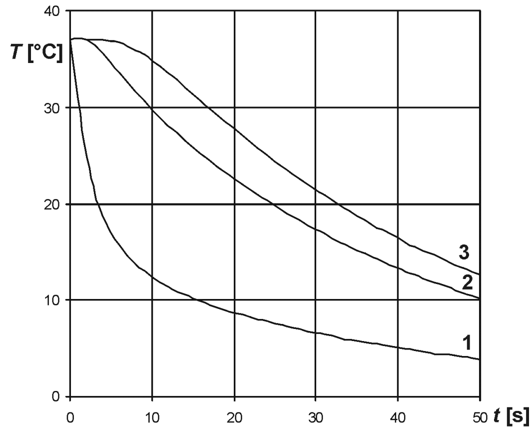

Fig. 2. Cooling curves at the points 0.002 m, 0.003 m and 0.005 m

In Figure 1 the temperature distribution for times 1 - 10, 2 - 20, 3 - 30, 4 - 40, 5 - 50 second is shown. Figure 2 presents the cooling curves at the points 1 - x = 0.002 m, 2 - x = 0.003 m and 3 - x = 0.005 m.

References

[1] Kącki E., Partial differential equations, WNT, Warsaw 1995 (in Polish).

[2] Majchrzak E., Mochnacki B., Sensitivity analysis and inverse problems in bio-heat transfer modelling, CAMES 2006, 13, 85-108.